Continuous, Nowhere Differentiable Functions

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Review

Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function.

Definition

- A function f is **continuous at a point** a if

$$\lim_{x \to a} f(x) = f(a).$$

- A function f is **continuous everywhere** if it is continuous for all points $a \in \mathbb{R}$.

Definition

- A function f is **differentiable at a point** a if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists at $x = a$.

- A function f is **differentiable everywhere** if it is differentiable at every point $a \in \mathbb{R}$.

Questions

1. How do you construct a function that is continuous everywhere but differentiable nowhere?
2. If you take a random continuous function, what is the probability that it is nowhere differentiable?

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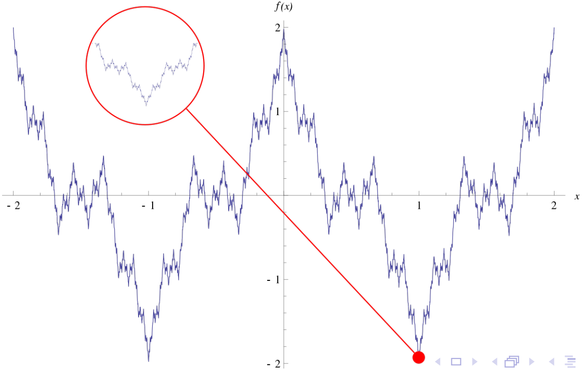
The Weierstrass Function (1875)

Theorem (Weierstrass, 1875)

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

for $0 < a < 1, ab > 1 + \frac{3\pi}{2}$, $b > 1$ *an odd integer is a continuous, nowhere differentiable function.*



Extensions

Theorem (Hertz, 1879)

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $a > 1$*,* b, p *odd integers, and* $ab > 1 + \frac{2}{3}p\pi$ *is a continuous, nowhere differentiable function.*

Theorem (Hardy, 1916)

$$f_1(x) = \sum_{k=0}^{\infty} a^k \sin(b^k \pi x), f_2 = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $0 < a < 1$*,* $b > 1$*, and* $ab \geq 1$ *are continuous, nowhere differentiable functions.*

Weierstrass-Type Functions

Definition

A **Weierstrass-type function** is of the form:

$$\mathbf{W}_{p,a,b,\boldsymbol{\theta}}(x) := \sum_{n=0}^{\infty} a^n \cos^p(2\pi b^n x + \theta_n), x \in \mathbb{R}$$

where $p \in \mathbb{N}, 0 < a < 1, ab \geq 1, \boldsymbol{\theta} := (\theta_n)_{n=0}^{\infty} \subset \mathbb{R}$

Theorem

Weierstrass-Type Functions are continuous, nowhere differentiable functions.

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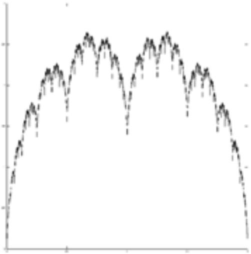
Takagi, 1903

Theorem (Takagi, 1903)

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \inf_{m\in\mathbb{Z}} |2^k x - m|$$

is a continuous, nowhere differentiable function.



Extensions

Theorem (Faber, 1907)

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k!} \inf_{m \in \mathbb{Z}} |2^{k!}x - m|$$

is a continuous, nowhere differentiable function.

Theorem (van der Waerden, 1930)

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{10^k} \inf_{m \in \mathbb{Z}} |10^k x - m|$$

is a continuous, nowhere differentiable function.

Generalization

Definition

Let $\psi(x) := \inf_{m\in\mathbb{Z}} |x - m|$. For $0 < a < 1, ab \geq 1, \boldsymbol{\theta}(x) = (\theta_n)_{n=0}^{\infty} \subset \mathbb{R}$, the **generalized Takagi–van der Waerden-Type function** is defined as

$$f(x) := \sum_{n=0}^{\infty} a^n \psi(b^n x + \theta_n), x \in \mathbb{R}$$

Theorem

The generalized Takagi–van der Waerden-Type functions are continuous, nowhere differentiable functions.

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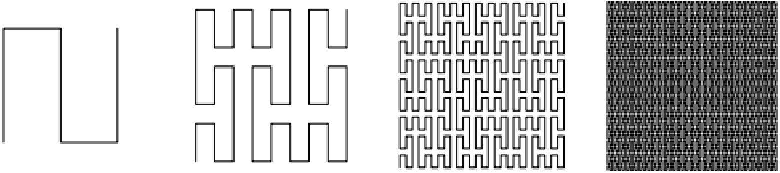
Peano's Curve, 1890

Definition

A space-filling curve is a "1 dimensional" curve that fills two-dimensional space.



(a) $n = 1$. (b) $n = 1$. (c) $n = 2$. (d) $n = 3$.

Theorem

Peano's curve is nowhere differentiable.

Hilbert's Curve, 1891

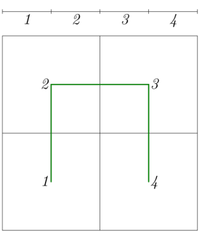


Fig. 1.

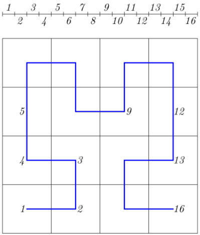


Fig. 2.

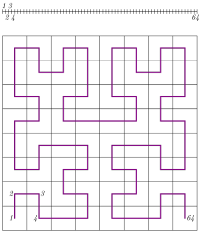
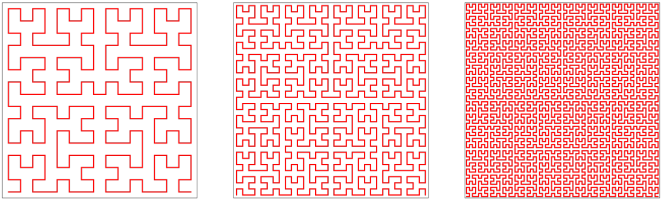


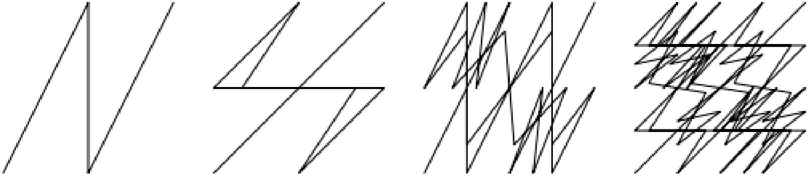
Fig. 3.



Theorem

Hilbert's function is nowhere differentiable.

Schoenberg's Function, 1938



Theorem

Schoenberg's function is nowhere differentiable.

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Wen's Function

Definition (Wen, 2002)

Wen's function is

$$f(x) := \prod_{n=1}^{\infty} (1 + a_n \sin(b_n \pi x))$$

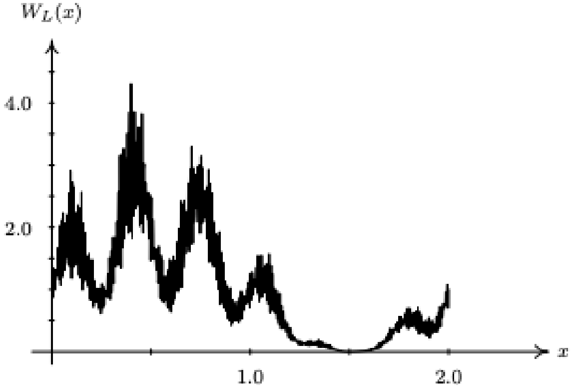
where the parameters a_n, b_n are chosen such that $0 < a_n < 1$ for all n, $\sum_{k=1}^{\infty} a_k < \infty$ and $b_n = \prod_{k=1}^{n} p_k$, and p_k is an even integer for all $k \in \mathbb{N}$. Moreover,

$$\lim_{n \to \infty} \frac{2^n}{a_n p_n} = 0.$$

Wen, Continued

Theorem

Wen's function is a continuous, nowhere differentiable function.



Schoenberg's Function

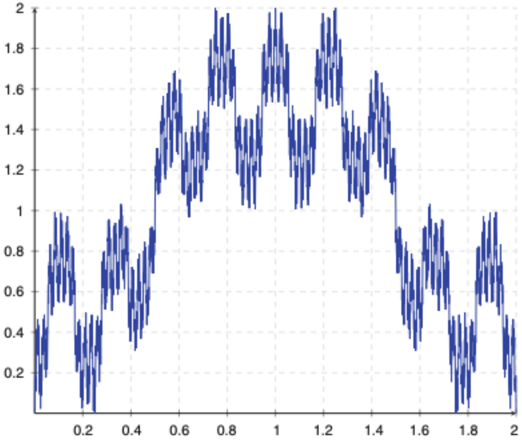
Definition (Schoenberg, 1981)

Schoenberg's function is

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n} p(3^{2n}x)$$

$$p(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{3}\right] \cup \left[\frac{5}{3}, 2\right] \\ 3x - 1 & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ 1 & \text{if } x \in \left[\frac{2}{3}, \frac{4}{3}\right] \\ 5 - 3x & \text{if } x \in \left[\frac{4}{3}, \frac{5}{3}\right] \end{cases}, p(x+2) = p(x), x \in \mathbb{R}$$

Schoenberg, Continued



Theorem

Schoenberg's function is a continuous, nowhere differentiable function.

Even More Examples

- Bolzano-type functions
- Topological approach
- Besicovitch functions

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The Answer

Theorem (Hunt, 1994)

Given a random continuous function, the probability that it is nowhere differentiable is 1.

Citations

- Continuous Nowhere Differentiable Functions by Jarnicki and Pflug
- Continuous Nowhere Differentiable Functions by Thim
- Google and Wikipedia for many images

Any Questions?
Thanks for coming!