Affine Quantum Groups and Category O

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Work in progress!

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1 Affine Lie Algebras and their Finite-Dimensional Representations

1.1 The Big Goal

Definition 1.1. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. The affine Lie algebra $\hat{\mathfrak{g}}$ is defined as the algebra of Laurent polynomials in the variable t with coefficients in $\mathfrak{g}$:*

$$\hat{\mathfrak{g}} := \mathfrak{g}[t^{\pm 1}],$$

with the Lie bracket given by:

$$[a(t), b(t)] = [a, b](t) + Res_{t=0}(a(t), b(t))\frac{dt}{t}K,$$

where $a(t), b(t) \in \mathfrak{g}[t^{\pm 1}]$, and K is a central element of the algebra.

This construction sets the stage for our main question of interest:

Problem 1.2. *What are the finite-dimensional representations of $\hat{\mathfrak{g}}$?*

The central element K acts trivially on all finite-dimensional representations of $\hat{\mathfrak{g}}$, as shown in the following lemma:

Lemma 1.3. *$K = 0$ on every finite-dimensional representation of $\hat{\mathfrak{g}}$.*

Proof. The affine Lie algebra $\hat{\mathfrak{g}} = \langle e_i, f_i, h_i \rangle$, where $i = 0, \dots, r$, is equipped with the central element $K = \sum k_i h_i$. For each root $\mathfrak{sl}_2$-triple $\langle e_i, h_i, f_i \rangle$, the commutation relation $[e_i, f_i] = h_i$ implies that the trace of h_i is zero on any finite-dimensional representation V, i.e., $\mathrm{tr}_V h_i = 0$. Thus, $\mathrm{Tr}_V(K) = 0$. Moreover, K is nilpotent on any indecomposable finite-dimensional representation. Since K is also semisimple, it follows that $K|_V = 0$. □

Thus, we reduce the problem to studying the finite-dimensional representations of the algebra $L\mathfrak{g} = \mathfrak{g}[t^{\pm 1}]$.

1.2 Tensor Products of Irreducible Representations

For each $z \in \mathbb{C}^\times$, define the evaluation map:

$$\mathrm{ev}_z : L\mathfrak{g} \to \mathfrak{g}, \quad a(t) \mapsto a(z),$$

which is surjective. For each finite-dimensional representation V of $\mathfrak{g}$, the corresponding representation of $L\mathfrak{g}$ is given by the pullback:

$$V(z) = \mathrm{ev}_z^* V.$$

In particular, the action of $a \in \mathfrak{g}$ on $V(z)$ is given by:

$$\pi_{V(z)}(a \otimes t^n) = \pi_V(a) z^n.$$

Thus, for each dominant weight $\lambda \in P_+$, there are irreducible representations $V_\lambda(z)$.

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Thus, for each dominant weight $\lambda \in P_+$, there are irreducible representations $V_\lambda(z)$.

Proposition 1.4. *The tensor product $V_{\lambda_1}(z_1) \otimes \cdots \otimes V_{\lambda_n}(z_n)$ is irreducible if and only if the z_i's are pairwise distinct.*

Proof. $\implies$: This reduces to the statement that if X, Y are irreducible representations of $\mathfrak{g}$ and both are nontrivial, then $X \otimes Y$ is reducible. To show this, we compute:

$$\dim \mathrm{Hom}_{\mathfrak{g}}(X \otimes Y, X \otimes Y) = \dim \mathrm{Hom}_{\mathfrak{g}}(X \otimes X^*, Y \otimes Y^*),$$

where $X \otimes X^* = \mathbb{C} \oplus \mathfrak{g} \oplus \cdots$ and $Y \otimes Y^* = \mathbb{C} \oplus \mathfrak{g} \oplus \cdots$. Thus, $\dim \mathrm{Hom} \geq 2$, implying that $X \otimes Y$ is reducible.

Let $a \in \mathfrak{g}$. Then:

$$a \otimes t^m \mapsto a_1 z_1^m + a_2 z_2^m + \cdots + a_n z_n^m = A(a)_m.$$

The Vandermonde determinant is:

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{pmatrix} = \prod_{i<j} (z_i - z_j) \neq 0,$$

so $a_1, a_2, \ldots, a_n$ are linear combinations of $A(a)_m$, where $m = 0, \ldots, n-1$. Therefore, $V_1(z_1) \otimes \cdots \otimes V_n(z_n)$ is irreducible.

$\Longleftarrow$: Exercise. Hint: $L\mathfrak{g} \twoheadrightarrow \mathfrak{g}^{\oplus k}$ via $(\mathrm{ev}_{z_1}, \ldots, \mathrm{ev}_{z_n})$. ∎

Problem 1.5. *Which tensor products in Proposition 1.2 are isomorphic?*

Proposition 1.6. *These tensor products are pairwise non-isomorphic.*

Proof. For $h \in \mathfrak{h} \subset \mathfrak{g}$, define $h_+(z) := -\sum_{n=0}^{\infty}(h \otimes t^{-n-1})z^n$. We can apply $h_+(z)$ to the vector $v := v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \in V_{\lambda_1}(z_1) \otimes \cdots \otimes V_{\lambda_n}(z_n)$. This vector is unique up to scaling and has weight $\lambda_1 + \cdots + \lambda_n$ for $\mathfrak{g} \subset L\mathfrak{g}$. Thus, we find:

$$h_+(z)v = \sum_{K,n} -\lambda_K(h)\left(\frac{z}{z_k}\right)^n = \sum_k \frac{\lambda_K(h)}{z - z_k},$$

which has poles at z_k with residues $-\lambda_k(h)$.

Let $n_{ik} := \lambda_k(h_i) \in \mathbb{Z}_{\geq 0}$. Then, we have:

$$h_{i+}(z)v = \left(\sum_k \frac{n_{ik}}{z - z_k}\right) v = \frac{P_i'(z)}{P_i(z)} v,$$

where $P_i(z) := \prod_k (z - z_k)^{n_{ik}}$ is the Drinfeld polynomial. □

As a consequence of these results, the highest weight of $V_{\lambda_1}(z_1) \otimes \cdots \otimes V_{\lambda_n}(z_n)$ with respect to $\mathfrak{h} \otimes \mathbb{C}[t^{-1}]$ is captured by the Drinfeld polynomials $P_1, \ldots, P_r$.

Finally, we conclude with a significant result that characterizes the finite-dimensional irreducible representations of $L\mathfrak{g}$:

Proposition 1.7. *These are the only irreducible finite dimensional representations of $L\mathfrak{g}$.*

Proof. **Claim:** I is an ideal.

Proof of Claim: Let $a, b \in \mathfrak{g}$, $q \in I$, and $p \in \mathbb{C}[t, t^{-1}]$. Then, we have the following calculation:

$$\pi_V([a, b] \otimes pq) = [\pi_V(ap), \pi_V(bq)] = \pi_V([a \otimes p, b \otimes q]) = [\pi_V(a \otimes p), \pi_V(b \otimes q)] = 0.$$

Since elements of the form $[a, b]$ span $\mathfrak{g}$, we conclude that for all $c \in \mathfrak{g}$, $\pi_V(c \otimes pq) = 0$, which implies that $pq \in I$. Therefore, $I = (q)$, where $q = \prod_{i=1}^{\alpha}(t - t_i)^{n_i}$.

The map $\mathfrak{g}[t, t^{-1}] \to \mathrm{End}_{\mathbb{C}}(V)$ factors through $\mathfrak{a} := \mathfrak{g} \otimes (\mathbb{C}[t^{\pm 1}]/(q))$, which is a finite-dimensional Lie algebra. This can be decomposed as:

$$\mathfrak{a} = \mathfrak{a}_{\text{semisimple}} \ltimes \mathrm{Rad}(\mathfrak{a}),$$

where $\mathfrak{a}_{\text{semisimple}} = \bigoplus_{i=1}^{\alpha} \mathfrak{g}$ and $\mathrm{Rad}(\mathfrak{a}) = t_1\mathfrak{g}[t]/t^{m_1} \oplus \cdots \oplus t_n\mathfrak{g}[t]/t^{m_n}$.

We now use the following standard fact:

Fact: In a finite-dimensional irreducible representation, $\mathrm{Rad} = 0$.

This implies that $m_i = 1$, so V is an irreducible representation of $\mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$. □

Remark 1.8.

- *The classification of irreducible representations extends to the case of* $\mathfrak{g} \otimes_{\mathbb{C}} A$ *for any finitely generated commutative* $\mathbb{C}$*-algebra* A*.*
- *The tensor product of simple representations is semisimple.*
- *Indecomposable representations of* $L\mathfrak{g}$ *remain an interesting topic of study.*

2 Introduction to Quantum Groups

2.1 The Basics

Consider the presentation of Kac-Moody Lie algebras, where $a_{ij} \in \mathbb{Z}$ satisfy $a_{ii} = 2$, $a_{ij} = 0 \iff a_{ji} = 0$, and $a_{ij} \leq 0$ for $i \neq j$. We assume that the Kac-Moody Lie algebras are symmetrizable, meaning there exist α_i such that $d_i a_{ij} = d_j a_{ji}$, which we fix.

The generators h_i, e_i, f_i satisfy the relations:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i,$$

along with the Serre relations:

$$(\text{ad } e_i)^{1-a_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-a_{ij}}(f_j) = 0.$$

Alternatively, the Serre relations can be omitted, and we can define $\tilde{\mathfrak{g}}(A)$ as the same Lie algebra without the Serre relations. This gives the triangular decomposition $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$, where $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are free in the generators e_i and f_i, respectively, and $\mathfrak{h} = \text{span}(h_i)$.

There exists a unique ideal $I \subset \tilde{\mathfrak{g}}(A)$, the largest graded ideal with $I \cap \mathfrak{h} = \{0\}$, such that the degree of f_i is -1, the degree of e_i is 1, and the degree of h is 0. This ideal decomposes as $I = I_+ \oplus I_-$, where $I_\pm \subset \tilde{\mathfrak{n}}_\pm$.

We define $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/I$, which admits a triangular decomposition:

$$\mathfrak{g}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-,$$

where $\tilde{\mathfrak{n}}_\pm / I_\pm$ corresponds to the respective subalgebras of $\mathfrak{g}(A)$.

Theorem 2.1 (Gabber-Kac Theorem)**.** *The ideals I_+ and I_- generate the Serre relations for e_i and f_i, respectively.*

Next, we discuss Drinfeld's quantization: Let $q \in \mathbb{C}^\times$ (not a root of unity) or work over $\mathbb{C}(q)$. We define $q_i = q^{\alpha_i}$ and $K_i = q_i^{h_i}$. Then, the quantum group $\mathcal{U}_q(\mathfrak{g}(A))$ is generated by $K_i^{\pm 1}, e_i, f_i$ with the following relations:

$$[K_i, K_j] = 0, \quad K_i e_j K_i^{-1} = q_i^{a_{ij}} e_j, \quad K_i f_j K_i^{-1} = q_i^{-a_{ij}} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad (\text{ad}_{q_i} e_i)^{1-a_{ij}} e_j = 0, \quad (\text{ad}_{q_i} f_i)^{1-a_{ij}} f_j = 0.$$

The last two relations are the quantum Serre relations, with $(\text{ad}_q x)(y) = xy - qyx$. Using the same method as before, we can bypass the Serre relations:

$$\mathcal{U}(\tilde{\mathfrak{g}}(A)) = \mathcal{U}_q(\tilde{\mathfrak{h}}_+) \otimes \mathcal{U}_q(\mathfrak{h}) \otimes \mathcal{U}_q(\tilde{\mathfrak{h}}_-).$$

We quotient by the same ideal I to get $\mathfrak{U}_q(\mathfrak{g}(A))$.

One important observation: $\mathcal{U}_q(\mathfrak{g}(A))$ is almost the Drinfeld double of $\mathcal{U}_q(\mathfrak{b}_+) = \langle K_i, e_i \rangle$ where $\mathfrak{b} = \mathfrak{h} \oplus h$. This leads to the universal R-matrix.

Proposition 2.2. *The algebra* $\mathcal{U}_q(\mathfrak{g}(A))$ *is a Hopf algebra, with comultiplication given by:*

$$\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i, \quad \Delta(K_i) = K_i \otimes K_i,$$

and the antipode given by:

$$S(e_i) = -e_i K_i^{-1}, \quad S(f_i) = -K_i f_i, \quad S(K_i) = K_i^{-1}.$$

2.2 The Quantum Double

Recall the concept of the quantum double. Let H be a finite-dimensional Hopf algebra. Its Drinfeld double $\mathcal{D}(H)$ is defined as:

$$\mathcal{D}(H) = H \otimes H^{*,\mathrm{co}},$$

where $H^{*,\mathrm{co}}$ is the dual Hopf algebra with the opposite coproduct. The algebras H and $H^{*,\mathrm{co}}$ are subalgebras of $\mathcal{D}(H)$, but they do not generally commute. Drinfeld's commutation law states that for $b \in H^{*,\mathrm{co}}$ and $a \in H$, the product is given by ba. In terms of the coproducts, we have $\Delta_3 a = a_1 \otimes a_2 \otimes a_3$ and $\Delta_3 b = b_1 \otimes b_2 \otimes b_3$. The product ba is then given by:

$$ba := (S^{-1}(a_1), b_1)(a_3, b_3) a_2 b_2.$$

Proposition 2.3. *The category Rep*$(\mathcal{D}(H))$ *is braided.*

Definition 2.4. *If* $\mathcal{C}$ *is a monoidal category, its Drinfeld center* $Z(\mathcal{C})$ *is the category whose objects are pairs* (X, φ_X)*, where* $X \in \mathcal{C}$ *and* $\varphi_X : X \otimes \bullet \xrightarrow{\sim} \bullet \otimes X$ *is an isomorphism satisfying the hexagonal identity:*

$$\begin{array}{ccc} X \otimes M \otimes N & & \\ \downarrow{\scriptstyle \varphi_{X,M} \otimes 1} & \searrow^{\varphi_{X,M\otimes N}} & \\ M \otimes X \otimes N & \xrightarrow{1_M \otimes \varphi_{X,N}} & M \otimes N \otimes X \end{array}$$

The hexagonal relation must hold for all objects in $\mathcal{C}$.

Then, $Z(\mathcal{C})$ is a monoidal category, and in fact, it is a braided monoidal category with the braiding maps $c_{X,Y} : X \otimes Y \to Y \otimes X$.

Theorem 2.5 (Drinfeld)**.** *The Drinfeld center of the representation category of a Hopf algebra is equivalent to the representation category of its Drinfeld double:*

$$Z(Rep(H)) \cong Rep(\mathcal{D}(H)),$$

where the braiding in Rep$(\mathcal{D}(H))$ *is given by the universal R-matrix* $\sum_i a_i \otimes a^i$*, where* a_i *is a basis of* H *and* a^i *is the dual basis. The braiding is explicitly given by:*

$$c_{X,Y} = \varphi_{X,Y} = P \circ R|_{X \otimes Y} : X \otimes Y \to Y \otimes X,$$

where P *denotes the permutation.*

Proposition 2.6. *For all $x \in \mathcal{D}(H)$, we have:*

$$R\Delta(x) = \Delta^{op}(x)R.$$

Proposition 2.7. *The hexagon relations imply the hexagon relations for the braiding:*

$$(\Delta \otimes 1)(R) = R_{13}R_{23},$$
$$(1 \otimes \Delta)(R) = R_{13}R_{12}.$$

2.3 Extension to Infinite Dimensional Cases

The Drinfeld double construction can be extended to infinite-dimensional cases, where the universal R-matrix R now belongs to the tensor product $\mathcal{D}(H)\widehat{\otimes}\mathcal{D}(H)$.

Example 2.8 ($\mathcal{U}_q(\mathfrak{sl}_2)$ as an almost Drinfeld double)**.** *Let $H := \mathcal{U}_q(\mathfrak{h}_+) = \langle K^{\pm 1}, e\rangle$. The relations are $KeK^{-1} = q^2 e$, and the comultiplication $\Delta(K), \Delta(e)$ are as usual. Consider the restricted dual $H^* = \mathcal{U}_q(\mathfrak{b}_-) = \langle \tilde{K}, f\rangle$, where $\tilde{K}f\tilde{K}^{-1} = q^{-2}f$, and the comultiplication $\Delta(\tilde{K}) = \tilde{K} \otimes \tilde{K}, \Delta(f) = f \otimes 1 + \tilde{K}^{-1} \otimes f$. The Drinfeld double is given by:*

$$\mathcal{D}(H) = H \otimes H^{*,co} = \langle e, f, K, \tilde{K}\rangle.$$

However, the element $C := \tilde{K}K^{-1}$ is central, so the quotient algebra $\overline{\mathcal{D}}(H) = \mathcal{D}(H)/(C-1)$ is isomorphic to $\mathcal{U}_q(\mathfrak{sl}_2)$.

The Drinfeld commutation relation is:

$$[e, f] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The universal R-matrix can be written as:

$$R = q^{\frac{h \otimes h}{2}} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{(q - q^{-1})^k}{[k]_q!} e^k \otimes f^k,$$

where $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ and $[k]_q! = [1]_q[2]_q \cdots [k]_q$.

Remark 2.9. *The universal R-matrix gives the braiding on the category $\mathcal{O}$ of $\mathcal{U}_q(\mathfrak{sl}_2)$-representations.*

The Drinfeld double construction can be extended to all Kac-Moody algebras, starting with $\mathcal{U}_q(\mathfrak{b}_+)$.

3 Representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$

3.1 Algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$

We begin by defining the algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$. Let $q \in \mathbb{C}^\times$ be not a root of unity, and let $\mathfrak{g} = \mathfrak{sl}_2$ with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The generators of the algebra are e_i, f_i, and $K_i^{\pm 1}$ where $i = 0, 1$, subject to the following relations:

$$\begin{aligned} K_i e_i K_i^{-1} &= q^2 e_i, \\ K_i f_i K_i^{-1} &= q^{-2} f_i, \\ K_i e_j K_i^{-1} &= q^{-2} e_j \quad \text{for } i \neq j, \\ K_i f_j K_i^{-1} &= q^2 f_j \quad \text{for } i \neq j, \\ K_i K_j &= K_j K_i, \\ [e_i, f_i] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ [e_i, f_j] &= 0 \quad \text{for } i \neq j, \end{aligned}$$

plus the quantum Serre relations.

Set $K = K_0 K_1$ to be central. We focus on finite-dimensional type 1 representations, where informally, $K_i = q^{h_i} w$, with h_i acting with integral eigenvalues.

Exercise 3.1. *In any finite-dimensional representation, $K = 1$.*

3.2 Evaluation and Twists by Loop Rotations

Consider the evaluation homomorphism $\mathcal{U}_q(\hat{\mathfrak{sl}}_2) \xrightarrow{\varphi} \mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ of algebras, defined by

$$\varphi(e_1) = \varphi(f_0) = e, \quad \varphi(f_1) = \varphi(e_0) = f, \quad \varphi(K_1) = \varphi(K_0^{-1}) = K.$$

Note that this is not a Hopf algebra homomorphism.

For any $\mathfrak{g}$, there exists a $\mathbb{Z}$-grading on $\mathcal{U}_q(\hat{\mathfrak{g}})$ (by energy), which gives rise to a loop rotation action $\mathbb{C}_m$ on $\mathcal{U}_q(\hat{\mathfrak{g}})$, denoted by $z \mapsto \tau_z$.

For $\mathfrak{sl}_2$ (and $\mathfrak{sl}_n$), define $\varphi_z := \varphi \circ \tau_z$. The induced map

$$\varphi_z^* : \text{Rep}\, \mathcal{U}_q(\mathfrak{sl}_2) \to \text{Rep}\, \mathcal{U}_q(\hat{\mathfrak{sl}}_2)$$

acts on a representation Y as $Y(z) = \varphi_z^* Y$ for $Y \in \text{Rep}\, \mathcal{U}_q(\mathfrak{sl}_2)$.

Remark 3.2. *For a general $\mathfrak{g}$, if W is a $\mathcal{U}_q(\hat{\mathfrak{g}})$-representation, then $W(z) := \tau_z^* W$.*

Proposition 3.3. *For all $W \in \text{Rep}\, \mathcal{U}_q(\mathfrak{g})$, the following relations hold:*

$$\begin{aligned} W(z)(u) &= w(zu), \\ (X \otimes Y)(z) &= X(z) \otimes Y(z), \\ Y(z)^* &= Y^*(z). \end{aligned}$$

3.3 Failure of Braiding/Semisimplicity

We now observe that if $V, W \in \text{Rep}\, \mathcal{U}_q(\mathfrak{sl}_2)$, then $(V \otimes W)(z) \not\simeq V(z) \otimes W(z)$ because φ is not a Hopf algebra homomorphism. Similarly, $V(z) \not\simeq V^*(z)$.

Remark 3.4. *The irreducible representations of $\mathcal{U}_q(\mathfrak{sl}_n)$ are of the form V_a with $\dim V_a = a + 1$, where $a \in \mathbb{Z}_{\geq 0}$, and give rise to $V_a(z)$.*

For $a = 1$, the representation $V_a(z)$ is expressed in matrices as:

$$\begin{aligned} e_0 &\mapsto \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}, \\ e_1 &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ f_0 &\mapsto \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}, \\ f_1 &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ K_0 &\mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \\ K_1 &\mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}. \end{aligned}$$

Exercise 3.5. *Any 2-dimensional nontrivial $\mathcal{U}_q(\mathfrak{sl}_2)$ representation is of the form $V_1(z)$ for a unique z.*

Corollary 3.6. *$V_1(z)^* \simeq V_1(w)$ for a unique w.*

Remark 3.7. *We have the relations:*

$$z = tr_{V_1(t)}(e_0 e_1)$$

and

$$\begin{aligned} w &= tr_{V_1(z)^*}(S(e_0)^* S(e_1)^*) \\ &= tr(S(e_1)S(e_0)) \\ &= tr(-e_1 K_1^{-1} \cdot (-e_0 K_0)) \\ &= q^2 tr(e_1 e_0) \\ &= q^2 z. \end{aligned}$$

This implies that $V(z)^{**} = V(q^4 z)$, so Rep $\mathcal{U}_q(\hat{\mathfrak{g}})$ is not braided.

In any rigid tensor category $\mathcal{C}$, if $X \in \mathcal{C}$, then the evaluation map $\mathrm{ev}_X : X^* \otimes X \to 1$ and coevaluation map $\mathrm{coev} : 1 \hookrightarrow X \otimes X^*$ exist.

Proposition 3.8. *If X is simple and either of these maps splits, then $X^{**} \simeq X$.*

Proof. Suppose ev_X splits. Then $X^* \otimes X \simeq Y \otimes 1$, and if $1 \overset{i}{\hookrightarrow} X^* \otimes X$, we have the commutative diagram:

$$ {}^*X \xhookrightarrow{i \otimes 1} X^* \otimes X \otimes {}^* X \longrightarrow X^* , \qquad \alpha_i : {}^*X \to X^* $$

□

Exercise 3.9. *This defines an isomorphism:*

$$ Hom(1, X^* \otimes X) \xrightarrow{\sim} Hom({}^*X, X^*), $$
$$ i \mapsto \alpha_i. $$

Since ${}^*X$ and X^* are isomorphic by Schur's lemma, we have ${}^*X \simeq X^*$.

Exercise 3.10.

$$ 1 \overset{coev}{\hookrightarrow} V_1(z) \otimes V_1(q^2 z) \to V_z(qz) \to 0 \qquad (*) $$

is nonsplit. If $Y \in Rep\, \mathcal{U}_q(\hat{\mathfrak{sl}}_2)$, then $Y|_{\mathcal{U}_q(\mathfrak{sl}_2)}$ is irreducible, so $Y \simeq V_a(z)$ for some z.

Dualize $(*)$: $0 \to V_z(qz) \to V(q^2 z) \otimes V(z) \to \mathbb{C} \to 0$, so $V(q^2 z) \otimes V(z) \not\simeq V(z) \otimes V(q^2 z)$.

However, if $w \neq q^2 z$, then $V(z) \otimes V(w)$ is irreducible and isomorphic to $V(w) \otimes V(z)$. This is defined by an R-matrix.

Remark 3.11. *For general $\mathfrak{g}$ and for all irreducible X, Y, $X(z) \otimes Y$ is irreducible and isomorphic to $Y \otimes X(z)$ for all but finitely many z.*

3.4 Double Dual

For a general $\mathfrak{g}$, if Y is a finite-dimensional representation of $\mathcal{U}_q(\hat{\mathfrak{g}})$, then $Y^{**} = Y(q^{2h^\vee})$, where $h^\vee$ is the dual Coxeter number (for $\mathfrak{sl}_2$, $Y^{**} \simeq Y(z^*)$).

Why $h^\vee$? For a q-triangular Hopf algebra (H, R) with $R = \sum_i a_i \otimes b_i$ and R invertible, the relations

$$ R\Delta(x) = \Delta^{\mathrm{op}}(x) R, \quad (\Delta \otimes 1)(R) = R_{12} R_{23}, \quad (1 \otimes \Delta)(R) = R_{13} R_{12} $$

lead to this structure.

Theorem 3.12 (Drinfeld)**.** *For $u = \sum_i S(b_i) a_i$, we have $uxu^{-1} = S^2(x)$, where $u : X \simeq X^{**}$.*

For $\mathcal{U}_q(\mathfrak{g})$, $u = vq^{2p}$, where v is the central ribbon element. For an affine Lie algebra, $\hat{p} = p + h^\vee \alpha$ gives $q^{2\hat{p}} - q^{2p} q^{2h^\vee \alpha}$. This shifts z.

3.5 Classification of Finite Dimensional Representations for $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$

Proposition 3.13. *All irreducible representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ are of the form $V_{a_1}(z_1) \otimes \cdots \otimes V_{a_n}(z_n)$.*

The key question is: when is this representation irreducible?

We can rule out cases such as $a_i = a_{i+1} = 1$, with $\frac{z_i}{z_{i+1}} = q^{\pm z}$, and similarly for $a_i = a_j = 1$ when $i - j > 1$.

To answer this question, we need a combinatorial construction: associate to each $V_a(z)$ a q^2-string $(q^{-a+1}z, q^{-a+3}z, \ldots, q^{a-1}z)$.

Definition 3.14. *A collection of strings $S_1, \ldots, S_n$ is in special position if there exist indices i, j such that $S_i \cup S_j \supsetneq S_i, S_j$ and $S_i \cup S_j$ is a q^2-string. Otherwise, we say that $S_1, \ldots, S_n$ is in general position.*

Theorem 3.15. *The tensor product $V_{a_1}(z_1) \otimes \cdots \otimes V_{a_n}(z_n)$ is irreducible if and only if the strings of factors are in general position. The product is independent of the order of the strings.*

This result generalizes the case $V(z) \otimes V(w)$, as the strings are z and w.

Proposition 3.16. *Any finite multi-subset of $\mathbb{C}^\times$ can be uniquely written as a union of strings in general position (up to permutation).*

Conclusion: the irreducible representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ correspond to multisubsets of $\mathbb{C}^\times$, which can be identified with polynomials with a nonzero constant term (up to scaling). These are called **Drinfeld polynomials**, usually normalized to have constant term 1.

3.6 R-Matrices With Spectral Parameter

The quotient $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)/(K-1)$ has a universal R-matrix, given by

$$R = \sum_i a_i \otimes a^i,$$

where $a_i \in \mathcal{U}^+$ and $a^i \in \mathcal{U}^-$. But can we understand $R|_{X \otimes Y}$ more clearly? Not in general.

Now, consider the tensor product $X(z) \otimes Y$ for a formal variable z:

$$R(z) = \sum_i \tau_z(a_i) \otimes a^i,$$

where τ contains only nonnegative powers of z. This implies that $R(z)|_{X \otimes Y} \in \mathrm{End}(X \otimes Y)[\![z]\!]$.

Theorem 3.17 (Drinfeld)**.** *For all $\mathfrak{g}$, this gives a convergent series in a neighborhood of 0, i.e., for $|z| < r$, where $r = r_{XY}$.*

The operator $R_{XY}(z) : X(z) \otimes Y \to X(z) \otimes Y$ extends meromorphically to $\mathbb{C}$.

Proposition 3.18. *This operator extends meromorphically to* $\mathbb{C}$*.*

For irreducible X and Y, the tensor product $X(z) \otimes Y$ is irreducible for generic z.

Proposition 3.19. $R_{XY}(z) = \overline{R}_{XY} f_{XY}(z)$*, where* $\overline{R}_{XY}$ *is a rational matrix function and* f_{XY} *is a scalar function. This* $\overline{R}_{XY}(z)$ *can be normalized to satisfy the following relations:*

$$\begin{aligned} \overline{R}(z)\overline{R}(z^{-1}) &= 1 \otimes 1, \\ \overline{R}_{XZ}(z)\overline{R}_{YZ}(z) &= \overline{R}_{X \otimes Y, Z}(z), \\ \overline{R}_{XZ}(z)\overline{R}_{XY}(z) &= \overline{R}_{X, Y \otimes Z}(z). \end{aligned}$$

This implies the braid relation:

$$\overline{R}^{12}_{XX}\left(\frac{z_1}{z_2}\right)\overline{R}^{13}_{XX}\left(\frac{z_1}{z_3}\right)\overline{R}^{23}_{XX}\left(\frac{z_2}{z_3}\right) = \overline{R}^{23}_{XX}\left(\frac{z_2}{z_3}\right)\overline{R}^{13}_{XX}\left(\frac{z_1}{z_3}\right)\overline{R}^{12}_{XX}\left(\frac{z_1}{z_2}\right).$$

Remark 3.20. *This structure can be thought of as commutative, similar to a vertex algebra.*

4 The BGG Category $\mathcal{O}$ and Highest Weight Structures

Notation: Let the base field be $\mathbb{C}$, G a connected reductive group, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $H \subset B \subset G$ denote the Cartan and Borel subgroups, and let $\Lambda = \mathrm{Hom}(H, \mathbb{C}^\times)$.

Definition 4.1. *Let $\nu \in \mathfrak{h}^*$, and view ν as an element of $\mathfrak{b}^*$ via the embedding $\mathfrak{h}^* \hookrightarrow \mathfrak{b}^*$. The subcategory $\mathcal{O}_\nu$ is the full subcategory in $\mathcal{U}(\mathfrak{g})$-mod$_{fg}$ consisting of all modules $\mathcal{M}$ such that the action of $\mathfrak{b}$ on $\mathcal{M}$, given by $x \cdot m = xm - \langle \nu, x\rangle m$, integrates to a B-action.*

Standard consequences:

- **Weight decomposition:** For $M \in \mathcal{O}_\nu$, we have $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, where $M_\lambda = \{m \in M \mid xm = \langle \lambda + \nu, x\rangle m \, \forall x \in \mathfrak{h}\}$ and $\dim M_\lambda < \infty$.
- The set $\{\lambda \mid M_\lambda \neq 0\}$ is bounded from above with respect to the usual order: $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1 \in \mathrm{Span}_{\mathbb{Z}_{\geq 0}}$ (i.e., $\lambda_2 - \lambda_1$ is a linear combination of positive roots).
- One can form the Verma module $\Delta_\nu(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda+\nu}$ and its simple quotient $L_\nu(\lambda)$, establishing an isomorphism $\Lambda \cong \mathrm{Irr}(\mathcal{O}_\nu)$, where $\lambda \mapsto L_\nu(\lambda)$.
- For $\mu \in \Lambda$, there is an equivalence $\mathcal{O}_\nu \cong \mathcal{O}_{\nu+\mu}$, with $L_\nu(\lambda) \mapsto L_{\mu+\nu}(\lambda - \mu)$.

4.1 And It's Siblings

The category $\mathcal{O}_\nu$ is a "finite type" category, controlled by the Hecke category associated with a subgroup of W, the Weyl group of G. There are also "affine" and potentially "double affine" analogs, which will be briefly mentioned now and hopefully elaborated on later.

Affine world: The affine world is populated by:

- Categories $\mathcal{O}$ over affine Lie algebras, which exhibit three possible behaviors: "negative", "positive", and "critical" level.
- Modular/quantum categories $\mathcal{O}$ at a root of unity.

Most of these (except for the critical affine category) are directly controlled by the affine Hecke category. Additionally, there are various geometric relatives of these categories.

Double affine world: While we haven't encountered many categories in this setting, one family that should be included is quantum categories at a root of unity, affine categories $\mathcal{O}$ at rational levels, and their modular counterparts. There are likely many more, though all of them, including the quantum affine ones, are very complicated.

4.2 Goals and Tools

Categories $\mathcal{O}$ (and their siblings) decompose into direct sums of blocks. Our goal is to establish derived equivalences between blocks of different categories $\mathcal{O}$. The most fundamental and crucial tool for this is the notion of highest weight structures, which will be discussed in the main part of this lecture.

4.3 Highest Weight Structures

Let $\mathbb{F}$ be a field and $\mathcal{C}$ be an $\mathbb{F}$-linear abelian category.

Definition 4.2. *The structure of a* ***highest weight category with finite poset*** *on $\mathcal{C}$ is given by a finite poset $\mathcal{J}$ and a collection of standard objects $\Delta(t) \in \mathcal{C}$, indexed by $\tau \in \mathcal{J}$, satisfying the following conditions:*

- $\dim_{\mathbb{F}} Hom_{\mathcal{C}}(\Delta(\tau), M) < \infty$ *for all* $\tau \in \mathcal{J}$ *and* $M \in \mathcal{C}$.
- $Hom_{\mathcal{C}}(\Delta(\tau), \Delta(\tau')) \neq 0 \implies \tau \leq \tau'$.
- $\mathbb{F} \cong End_{\mathcal{C}}(\Delta(\tau))$ *for all* $\tau \in \mathcal{J}$.
- *For every $M \in \mathcal{C}$, $M \neq 0$, there exists $\tau \in \mathcal{J}$ such that $Hom_{\mathcal{C}}(\Delta(\tau), M) \neq 0$.*
- *For every $\tau \in \mathcal{J}$, there exists a projective $P_\tau \in \mathcal{C}$ such that $P_\tau \twoheadrightarrow \Delta(\tau)$, and the kernel of the map $P_\tau \to \Delta(\tau)$ admits a finite filtration by objects $\Delta(\tau')$ with $\tau' > \tau$.*

Exercise 4.3.

1. *Let $A := End_{\mathcal{C}}\left(\bigoplus_\tau P_\tau\right)$ be finite. Then, the functor $Hom_{\mathcal{C}}\left(\bigoplus_\tau P_\tau, \cdot\right) : \mathcal{C} \to A^{opp}\text{-}mod_{fd}$ is an equivalence.*
2. *Each $\Delta(\tau)$ has a unique simple quotient, $L(\tau)$, and the map $\tau \mapsto L(\tau)$ is a bijection $\mathcal{J} \cong Irr(\mathcal{C})$.*

4.4 Infinitesimal Blocks of $\mathcal{O}$

The category $\mathcal{O}_\nu$ itself is not a highest weight category in the sense defined above, but it is the direct sum of such categories. Recall the Harish-Chandra isomorphism:

$$\mathrm{HC} : Z(\mathcal{U}(\mathfrak{g})) \cong \mathbb{C}[\mathfrak{h}^*]^{(w,\cdot)},$$

where $w \cdot \lambda = w(\lambda + p) - p$, and $z \in Z(\mathcal{U}(\mathfrak{g}))$ acts on $\Delta_\nu(\lambda)$ by $\mathrm{HC}_z(\lambda + \nu)$. Consider the equivalence relation $\sim_\nu$ on Λ: $\lambda_1 \sim_\nu \lambda_2$ if $\lambda_1 + \nu = w \cdot (\lambda_2 + \nu)$.

This gives the decomposition $\mathcal{O}_\nu = \bigoplus_\Xi \mathcal{O}_{\nu,\Xi}$, where Ξ runs over the equivalence classes for $\sim_\nu$.

Exercise 4.4. *Each $\mathcal{O}_{\nu,\Xi}$ is a highest weight category with standard objects $\Delta_\nu(\lambda)$, where $\lambda \in \Xi$, and the order on Ξ is inherited from the usual order.*

4.5 Deformation

Let R be a Noetherian ring, and let $\mathcal{C}_R$ be an R-linear abelian category. For $M \in \mathcal{C}_R$, we define a right exact functor $M \otimes_R ? : R\text{-mod}_{fg} \to \mathcal{C}_R$. We say that M is *R-flat* if this functor is exact.

The definition of a highest weight category can be generalized to $\mathcal{C}_R$. We require that $\Delta_R(\tau)$ are flat over R and modify (1) and (5) from Definition 3.2 as follows:

- $\mathrm{Hom}_{\mathcal{C}_R}(\Delta_R(\tau), M)$ is finitely generated over R.
- The kernel of the map $P_\tau \twoheadrightarrow \Delta_R(\tau)$ is filtered by objects of the form $R^{\tau'} \otimes_R \Delta_R(\tau')$ for $\tau' > \tau$, where $R^{\tau'}$ is a finitely generated projective R-module.

Exercise 4.5. *$End_{\mathcal{C}_R}(\bigoplus_\tau P_\tau)$ is a finitely generated projective R-module.*

Example 4.6. *Let $R := \mathbb{C}[\mathfrak{h}^*]^\wedge$ be the completion at O. Let ι be the composition $\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \hookrightarrow R$. Then $\mathcal{O}_{\nu,R}$ is the full subcategory in $\mathcal{U}(\mathfrak{g}) \otimes R$-mod$_{fg}$ consisting of all M such that the action of $\mathfrak{b}$ on M is given by*

$$x \cdot m = xm - (\langle \lambda, \nu \rangle + \iota(x))m,$$

and this integrates to a B-action.

The same properties hold for $\mathcal{O}_\nu$ as for $\mathcal{O}_{\nu,R}$: the weight decomposition $M = \bigoplus_\lambda M_\lambda$ with finitely generated R-modules M_λ and weights bounded from above. Verma modules $\Delta_{\nu,R}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} R_{\lambda+\nu}$ can also be formed, where $R_{\lambda+\nu} \simeq R$ with $\mathfrak{h}$ acting on R by $x \mapsto \iota(x) + \langle \lambda + \nu, x \rangle$.

Exercise 4.7. *$\mathcal{O}_\nu$ is identified with the full subcategory of $\mathcal{O}_{\nu,R}$ consisting of all objects where R acts via $R \twoheadrightarrow \mathbb{C}$.*

Remark 4.8. *Informally, one can view R as the algebra of functions on a tiny neighborhood around ν. Then, $\mathcal{O}_{\nu,R}$ is a family of categories over this neighborhood, with the fiber at a point ν' being $\mathcal{O}_{\nu'}$ (note that, strictly speaking, Spec(R) only has one $\mathbb{C}$-point).*

We can extend the infinitesimal block decomposition for $\mathcal{O}_\nu = \bigoplus_\Xi \mathcal{O}_{\nu,\Xi}$ to $\mathcal{O}_{\nu,R}$. Let $m \subset R$ denote the maximal ideal, and define:

$$\mathcal{O}_{\nu,R,\Xi} := \{M \in \mathcal{O}_{\nu,R} \mid M/m^*M \text{ is filtered by objects in } \mathcal{O}_{\nu,\Xi} \text{ for all } R\}.$$

Exercise 4.9.

1. *$\mathcal{O}_{\nu,R} = \bigoplus_\Xi \mathcal{O}_{\nu,R,\Xi}$.*
2. *$\mathcal{O}_{\nu,R,\Xi}$ is a highest weight category with standard objects $\Delta_{\nu,R}(\lambda)$, where $\lambda \in \Xi$.*

Definition 4.10. *An object in $\mathcal{C}_R$ is called **standardly filtered** if it admits a finite filtration by $R^{\tau'} \otimes_R \Delta_R(\tau')$, where $\tau' \in \mathcal{J}$ and $R^{\tau'}$ is a finitely generated projective R-module. The full subcategory of standardly filtered objects will be denoted by $\mathcal{C}_R^\Delta$.*

The following propositions require introducing "costandard" objects, which we leave for the reader to explore.

Proposition 4.11.

- *Every projective in $\mathcal{C}_R$ is in $\mathcal{C}_R^\Delta$.*

- *If $M, N \in \mathcal{C}_R^\Delta$ and $\varphi : M \twoheadrightarrow N$, then $Ker\varphi \in \mathcal{C}_R^\Delta$.*

Corollary 4.12. *For $M \in \mathcal{C}_R^\Delta$, the following are equivalent:*

- *M is projective.*
- *$Ext^1_{\mathcal{C}_R}(M, N) = 0$ for all $N \in \mathcal{C}_R^\Delta$.*
- *$Ext^1_{\mathcal{C}_R}(M, \Delta_R(\tau)) = 0$ for all $\tau \in \mathcal{J}$.*

The importance of this corollary is as follows: $\mathcal{C}_R^\Delta$ is an exact category (an additive category with a good notion of short exact sequences). The first point of Proposition 3.11 shows that the additive category of projectives $\mathcal{C}_R$-proj is contained within $\mathcal{C}_R^\Delta$, and the corollary allows us to recover $\mathcal{C}_R$-proj inside $\mathcal{C}_R^\Delta$. Once we know $\mathcal{C}_R$-proj, we can recover the abelian category $\mathcal{C}_R$.

4.6 What's Next?

Here's the "lazy approach" to understand the categories $\mathcal{O}_{\nu,\Xi}$ (the most interesting case is $\nu = 0$). We will construct a "nice" right exact functor $\mathbb{V} : \mathcal{O}_{\nu,R,\Xi} \to \underline{\mathcal{C}}_R$, where $\underline{\mathcal{C}}_R$ is a "simplified" category that roughly depends on the combinatorics of $\mathcal{O}_{\nu,R,\Xi}$. We will show that $\mathbb{V}$ is acyclic on the standard objects and fully faithful on $\mathcal{O}^\Delta_{\nu,R,\Xi}$. Therefore, we only need to understand the localizations of the categories and functors around prime ideals (which corresponds to understanding cases when ν is generic on a root hyperplane).

This approach, while implicit, provides a path to proving equivalences between different such categories.

5 The Quantum Group $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$

5.1 Drinfeld-Jimbo Presentation

Cartan Matrix:

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Generators: E_0, E_1, K_0, K_1, F_0, F_1

Relations:

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \\ K_i E_j &= q^{a_{ij}} E_j K_i \\ K_i F_j &= q^{-a_{ij}} F_j K_i \\ K_i K_i^{-1} &= K_i^{-1} K_i = 1 \\ K_i K_j &= K_j K_i \end{aligned}$$

$$\begin{aligned} E_i^3 E_j - (q^{-2} + q^2) E_i^2 E_j E_i + (q^{-2} + 1 + q^2) E_i E_j E_i^2 - E_j E_i^3 = 0 \\ F_i^3 F_j - (q^{-2} + 1 + q^2) F_i^2 F_j F_i + (q^{-2} + 1 + q^2) F_i F_j F_i^2 - F_j F_i^3 = 0 \end{aligned}$$

Coproduct:

$$\begin{aligned} \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\ \Delta(K_i) &= K_i \otimes K_i \\ \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i \end{aligned}$$

The element $K = K_0 K_1$ is central.

One can introduce an element d or q^{2d} with the following commutation relations:

$$[d, E_1] = [d, F_1] = [d, K_1] = 0, \quad [d, E_0] = E_0, \quad [d, F_0] = -F_0$$

In the non-q-deformed setting, there are two main presentations:

1. Kac-Moody presentation: f_0, h_0, e_0, f_1, h_1, e_1
2. Loop presentation: X_n^-, X_n^0, X_n^+, x for $n \in \mathbb{Z}$, where $X^+ = e$, $X^0 = h$, and $X^- = f$, with the commutation relation:

$$X_n^\epsilon, X_{n'}^{\epsilon'} = [X^\epsilon, X^{\epsilon'}]_{n+n'} + n(X^\epsilon, X^{\epsilon'}) K \delta_{n+n',0}$$

3. Presentation P_2 by h_i, e_i, f_i

The advantage of this formulation is that it provides a PBW basis.

5.2 Braid Group Action

Definition 5.1 (Lusztig's Braid Group)**.** *The braid group action on the generators is defined as follows:*

$$
\begin{aligned}
T_i(E_i) &= -F_i K_i \\
T_i(F_i) &= -K_i^{-1} E_i \\
T_i(K_j) &= K_j K_i^{-a_{ij}} \\
T_i(E_j) &= \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)} \\
T_i(F_j) &= \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}
\end{aligned}
$$

where $E_i^{(r)} = \frac{E_i^r}{[r]_q!}$.

Remark 5.2. *The braid group action can also be expressed as:*

$$T_i(E_j) = ad_{\Delta^{op}, E_i^{-a_{ij}}} E_j = \frac{1}{[-a_{ij}]_q!} ad_{q,E_i}^{-a_{ij}} E_j$$

where $ad_{q,x}(y) = xy - q^{\langle wt\ X, wt\ Y \rangle} yx$.

For example, in the case of T_1 acting on E_0, we have:

$$T_1(E_0) = E_1^{(2)} E_0 - qE_1 E_0 E_1 + q^2 E_0 E_1^{(2)}.$$

Theorem 5.3. *The operators T_i define automorphisms of the quantum group, and they satisfy the braid group relations as an algebra.*

The following additional transformation is introduced:

$$
\begin{aligned}
\tau : E_0 \mapsto E_1, \quad K_0 \mapsto K_1, \quad F_0 \mapsto F_1 \\
E_1 \mapsto E_0, \quad K_1 \mapsto K_1, \quad F_1 \mapsto F_0
\end{aligned}
$$

This defines the braid group relation:

$$Br^{ae} = \langle T_0, T_1, \tau \mid \tau T_0 \tau^{-1} = T_1,\ \tau T_1 \tau^{-1} = T_0,\ \tau^2 = e \rangle.$$

(Note that the braid relation does not hold here.)

The braid group is generated by the elements $\{T_0, T_1\}$, with the relation $E_1 T_0 E_1 = T_0 E_1 T_0$ (again, without the braid relation).

Inverse Map: The inverse map for T_i is given by:

$$\begin{aligned}
T_i^{-1}(E_i) &= -K_i^{-1}F_i,\\
T_i^{-1}(F_i) &= -E_iK_i,\\
T_i^{-1}(K_j) &= K_jK_i^{-a_{ij}},\\
T_i^{-1}(E_j) &= \sum_{r=0}^{-a_{ij}}(-1)^{r-a_{ij}}q_i^{-r}E_i^{(r)}E_jE_i^{(-a_{ij}-r)},\\
T_i^{-1}(F_j) &= \sum_{r=0}^{-a_{ij}}(-1)^{r-a_{ij}}q_i^{r}F_i^{(-a_{ij}-r)}F_jF_i^{(r)}.
\end{aligned}$$

Weyl Group: Consider the Weyl group generated by the elements s_0, s_1, and τ, with the following defining relations:

$$\langle s_0, s_1, \tau \mid \tau s_0\tau^{-1} = s_1,\ \tau s_1\tau^{-1} = s_0,\ s_1^2 = s_0^2 = \tau^2 = e\rangle.$$

Translations:

- s_0s_1 corresponds to a root translation.
- τs_0 and τs_1 correspond to weight translations.

5.3 Definition and Relations

Definition 5.4. *For $n \geq 0$, define the following elements:*

$$\begin{aligned}
E_{2+n\delta} &= (\tau T_1)^{-n}E_1,\\
E_{-2+(n+1)\delta} &= (\tau T_n)^nE_0.
\end{aligned}$$

Question: How do we define E_δ, the q-analog of $[e_1, e_0]$?

Natural choices:

$$\begin{aligned}
\mathrm{ad}_{q,E_1}(E_0) &= E_1E_0 - q^{-2}E_0E_1,\\
\mathrm{ad}_{q,E_0}(E_1) &= E_0E_1 - q^{-1}E_1E_0.
\end{aligned}$$

Lemma 5.5.

$$(\tau T_1)(E_0E_1 - q^{-2}E_1E_0) = E_0E_1 - q^{-2}E_1E_0.$$

Definition 5.6. *Define $E_{n\delta}$ by the following relation:*

$$E_{n\delta} = E_{-2+\delta}E_{2+(n-1)\delta} - q^{-2}E_{2+(n-1)\delta}E_{-2+\delta}.$$

Lemma 5.7. *The commutation relations for E_δ are:*

$$\begin{aligned}
[E_\delta, E_{2+nd}] &= [2]_qE_{2+(n+1)\delta},\\
[E_\delta, E_{-2+nd}] &= -[2]_qE_{-2+(n+1)\delta}.
\end{aligned}$$

Proof. For $n = 0$, the computation uses τT_1. $\square$

Let $\mathcal{U}_q(\widehat{\mathfrak{n}}_+)$ denote the subalgebra generated by E_0, E_1.

Corollary 5.8. *The elements $E_{2+n\delta}, E_{(n+1)\delta}, E_{-2+(n+1)\delta}$ lie in $\mathcal{U}_q(\widehat{\mathfrak{n}}_+)$ for $n \geq 0$.*

Relations:

Lemma 5.9. *The following relation holds:*

$$E_{2+(n+1)\delta}E_{2+m\delta}-q^2E_{2+n\delta}E_{2+(m+1)\delta}+E_{2+(m+1)\delta}E_{2+n\delta}-q^2E_{2+m\delta}E_{2+(n+1)\delta} = 0.$$

Definition 5.10 (Half-current)**.** *Define the half-current $e^+(z)$ by the series:*

$$e^+(z) = \sum_{n\geq 0} E_{2+n\delta}z^{-n}.$$

The relation for $e^+(z)$ is:

$$e^+(z)e^+(w)(z-q^2w) + e^+(w)e^+(z)(w-q^2z) = (1-q^2)(ze^+(w)^2 + we^+(z)^2).$$

Definition 5.11 (Half-currents)**.** *Define the half-currents $e^-(z)$ and e_δ as:*

$$e^-(z) = \sum_{n\geq 0} E_{-2+n\delta}z^{-n},$$
$$e_\delta = (q-q^{-1})\sum_{n>0} E_{n\delta}z^{-n}.$$

The following relations hold:

$$(z-q^2w)e_\delta(z)e^+(w) = (z-q^{-2}w)e^+(w)e_\delta(z),$$
$$(z-q^{-2}w)e_\delta(z)e^-(w) = (z-q^2w)e^+(w)e_\delta(z).$$

Additionally, the relation for $e^-(z)$ is:

$$e^-(z)e^-(w)(z-q^{-2}w)+e^-(w)e^-(z)(w-q^{-2}z) = (1-q^{-2})(ze^-(w)^2-we^-(z)^2).$$

The commutation relation $[E_{n\delta}, E_{m\delta}] = 0$ holds, and the following identity is true:

$$E_{-2+(p-r)\delta}E_{2+r\delta} - q^{-1}E_{2+r\delta}E_{-2+(P-r)\delta} = E_{p\delta}.$$

Theorem 5.12 (PBW)**.** *The elements*

$$\{E^{a_1}_{-2+\delta}E^{a_2}_{-2+2\delta}\cdots E^{b_1}_{\delta}E^{b_2}_{2\delta}\cdots E^{c_2}_{2+2\delta}E^{c_1}_{2+\delta}E^{c_0}_{2}\}$$

form a basis in $\mathcal{U}_q(\widehat{\mathfrak{n}}_+)$.

Remark: The elements are arranged in convex order:

$$-2-\delta < -2+2\delta < \cdots < 2\delta < \cdots < 2+\delta < 2.$$

Proof. The generating set follows from the relations, and linear independence follows from the limit $q \to 1$. □

Next, consider $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$ with an automorphism ϕ such that:

$$\begin{aligned} \phi(E_i) &= F_i, \\ \phi(F_i) &= E_i, \\ \phi(K_i) &= K_i, \\ \phi(q) &= q^{-1}. \end{aligned}$$

Definition 5.13. *The following relations hold for $\tau\phi$:*

$$\begin{aligned} \tau\phi(E_{2+n\delta}) &= (\tau T_1)^n F_0 = F_{2-(n+1)\delta}, \\ \tau\phi(E_{-2+(n+1)\delta}) &= (\tau T_1)^{-n} F_1 = F_{-2-n\delta}, \\ \tau\phi(E_{n\delta}) &= F_{-n\delta}. \end{aligned}$$

These imply the PBW property.

5.4 Full Currents

Definition 5.14. *Define the full currents X_n^+ and X_n^- by:*

$$\begin{aligned} X_n^+ &= (\tau T_1)^{-n} E_1, \\ X_n^- &= (\tau T_1)^n F_1, \quad \text{for} \quad n \in \mathbb{Z}. \end{aligned}$$

Remark 5.15. *For $n \geq 0$, we have:*

$$X_n^+ = E_{2+n\delta}, \quad X_{-n}^- = F_{-2-n\delta}.$$

However, for $n > 0$, the following expressions do not belong to $\mathcal{U}_q(\hat{\mathfrak{n}}_-)$ or $\mathcal{U}_q(\hat{\mathfrak{n}}_+)$:

$$X_n^+ = -(F_{2-n\delta} K^n) K_n^{-1}, \quad X_n^+ = -K_1 K^{-n} E_{-2+n\delta}.$$

Definition 5.16. *The full currents in z-representation are defined as:*

$$\begin{aligned} X^+(z) &= \sum_{n\in\mathbb{Z}} X_n^+ z^{-n} = e^+(z) - f^+(Kz) K_1^{-1}, \\ X^-(z) &= \sum_{n\in\mathbb{Z}} X_n^- z^{-n} = -K_1 e^-(Kz) - f^-(z). \end{aligned}$$

where

$$K_1^{-1}\psi^+(z) = 1 + (q-q^{-1})\sum_{n>0} E_{n\delta} z^{-n} = \exp\left(\sum_{n>0}(q-q^{-1})h_n z^{-n}\right),$$
$$K_1\psi^-(z) = 1 + (q^{-1}-q)\sum_{n>0} F_{-n\delta} z^{n} = \exp\left(\sum_{n>0}(q^{-1}-q)h_{-n} z^{n}\right).$$

Theorem 5.17. *The algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ has the following presentation:*

$$\mathcal{U}_q(\widehat{\mathfrak{sl}}_2) = \langle X_n^+, X_n^-, h_r, h_{-r}, K^{\pm 1}, K_1^{\pm 1} \mid n \in \mathbb{Z}, r \in \mathbb{Z}_{>0}\rangle,$$

with the following relations:

- K *is central.*
- $K_1 X_n^+ = q^x X_n^+ K_1$.
- $K_1 X_n^- = q^{-2} X_n^- K_1$.
- $[h_r, h_s] = \frac{[2r]}{r}\frac{K^r - K^{-r}}{q-q^{-1}}\delta_{r+s,0}$.
- $[h_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w)$.
- $[h_{-r}, X^+(w)] = \frac{[2r]}{r} K^{-r} w_{-r} X^+(w)$.
- $[h_r, X^-(w)] = -K^r \frac{[2r]}{r} w^r X^-(w)$.
- $[h_{-r}, X^-(w)] = -\frac{[2r]}{r} w^{-r} X^-(w)$.
- $[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}}\left(\psi^+(z)\delta\left(\frac{Kw}{z}\right) - \psi^-(w)\delta\left(\frac{w}{Kz}\right)\right)$.
- $X^+(z)X^+(w)(z-q^2w) + X^+(w)X^-(z)(w-q^2z) = 0$.
- $X^-(z)X^-(w)(z-q^{-2}w) + X^-(w)X^-(z)(w-q^{-2}z) = 0$.

where $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$.

Remark 5.18. *This construction works for q a root of unity (possibly for $q^4 \neq 1$).*

In general, the affine KM algebra is related to the $x_n^{(K)}$ structure. Let $\overline{I}$ be the set of vertices of X_n.

5.5 General Affine KM Algebra

Definition 5.19. *The algebra $\mathcal{U}^D(X_n^{(K)})$ (for simplicity, let $k=1$, $X = ADE$) is the $\mathbb{C}(q)$-algebra with:*

Generators: $X_{i,n}^+, X_{i,n}^-$, $h_{i,r}$, $h_{i,-r}$, $K_i^{\pm 1}$, $K^{\pm 1}$ *where* $i \in \overline{I}$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$, *and* $i \in I$, $n \in \mathbb{Z}$.

Relations:

$$
\begin{aligned}
K_iK_j &= K_jK_i \quad \textit{(K is central)},\\
K_iX^+_{2,n} &= q^{a_{ij}}X^+_{2,n}K_i,\\
K_iX^-_{2,n} &= q^{-a_{ij}}X^-_{2,n}K_i,\\
[h_r, X^+(w)] &= \frac{[ra_{ij}]}{r}w^rX^+(w),\\
[h_{-r}, X^+(w)] &= \frac{[ra_{ij}]}{r}K^{-r}w^{-r}X^+(w),\\
[h_r, X^-(w)] &= -K^r\frac{[ra_{ij}]}{r}w^rX^-(w),\\
[h_{-r}, X^-(w)] &= -\frac{[ra_{ij}]}{r}w^{-r}X^-(w),\\
[h_{i,r}, h_{2,s}] &= \frac{[ra_{ij}]}{r}\frac{K^r-K^{-r}}{q-q^{-1}}\delta_{r+s,0},\\
[X^+_i(z), X^-_j(w)] &= \frac{\delta_{ij}}{q-q^{-1}}\left(\psi^+_i(z)\delta\left(\frac{Kw}{z}\right)-\psi^-_i\delta\left(\frac{w}{Kz}\right)\right),\\
X^+_i(z)X^+_j(w)(z-q^{a_{ij}}w)+X^+_jX^+_i(z)(w-q^{a_{ij}}z) &= 0,\\
X^-_i(z)X^-_j(w)(z-q^{-a_{ij}}w)+X^-_jX^-_i(z)(w-q^{-a_{ij}}z) &= 0.
\end{aligned}
$$

Finally, the symmetrization over $n_1,\ldots,n_{1-a_{ij}}$ is given by:

$$
Sym\left[\sum_{p=0}^{1-a_{ij}}(-1)^p\begin{bmatrix}1-a_{ij}\\ p\end{bmatrix}_q X^+_{in_1}\cdots X^+_{in_p}X^+_{2m}X^+_{in_{p+1}}\cdots X^+_{in_{1-a_{ij}}}\right].
$$

Theorem 5.20 (Drinfeld, Beck, Damiani)**.**

$$
\mathcal{U}_q^{DJ}\simeq\mathcal{U}_q^{D}.
$$

Corollary 5.21. *Let $\overline{J}\subset\overline{I}$, then there is an embedding $\mathcal{U}_q(\hat{\mathfrak{g}}_a)\hookrightarrow\mathcal{U}_q(\hat{\mathfrak{g}}_I)$.*

In particular, if $i\in I$, then:

$$
\mathcal{U}_q(\hat{\mathfrak{sl}}_2)_i\hookrightarrow\mathcal{U}_q(\hat{\mathfrak{g}}).
$$

6 Lazy approach to categories

6.1 Recap

Let $\nu \in \mathfrak{h}^*$. We define $R := \mathbb{C}[\mathfrak{h}^*]^0$, the completion at 0. Let ι denote the composition

$$\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*] \hookrightarrow R.$$

The category $\mathcal{O}_{\nu,R}$ is the full subcategory of $\mathcal{U}(\mathfrak{g}) \otimes R\text{-mod}_{fg}$ consisting of all $\mathcal{M}$ such that the action of $\mathfrak{b}$ on $\mathcal{M}$ is given by

$$x \cdot m = xm - (\langle \nu, x\rangle + \iota(x))m,$$

and integrates to a B-action.

Remark 6.1. *Let S be an R-algebra. Analogous to the definition of $\mathcal{O}_{\nu,R}$, we can define the category $\mathcal{O}_{\nu,S}$, which is the full subcategory of $\mathcal{U}(\mathfrak{g}) \otimes S$-mod with the same integrability condition, where we replace ι by the composition $\mathfrak{h} \overset{\iota}{\hookrightarrow} R \to S$.*

Recall the equivalence $\sim_\nu$ on the root lattice Λ: $\lambda_1 \sim_\nu \lambda_2$ if $\lambda_1 + \nu \in W \cdot (\lambda_2 + p)$ for some $p \in \Lambda$. Then, we have the decomposition

$$\mathcal{O}_{\nu,R} = \bigoplus_{\Xi} \mathcal{O}_{\nu,R,\Xi},$$

where $\mathcal{O}_{\nu,R,\Xi}$ is the Serre span of the standard modules $\Delta_{\nu,R}(\lambda)$ for $\lambda \in \Xi$. Later, we will explore the possibility that each $\mathcal{O}_{\nu,R,\Xi}$ may decompose further.

Additionally, recall that $\mathcal{O}_{\nu,R,\Xi}$ is the highest weight category with poset Ξ and standards $\Delta_{\nu,R}(\lambda)$ for $\lambda \in \Xi$.

Our goal is to describe the category $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$ of standardly filtered objects.

6.2 Sub-Generic Behavior

Exercise 6.2.

1. *If $\mathcal{O}_\nu$ is not semisimple, then there exists a root α such that $\langle \nu, \alpha^\vee\rangle \in \mathbb{Z}$.*
2. *Let $\mathbb{K} = Frac(R)$. Then $\mathcal{O}_{\nu,\mathbb{K}}$ is semisimple.*

Next, consider a very generic element ν on the hyperplane $\langle \nu, \alpha^\vee\rangle = n$ (for $n \in \mathbb{Z}$). We require that each equivalence class Ξ for $\sim_\nu$ contains at most two elements, and the corresponding locus is the complement of countably many hyperplanes.

- If $|\Xi| = 1$, then $\mathcal{O}_{\nu,\Xi} \simeq \text{Vect}$.
- If $|\Xi| = 2$, then $\Xi = \{\lambda_- < \lambda_+\}$.

Proposition 6.3 (Chapter 4 in Humphreys)**.**

$$\dim Hom(\Delta_\nu(\lambda_-), \Delta_\nu(\lambda_+)) = 1.$$

Proposition 6.4. *BGG reciprocity holds: the indecomposable projective* $P(\lambda_-)$ *fits into the short exact sequence*

$$0 \to \Delta_\nu(\lambda_+) \to P_\nu(\lambda_-) \to \Delta_\nu(\lambda_-) \to 0.$$

Exercise 6.5. *Use the previous results and observations to establish an equivalence of highest weight categories between* $\mathcal{O}_{\nu,\Xi}$ *and the principal block of the category* $\mathcal{O}$ *for* $\mathfrak{sl}_2$*.*

Remark 6.6. *A similar but more technical statement holds in a deformed setup. Very informally, near a point generic with* $\langle \nu, \alpha^\vee \rangle = n$*, as described above, the category* $\mathcal{O}$ *behaves like the category* $\mathcal{O}$ *for* $\mathfrak{sl}_2$ *near 0.*

6.3 Whittaker Coinvariants

6.3.1 Construction of the Functor

Let $\mathfrak{n}^-$ denote the opposite maximal nilpotent subalgebra. Fix a non-degenerate character $\psi : \mathfrak{n}^- \to \mathbb{C}$, given by

$$\psi(x) = \left(\sum_{i=1}^{\text{rank } \mathfrak{g}} e_i, x \right).$$

Definition 6.7. *For* $M \in \mathcal{U}(\mathfrak{g})$*-mod, we define its* ***Whittaker coinvariants*** *as*

$$Wh(M) = M/\{x - \psi(x) \mid x \in \mathfrak{n}^-\}M.$$

Note that the center $Z(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ acts on $\mathrm{Wh}(M)$, giving a right exact functor

$$\mathrm{Wh} : \mathcal{U}(\mathfrak{g})\text{-mod} \to Z(\mathfrak{g})\text{-mod}.$$

For $M \in \mathcal{O}_{\nu,R}$, we have commuting R-actions, so the Whittaker functor extends to

$$\mathrm{Wh} : \mathcal{O}_{\nu,R} \to Z(\mathfrak{g}) \otimes R\text{-mod}.$$

Exercise 6.8.

1. *Show that* $Wh(\Delta_\nu(\lambda)) \simeq \mathbb{C}$ *as a vector space (hint:* $\Delta_\nu(\lambda) \overset{\mathfrak{n}^-}{\simeq} U(\mathfrak{h}^-)$*), with the action of* $Z(\mathfrak{g}) = \mathbb{C}[\mathfrak{h}]^{(W,\cdot)}$ *given by evaluation at* $\lambda + \nu$*.*
2. *Show that* $Wh(\Delta_{\nu,R}(\lambda)) \simeq R$ *as right* R*-modules, with* $Z(\mathfrak{g}) = \mathbb{C}[\mathfrak{h}^+]^{(W,\cdot)}$ *acting via* $\mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)} \hookrightarrow S(\mathfrak{h}) \overset{(\sim)}{\hookrightarrow} R = S(\mathfrak{h})^{\Lambda_0}$*, with the map*
$$(*) : x \in \mathfrak{h} \mapsto \iota(x) + \langle \lambda + \nu, x \rangle \in \mathbb{R}.$$
3. *Show that* Wh *is acyclic on* $\Delta_1(\lambda)$ *and* $\Delta_{\nu,R}(\lambda)$*.*

6.3.2 Faithfulness

We now aim to prove the following result:

Theorem 6.9.

1. *The functor $Wh : \mathcal{O}_\nu \to Vect$ is faithful (injective on Homs between standardly filtered objects).*
2. *The functor $Wh : \mathcal{O}^{\Delta}_{\nu,R} \to Z(\mathfrak{g}) \otimes R$-mod is fully faithful (bijective on Homs between standardly filtered objects).*

There are two main approaches to proving (1): geometric and representation-theoretic. We will adopt the geometric approach, which requires a connection between category $\mathcal{O}$ and Whittaker modules.

Proof of (1). Consider the algebra $U_\hbar(\mathfrak{g}) = T(\mathfrak{g})[\hbar]/(x \otimes y - y \otimes x - \hbar[x, y])$, which is the Rees algebra of $\mathcal{U}(\mathfrak{g})$ under the PBW filtration. This is a graded flat $\mathbb{C}[\hbar]$-algebra, with the quotient map $U_\hbar(\mathfrak{g})/(\hbar) \xrightarrow{\sim} S(\mathfrak{g})$.

Next, consider the category $\mathcal{O}_{\nu,\hbar}$ of graded finitely generated $U_\hbar(\mathfrak{g})$-modules that are equipped with a rational B-action such that:

- The map $U_\hbar(\mathfrak{g}) \otimes M \to M$ is B-equivariant.
- For each $x \in \mathfrak{b}$, we write $x_M \in \mathrm{End}(M)$ for the element corresponding to the differential of the B-action. Then we have $\hbar x_M m = xm - \hbar\langle v, x\rangle m$ for all $x \in \mathfrak{b}$ and $m \in M$.

In particular, $M/(\hbar - 1)M \in \mathcal{O}_\nu$, while $M/\hbar M \in \mathrm{Coh}^{B\times\mathbb{G}_m}[(\mathfrak{g}/\mathfrak{b})^*]$.

We still have the functor $\mathrm{Wh} : \mathcal{O}_{\nu,\hbar} \to \mathbb{C}[\hbar]$-mod, as defined earlier. Moreover, $\mathrm{Wh}(M)$ is naturally graded. Namely, let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be the principal grading.

We can define a modified grading on $\mathcal{U}(\mathfrak{g})$ by putting $\mathfrak{g}(i)$ in degree $i+1$ (while $\hbar$ is still in degree 1). Then $\{x - \psi(x) \mid x \in \mathfrak{h}^-\}$ is homogeneous, and we modify the grading on any T-equivariant graded $\mathcal{U}_\hbar(\mathfrak{g})$-module, N, to make it graded with respect to this modified grading.

This upgrades Wh to a functor

$$\mathcal{O}_{\nu,\hbar} \to \mathbb{C}[\hbar]\text{-grmod}.$$

Consider the full subcategory of $\mathcal{O}_{\nu,\hbar}$ consisting of objects where $\hbar$ acts by 0. This subcategory is identified with $\mathrm{Coh}^{B\times\mathbb{G}_m}((\mathfrak{g}/\mathfrak{b})^*)$. The restriction of Wh to this subcategory is given by

$$\mathrm{Wh}(N) \mapsto N_\psi,$$

the fiber at ψ, where we view ψ as a point of $(\mathfrak{g}/\mathfrak{b})^*$.

Exercise 6.10.

1. *Show that $B\psi$ is dense in $(\mathfrak{g}/\mathfrak{b})^*$.*
2. *Deduce that the functor $M \mapsto M_\psi$ is fully faithful on the full subcategory of $Coh^{B\times\mathbb{G}_m}((\mathfrak{g}/\mathfrak{b})^*)$ consisting of torsion-free modules.*

Now, for $\lambda \in \Lambda$ and $m \in \mathbb{Z}$, we consider the Verma module $\Delta_{\nu,\hbar}(\lambda, m) = \mathcal{O}_\hbar$ with highest weight vector of weight λ in degree m. The following exercise completes the proof:

Exercise 6.11.

1. *Use (2) of Exercise 1 to show that Wh is faithful on the full subcategory of $\mathcal{O}_{\nu,\hbar}$ whose objects are $\Delta_{\nu,\hbar}(\lambda, m)$.*
2. *Deduce that Wh is faithful on the full subcategory of $\mathcal{O}_\nu$ with objects $\Delta_\nu(\lambda)$ (hint: use the Rees construction) and hence on $\mathcal{O}_\nu^\Delta$.*

□

Sketch of proof of (2). Let $\mathbb{K} = \mathrm{Frac}(R)$. As noted in Section 0, we can consider the $\mathbb{K}$-linear category $\mathcal{O}_{\nu,\mathbb{K}}$, which is semisimple by Exercise 1 in Section 1. Next, it is straightforward to show that $\mathrm{Wh} : \mathcal{O}_{\nu,\mathbb{K}} \to Z(\mathfrak{g}) \otimes \mathbb{K}$-mod is fully faithful. The following formal exercise completes the proof:

Exercise 6.12. *Deduce that $Wh : \mathcal{O}_{\nu,R} \to Z(\mathfrak{g}) \otimes R$-mod is fully faithful from the facts:*

- *$Wh : \mathcal{O}_\nu^\Delta \to Vect$ is faithful,*
- *$Wh : \mathcal{O}_{\nu,\mathbb{K}}^\Delta \to Z(\mathfrak{g}) \otimes \mathbb{K}$-mod is fully faithful.*

Hint: Prove that $Wh : \mathcal{O}_{\nu,S} \to Z(\mathfrak{g}) \otimes S$-mod is faithful for S being any localization of any quotient of R.

□

Remark 6.13. *The category $Coh^{B\times\mathbb{G}_m}((\mathfrak{g}/\mathfrak{b})^*)$, which appeared in the proof of (1), is an example of a category from the affine world.*

Exercise 6.14 (Premium)**.** *Show that $Wh : \mathcal{O}_\nu \to Vect$ is exact.*

7 Description of $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$

7.1 Recap

Let $\nu \in \mathfrak{b}^*$, $R = \mathbb{C}[\mathfrak{h}^*]^{\Lambda_0}$, $\mathbb{K} = \mathrm{Frac}(R)$, and $\iota : \mathfrak{h} \hookrightarrow R$ be the natural inclusion. Earlier, we constructed a functor $\mathrm{Wh} : \mathcal{O}_{\nu,R} \to Z(\mathfrak{g}) \otimes R\text{-mod}$, and demonstrated that it is faithful on $\mathcal{O}^{\Delta}_{\nu}$ and fully faithful on $\mathcal{O}^{\Delta}_{\nu,R}$.

Our goal now is to describe the full subcategory $\mathrm{Wh}(\mathcal{O}^{\Delta}_{\nu,R,\Xi}) \subset Z(\mathfrak{g}) \otimes R$-mod. An additional ingredient is the analysis of subgeneric behavior, which was discussed earlier.

7.2 Target Category

Recall that $\mathrm{Wh}(\Delta_{\nu,R}(\lambda)) \simeq R$, where $Z(\mathfrak{g})$ acts via the following diagram:

$$
\begin{array}{ccccccc}
Z(\mathfrak{g}) & \simeq & \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)} & \hookrightarrow & S(\mathfrak{h}) & \longrightarrow & R \\
 & & & & \cup & & \Cup \\
 & & & & \mathfrak{h} \in x & \longmapsto & \iota(x) + \langle \lambda + \nu, x \rangle
\end{array}
$$

In particular, let $\mathfrak{m}_{\Xi} \subset Z(\mathfrak{g})$ denote the maximal ideal corresponding to $\lambda + \nu$ for $\lambda \in \Xi$ (which is the same for all such λ). We see that

$$\mathfrak{m}_{\Xi} \, \mathrm{Wh}(\Delta_{\nu,R}(\lambda)) \subset \mathrm{Wh}(\Delta_{\nu,R}(\lambda)) \cdot m.$$

Since every object $M \in \mathcal{O}_{\nu,R,\Xi}$ has a finite filtration by quotients of $\Delta_{\nu,R}(\lambda)$ for $\lambda \in \Xi$, it follows that $m_{\Xi}^{R} \, \mathrm{Wh}(M) \subset \mathrm{Wh}(M) \cdot m$, where k is the length of the filtration.

Hence, $Z(\mathfrak{g})$ acts on $\mathrm{Wh}(M)$ canonically, and this action extends to the completion $Z(\mathfrak{g})^{\Lambda_{\Xi}}$ at m_{Ξ}.

Now, consider the structure of $\Xi = W \cdot (\lambda + \nu) \cap \nu + \Lambda$, where Λ is the root lattice. Note that for $\lambda \in \Lambda$, we have the following equivalence:

$$w \cdot (\lambda + \nu) \in \nu + \Lambda \quad \Leftrightarrow \quad w\nu - \nu \in \Lambda \quad \Leftrightarrow \quad w \in \mathrm{im}[\mathrm{Stab}_{W \ltimes \Lambda}(\nu)] \subset W.$$

Since $W \ltimes \Lambda$ is a reflection group, the stabilizer Stab and its image are reflection subgroups, which we denote by $W_{[\nu]}$. Every Ξ is a $W_{[\nu]}$-orbit, and hence contains a unique element $\lambda^- = \lambda^-_{\Xi}$ such that $\lambda^- + \nu$ is anti-dominant for $W_{[\nu]}$ with respect to the positive root system of W. Let $W^0 = \mathrm{Span}_{W_{[\nu]}}(\lambda^- + \nu)$.

It follows that $Z(\mathfrak{g})^{\Lambda_{\Xi}}$ is isomorphic to R^{W^0}. More precisely, we have the following important elementary result:

Exercise 7.1. *1. The action of $Z(\mathfrak{g})^{\Lambda_\Xi}$ on $Wh(\Delta_{\nu,R}(\lambda^-)) \simeq R$ is via an embedding $Z(\mathfrak{g})^{\Lambda_\Xi} \hookrightarrow R$ whose image is R^{W^0}. Denote this embedding by η.*

2. The action of $Z(\mathfrak{g})^{\Lambda_\Xi}$ on $Wh(\Delta_{\nu,R}(w\lambda^-))$ for $w \in W_{[\nu]}$ is via $w \circ \eta$, where w is viewed as an automorphism of R.

Next, we must shrink the target category, which involves a technical step:

Exercise 7.2. *Use (2) and the fact that $\mathcal{O}_{\nu,R,\Xi}$ is a highest weight category to show the existence of an ideal $I \subset R^{W^0} \otimes R$ such that:*

1. *$Wh(\mathcal{O}_{\nu,R,\Xi}) \subset (R^{W^0} \otimes R)/I$-mod,*
2. *$R^{W^0} \otimes R/\sqrt{I} = R^{W^0} \otimes_{R^W} R$, implying that $R^{W^0} \otimes R/I$ is finitely generated over R, and that I is generically radical. This implies that $[R^{W^0} \otimes R/I] \otimes_R \mathbb{K} \simeq \mathbb{K}^{\otimes |W_\nu/W^0|}$.*

A more precise and elegant statement can be made (especially by Soergel):

Proposition 7.3. *We can take $(R^{W^0} \otimes R)/I = R^{W^0} \otimes_{R^W} R$.*

Conclusion: We have established that the target category for Wh, as well as the images of standard modules, are determined by a reflection group $W_{[\nu]}$ and its parabolic subgroup W^0 (and the corresponding reflection representation of $W_{[\nu]}$).

Later, we will demonstrate that a similar result holds for $\mathrm{Wh}(\mathcal{O}^{\Delta}_{\nu,R,\Xi})$.

7.3 Abstract nonsense

Suppose:

- R is a regular complete Noetherian local ring $\mathbb{F} := R/m$.
- $\mathcal{C}_R$ is a highest weight category over R.
- $\underline{\mathcal{C}}_R$ is an R-linear abelian category equivalent to $\underline{A}_R$-mod$_{\mathrm{fg}}$, where $\underline{A}_R$ is an associative R-algebra that is a finitely generated R-module.
- $\pi_R : \mathcal{C}_R \to \underline{\mathcal{C}}_R$ is a right exact R-linear functor.

Note that π_R is given by $B_R \otimes_{A_R} \cdot$, where B_R is an $\underline{A}_R$-A_R-bimodule (with $\mathcal{C}_R \simeq A_R$-mod$_{\mathrm{fg}}$). For an R-algebra S, we can then consider the following:

$$A_S := S \otimes_R A_R, \quad \underline{A}_S := S \otimes_R \underline{A}_R, \quad \mathcal{C}_S = A_S\text{-mod}_{\mathrm{fg}}, \quad \underline{\mathcal{C}}_S, \quad \pi_S := B_S \otimes_{A_S} \cdot, \ldots$$

The functor π_R is supposed to satisfy the following conditions:

1. $\mathcal{C}_{\mathbb{K}}, \underline{\mathcal{C}}_{\mathbb{K}}$ are split semisimple $\mathbb{K}$-linear categories, and $\pi_{\mathbb{K}} : \mathcal{C}_{\mathbb{K}} \xrightarrow{\sim} \underline{\mathcal{C}}_{\mathbb{K}}$ is an equivalence.
2. $\pi_R(\Delta_R(\tau))$ is flat over R and $L_i \pi_R(\Delta_R(\tau)) = 0$ for all $i > 0$, for all τ.

3. $\pi_{\mathbb{F}}$ is faithful on $\mathcal{C}_{\mathbb{F}}^{\Delta}$.

We call such a functor π_R a **Rouquier-Soergel functor**. For example, take $\mathcal{C}_R = \mathcal{O}_{\nu,R,\Xi}$, let $\underline{\mathcal{C}}_R = R^{W^0} \otimes R/I$-mod, and $\pi_R = \mathrm{Wh}$.

Now we discuss the consequences of the axioms.

Here are consequences of the axioms (a)-(c). First, by conditions (a)-(c), we have that π_R is fully faithful on $\mathcal{C}_R^{\Delta}$. The Yoneda description of Ext^1 then implies that $\pi_R : \mathcal{C}_R^{\Delta} \hookrightarrow \underline{\mathcal{C}}_R$ is injective on Ext^1's.

Moreover, we can recover Ext^1 between objects of $\mathcal{C}_R^{\Delta}$. Since $\underline{\mathcal{C}}_{\mathbb{K}}$ is semisimple, there exists a divisor $D \subset \mathrm{Spec}(R)$ such that, for $\underline{M}_R, \underline{N}_R \in \mathcal{C}_R$ that are flat over R, the Ext group $\mathrm{Ext}^1_{\underline{\mathcal{C}}_R}(\underline{M}_R, \underline{N}_R)$ is supported on D. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \subset R$ be the prime ideals corresponding to the components of D. Define $L(R) := \bigoplus_{i=1}^k R_{\mathfrak{p}_i}$ as the localization of R. We have the maps

$$\pi_R : \mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R) \hookrightarrow \mathrm{Ext}^1_{\underline{\mathcal{C}}_R}(\pi_R M_R, \pi_R N_R)$$

for all $M_R, N_R \in \mathcal{C}_R^{\Delta}$, and similarly for $\pi_{L(R)}$.

We also have natural maps induced by the localization functor L:

$$L : \mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R) \to \mathrm{Ext}^1_{\mathcal{C}_{L(R)}}(M_{L(R)}, N_{L(R)}),$$

and similar maps for $\underline{\mathcal{C}}_R$.

Now we describe $\mathrm{Ext}^1_{\mathcal{C}_R}(M_R, N_R)$:

Theorem 7.4. *The following diagram is Cartesian:*

$$\begin{array}{ccc} Ext^1_{\mathcal{C}_R}(M_R, N_R) & \xrightarrow{\quad L \quad} & Ext^1_{\mathcal{C}_L}(M_L, N_L) \\ \Big\downarrow{\scriptstyle \pi_R} & & \Big\downarrow{\scriptstyle \pi_{L(R)}} \\ Ext^1_{\mathcal{C}_R}(\underline{M}_R, \underline{N}_R) & \xrightarrow[L]{\quad} & Ext^1_{\mathcal{C}_{L(R)}}(\underline{M}_{L(R)}, \underline{N}_{L(R)}) \end{array}$$

where $\underline{M}_R := \pi_R(M_R)$, and similarly for $\underline{N}_R$, with $M_R, N_R \in \mathcal{C}_R^{\Delta}$.

Note that the bottom arrow depends only on $\mathcal{C}_R$, while the right arrow depends only on the inclusions $\mathcal{C}^{\Delta}_{R_{\mathfrak{p}_i}} \hookrightarrow \underline{\mathcal{C}}_{R_{\mathfrak{p}_i}}$. Informally, once we have an RS functor, $\mathcal{C}_R$ can be recovered from the target category and its subgeneric behavior.

7.4 Back to $\mathcal{O}$

We now provide a proof of the following result due to Soergel:

Theorem 7.5. *A regular block of $\mathcal{O}_{\nu,\Xi}$ (one with $W^0 = \{1\}$) is determined up to an equivalence of highest weight categories by $W_{[\nu]}$.*

There is an immediate generalization to singular blocks, which can be proved similarly (left as an exercise).

Sketch of proof. For $w \in W_{[\nu]}$, we define R_w as the R-bimodule R, where R acts from the right by $r \mapsto r$ and from the left by $r \mapsto w(r)$, so that $\mathrm{Wh}(\Delta_R(w \cdot \lambda)) = R_w$.

Exercise 7.6. *$Ext^1_{\mathcal{C}_R}(R_u, R_v) \neq 0 \implies u^{-1}w = 1$ or s_α. Moreover, in the latter case, this R-bimodule is $R_w/R_w\alpha \simeq R_u/R_u\alpha$.*

Using this exercise, we can take $D = \bigcup \mathrm{Spec}(R/(\alpha))$, where the union is over the positive roots of $W_{[\nu]}$. Consider the corresponding localization $\mathcal{O}^{\Delta}_{\nu,R_{(\alpha)},\Xi}$. This splits into $|W|/2$ blocks, and so does $\underline{\mathcal{C}}_{R_{(\alpha)}}$. The blocks correspond to s_α-orbits in Ξ. The functor $\pi_{R_{(\alpha)}}$ acts between blocks. Let $\mathbb{F}$ be the residue field of $R_{(\alpha)}$.

Exercise 7.7. *Let $\lambda \in \Xi$ satisfy $\langle \lambda + \rho, \alpha^\vee \rangle < 0$. Then*

$$Ext_{\mathcal{O}_{\nu,R_{(\alpha)}}}(\Delta_{R_{(\alpha)}}(\lambda), \Delta_{R_{(\alpha)}}(s_\alpha \cdot \lambda)) \neq 0,$$

and hence Wh induces an isomorphism $Ext_{\underline{\mathcal{C}}_{R_{(\alpha)}}}(R_{w,(\alpha)}, R_{ws_\alpha,(\alpha)}) = \mathbb{F}_\alpha$ for $\lambda = w \cdot \lambda^-$.

This implies the following characterization of the image of the block: it consists of all objects M such that the short exact sequence

$$0 \to R^{\oplus ?}_{ws_\alpha,(\alpha)} \to M \to R^{\oplus ?}_{w,(\alpha)} \to 0$$

(with $w \in W_{[\nu]}$ shortest in its s_α-coset) holds. Informally, we recover all extensions in the "right direction" and none in the "wrong direction".

Thus, the result in Section 2 shows that Ext^1 between two objects in $\mathrm{Wh}(\mathcal{O}^{\Delta}_{\nu,R,\Xi})$ can be fully recovered inside their Ext^1 in $\underline{\mathcal{C}}_R$, without directly needing to know $\mathcal{O}^{\Delta}_{\nu,R,\Xi}$. The completion of the proof is left as an exercise. □