Fiebig's Correspondence Between Soergel Bimodules and Braden-MacPherson Sheaves

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Abstract

This expository paper provides an introduction to the theories of Soergel bimodules and Braden-MacPherson sheaves, culminating in a brief sketch of Fiebig's correspondence between the two seemingly distinct frameworks. We begin with a brief introduction to Coxeter groups and Hecke algebras. Eventually, this leads to Soergel's construction of bimodules that categorify the Hecke algebra, offering an algebraic pathway to understanding Kazhdan-Lusztig polynomials. Afterwards, we introduce moment graphs and Braden-MacPherson sheaves, combinatorial objects inspired by intersection cohomology that provide a local perspective. Finally, we dive into Fiebig's correspondence that demonstrates the equivalence between additive categories of Soergel bimodules and Braden-MacPherson sheaves.

Contents

1 Introduction

1.1 Motivation

The representation theory of semisimple Lie algebras has long been influenced by deep connections to geometry and combinatorics. A central example is the celebrated Kazhdan-Lusztig conjecture, which states that there is a correspondence:

Jordan-Hölder multiplicities of simple modules in Verma modules	$\leftrightarrow$	Kazhdan-Lusztig polynomials evaluated at $v = 1$

Although formulated as a purely algebraic statement in the 1970s, the conjecture was not proved until 1981, when Beilinson–Bernstein and Brylinski–Kashiwara independently established it using a wide array of geometric techniques. These proof techniques have heavily influenced the development of geometric representation theory in the past few decades, but this perspective is not the focus of this writing.

Seeking an algebraic formulation, Soergel introduced Soergel bimodules. Let (W, S) be a Coxeter system. Associated to W is the Hecke algebra $\mathcal{H}$, which is a deformation of the group algebra of W and plays a central role in representation theory and the theory of Kazhdan–Lusztig polynomials. Soergel bimodules provide a categorification of $\mathcal{H}$: they form a monoidal category whose Grothendieck group (with a grading) is isomorphic to $\mathcal{H}$, with the indecomposable objects mapping to the Kazhdan-Lusztig basis elements in the Hecke algebra. Using this framework, key results in Kazhdan-Lusztig theory were reproven, this time through more algebraic methods.

At the same time, intersection cohomology inspired the development of sheaves on moment graphs, which offer a powerful local perspective on Soergel bimodules. Let V be a finitely-generated free $\mathbb{R}$-module. On V, one can construct a moment graph $(\mathcal{G}, \mathcal{E}, \alpha, \leq)$. Using the Braden-MacPherson algorithm, we can construct a sheaf $\mathcal{M}$ on the moment graph called the Braden-MacPherson sheaf. These sheaves allow for simpler ways to compute rather complicated topological invariants.

To link these two stories, quite remarkably, Fiebig showed that we have the following correspondence

additive categories of Soergel bimodules	$\leftrightarrow$	Braden-MacPherson Sheaves

where the indecomposable Soergel bimodules are sent to the indecomposable normalized Braden-MacPherson sheaves.

The purpose of this essay is to give an introduction to the theory of Soergel Bimodules and Braden-MacPherson sheaves, and then to rigorously formulate Fiebig's correspondence.

1.2 Conventions

We work over a base field of characteristic zero, which we may assume to be $\mathbb{C}$ for convenience. Recently, there has been a lot of progress and interest in the characteristic p case - we will not discuss that here.

Graded objects are a key feature in the theory of Soergel bimodules. We establish the following conventions:

- **Grading:** Graded always refers to $\mathbb{Z}$-grading. A graded vector space (or module, ring) M is a direct sum $M = \bigoplus_{i \in \mathbb{Z}} M^i$. An element $m \in M^i$ is called homogeneous of degree i, denoted $\deg(m) = i$. Morphisms between graded objects are typically assumed to be degree-preserving unless otherwise specified (e.g., $\mathrm{Hom}^\bullet$ denotes the space of all graded homomorphisms).
- **Shifts:** For a graded object M and an integer $n \in \mathbb{Z}$, the *shifted object* $M(n)$ is defined by setting its degree j component to be $(M(n))^j := M^{n+j}$. This convention means that the shift (1) lowers the degrees of homogeneous elements by 1. A degree k map $f : M \to N$ is equivalent to a degree 0 map $M \to N(k)$ or $M(-k) \to N$. In the

context of the Grothendieck group, the shift (1) corresponds to multiplication by a formal variable v, i.e., $v[M] := [M(1)]$.

- **Graded Modules/Algebras:** A graded ring $R = \bigoplus R^i$ satisfies $R^i R^j \subseteq R^{i+j}$. A graded (left) module $M = \bigoplus M^i$ over R satisfies $R^i M^j \subseteq M^{i+j}$. Similar conditions hold for right modules and bimodules. Submodules and direct summands are assumed to be graded unless stated otherwise.
- **Graded Rank:** A graded module M over a graded ring R is called *graded free* if it possesses a basis consisting of homogeneous elements. If M is finitely generated and graded free over R, it is isomorphic to a direct sum of shifted copies of R. This isomorphism can be uniquely encoded by a Laurent polynomial $p = \sum p_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$, where p_i is the number of basis elements of degree $-i$ (consistent with $v[M] = [M(1)]$). We write $M \cong R^{\oplus p} := \bigoplus_{i \in \mathbb{Z}} R(i)^{\oplus p_i}$. The polynomial p is called the *graded rank* of M over R, denoted $\underline{\mathrm{rk}}_R M = p$.

1.3 Organization

Our exposition largely follows the first six chapters of [13], supplemented with additional commentary and clarifying remarks.

Section 2: Coxeter Groups and Hecke Algebras introduces the foundational algebraic and combinatorial concepts, setting the stage for Section 3. In the Coxeter groups subsection, we first define Coxeter systems, motivate the Bruhat order through matrix groups, explore root systems in the geometric representation, and conclude with an alternative perspective on root systems through Borel subgroups. In the Hecke algebras subsection, we provide an overview of the field, introduce the Hecke algebra and its two bases, and finish with a discussion on computing Kazhdan-Lusztig polynomials.

Section 3: Soergel Bimodules focuses on the first central object of study. Beginning with polynomial rings and Demazure operators, we develop the theory from basic bimodules all the way to Soergel bimodules. The section concludes with Soergel's Categorification Theorem, which shows that Soergel bimodules serve as a categorification of the Hecke algebra.

Section 4: Braden-MacPherson Sheaves introduces the second key framework, approaching it through the lens of intersection cohomology on varieties with torus actions. We cover the necessary geometric preliminaries (torus actions, fixed points, Whitney stratifications, etc.), define the moment graph, introduce sheaves on these graphs, and describe the Braden-MacPherson algorithm for constructing the canonical sheaf that computes intersection cohomology.

Section 5: Fiebig's Correspondence bridges the previous two sections by demonstrating how the additive categories of Soergel bimodules correspond to Braden-MacPherson sheaves.

1.4 Acknowledgements

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2 Coxeter Groups and Hecke Algebras

2.1 Coxeter Groups

This section introduces Coxeter groups, exploring their structure, classification, and connections to geometry and algebraic groups.

2.1.1 The Big Picture

We have already presented one motivation at the start of the paper, which guides the topics and the purpose of this paper. However, there are two other motivations for Coxeter groups that are worth knowing: one historical, the other is simply too fascinating to ignore.

Historical: The most natural starting point to study symmetry from a mathematically rigorous viewpoint is to study groups generated by geometric reflections. Such groups, acting on Euclidean space, describe the symmetries of familiar objects like regular polygons (whose symmetry groups are the dihedral groups) and regular polyhedra. These *reflection groups* capture the transformations preserving an object through mirror-like operations across hyperplanes. The inherent structure within these geometric examples motivated a more abstract algebraic formulation.

In 1934, H.S.M. Coxeter introduced a class of groups defined purely by generators and relations, abstracting the essential properties observed in geometric reflection groups. These *Coxeter groups* provide a unified framework encompassing not only the finite Euclidean reflection groups (indeed, the finite Coxeter groups correspond precisely to these geometric groups) but also infinite groups. Important examples of infinite Coxeter groups include affine Weyl groups, associated with tessellations of Euclidean space, and hyperbolic reflection groups, related to tessellations of hyperbolic space.

This abstraction has proved to be remarkably fruitful. Perhaps the biggest strength of Coxeter groups is its simplicity and versatility. Deattaching the study of Coxeter groups from specific geometric interpretations allows for generalizations across many geometric settings, including Euclidean, spherical, affine, and hyperbolic geometries, as well as combinational structures with no immediate geometric realization.

We've already noted that Coxeter groups are central to many mathematical structures. As a second motivation, we present an example that illustrates the kinds of insights and constructions these groups make possible.

Coxeter groups and q-Polynomials: This section is based on [2].

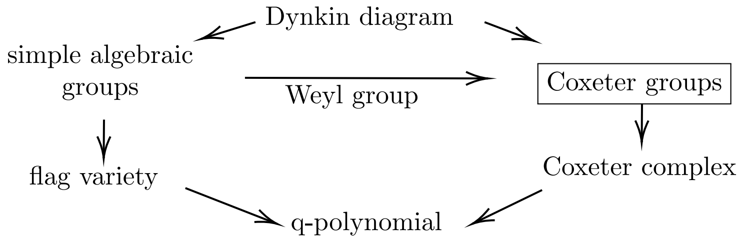
One compelling motivation for studying Coxeter groups stems from their deep connection to Dynkin diagrams, which appear in various mathematical contexts, notably Lie theory. Starting from a Dynkin diagram, one can embark on two seemingly distinct paths that remarkably converge on the same invariant, a q-polynomial.

- **Algebraic Path:** Given a Dynkin diagram and a field $\mathbb{F}$, one can construct a corresponding simple algebraic group G. Associated with G is its flag variety $\mathcal{FL}(G)$, a fundamental geometric object. The flag variety admits a stratification known as the Bruhat decomposition, $\mathcal{FL}(G) = \bigsqcup_w X_w$, where the cells X_w are indexed by elements w of the Weyl group W of G. The dimensions of these cells (related to the lengths of elements in W) can be encoded in a q-polynomial, often defined via the cohomology or intersection cohomology of the flag variety.
- **Combinatorial Path:** A Dynkin diagram also directly defines a finite Coxeter group W (which coincides with the Weyl group from the algebraic path). Associated with W is a simplicial complex called the Coxeter complex, $\Sigma(W)$. The combinatorial structure of this complex, specifically the enumeration of its simplices based on a notion of distance or rank, also yields a q-polynomial.

The remarkable fact is that these two approaches yield the same q-polynomial, which contains a lot of important information:

- Its degree equals the dimension of the flag variety $\mathcal{FL}(G)$ and also the length of the longest element in the Coxeter group W.
- Its value at $q = 1$ gives the order $|W|$, the number of elements in the Coxeter group.
- When q is a prime power, its value counts the number of $\mathbb{F}_q$-rational points on the flag variety $\mathcal{FL}(G)$.
- Its value at $q = -1$ yields the Euler characteristic of the real points of the flag variety.

This story can be briefly summarized in a diagram:



Today

Coxeter groups now occupy a central position in mathematics, with deep connections to Lie theory (where Weyl groups are fundamental examples), the theory of algebraic groups, geometric group theory, the study of polytopes and buildings, algebraic combinatorics, and representation theory. Standard comprehensive references include [4], [5], [11], and [22].

2.1.2 Coxeter Systems

We begin with the formal algebraic definition of a Coxeter system.

Definition 2.1 (Coxeter System)**.** A *Coxeter system* is a pair (W, S), where W is a group generated by a finite set S satisfying relations:

$$W = \langle S \mid (st)^{m(s,t)} = 1, \;\; s, t \in S \rangle$$

with $m(s,t) \in \mathbb{N} \cup \{\infty\}$ satisfying:

1. $m(s,s) = 1$, implying $s^2 = 1$.
2. $m(s,t) = m(t,s)$.
3. $m(s,t) \geq 2$ if $s \neq t$.
4. $m(s,t) = \infty$ if there is no relation $(st)^k = 1$ with $k \geq 1$.

These relations are called *braid relations.*

A fundamental, powerful, and non-trivial result concerning Coxeter groups is the following proposition, which connects the abstract relations to the group structure:

Proposition 2.2. *$m(s,t)$ is the order of the product st in W.*

The information defining a Coxeter system can be encoded succinctly in two equivalent ways, which are often more convenient to work with:

Definition 2.3 (Coxeter Matrix)**.** Given (W, S), define the *Coxeter matrix* $M = (m(s,t))_{s,t \in S}$, symmetric with entries from $\{1, 2, \ldots, \infty\}$.

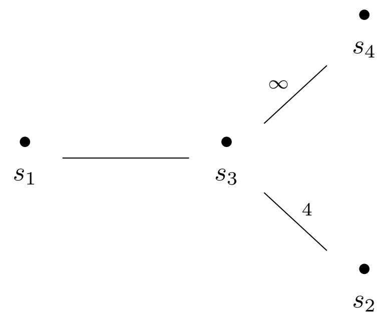
Definition 2.4 (Coxeter Graph)**.** The *Coxeter graph* Γ has vertices indexed by S and edges as follows:

- Vertices $s \neq t$ connected if $m(s,t) \geq 3$.
- Edges labeled by $m(s,t)$ if $m(s,t) > 3$, unlabeled if $m(s,t) = 3$.
- No edge if $m(s,t) = 2$.

To solidify the understanding of these definitions, let's take a look at a simple example:

Example 2.5. Consider the Coxeter matrix and its corresponding Coxeter graph shown below. The generators are s_1, s_2, s_3, s_4.

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix} \qquad \longleftrightarrow$$



From this representation, we can directly read off the defining relations of the Coxeter group W:

- The diagonal entries of the matrix are 1, so $s_1^2 = s_2^2 = s_3^2 = s_4^2 = \text{id}$.
- $m(s_1, s_2) = 2 \Rightarrow (s_1 s_2)^2 = s_1 s_2 s_1 s_2 = \text{id}$, which means $s_1 s_2 = s_2 s_1$ (they commute). Similarly, $s_1 s_4 = s_4 s_1$ and $s_2 s_4 = s_4 s_2$.
- $m(s_1, s_3) = 3 \Rightarrow (s_1 s_3)^3 = \text{id}$.
- $m(s_2, s_3) = 4 \Rightarrow (s_2 s_3)^4 = \text{id}$.
- $m(s_3, s_4) = \infty \Rightarrow$ there is no finite order relation between s_3 and s_4.

As mentioned earlier, the Coxeter matrix and the Coxeter graph are equivalent ways of encoding the structure of a Coxeter system:

Proposition 2.6. Up to isomorphism, there exists a bijective correspondence between Coxeter matrices and Coxeter graphs. Given one, the other can be uniquely determined.

While the Coxeter matrix might seem more directly tied to the algebraic definition, the Coxeter graph offers several advantages, particularly for visualization and understanding certain properties of the Coxeter group. For instance, the graph readily reveals which generators commute (absence of an edge) and the order of the product of non-commuting generators (edge labels). Extracting this information from a large matrix can be more annoying to do. Furthermore, the Coxeter graph provides a direct criterion for determining whether the corresponding Coxeter group is finite:

Theorem 2.7 (Classification of Finite Coxeter Groups)**.** A Coxeter group is finite if and only if the Coxeter graph is a finite disjoint union of Coxeter graphs from the list:

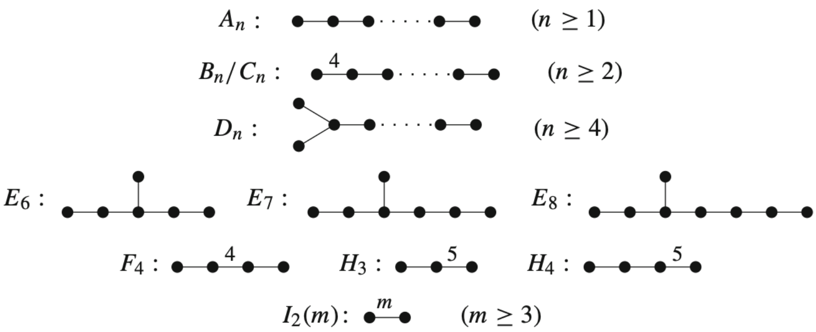


Figure 1: Classification of Finite Coxeter Groups

Proof. We sketch the proof presented in [24].

Every Coxeter group decomposes as a direct product over the connected components of its diagram, so W is finite if and only if each irreducible component is finite. Consider the real vector space $V = \mathbb{R}^S$ with basis $\{e_s\}_{s\in S}$, and define a symmetric bilinear form B by

$$B(e_s, e_s) = 2, \quad B(e_s, e_t) = -2\cos(\pi/m_{st})$$

for $s \neq t$. Each generator $s \in S$ then acts linearly on V by $s(v) = v - B(v, e_s)e_s$, which is a reflection fixing the hyperplane orthogonal to e_s. This yields a homomorphism $\rho : W \to \mathrm{GL}(V)$, called the reflection representation, whose image is a group generated by reflections preserving B. If B is positive definite, then $\rho(W) \subset O(V, B)$ is a finite subgroup of the orthogonal group, and hence W is finite.

Conversely, if W is finite, then ρ must preserve some positive-definite inner product; since B is invariant and defined by the group relations, it must coincide with the inner product up to scaling, and hence is positive definite. Therefore, for irreducible W, finiteness is equivalent to positive definiteness of B. The classification now reduces to finding all connected Coxeter diagrams for which the Gram matrix $(B(e_s, e_t))$ is positive definite. One proceeds by case analysis: trees with small edge labels yield positive definite forms, while cycles or multiple large edge weights introduce indefinite directions. Explicitly, one tests positive definiteness of B by constructing vectors $x \in V$ for which $B(x, x) \leq 0$ to eliminate infinite types, or shows that $B(x, x) > 0$ for all $x \neq 0$ to retain finite types. This leads to a finite list of diagrams whose Gram matrices are positive definite: the simply-laced types A_n, D_n, E_6, E_7, E_8, and the non-simply-laced types $B_n, F_4, H_3, H_4, I_2(m)$. Since all other connected diagrams fail positive definiteness, this exhausts the classification.

Therefore, a Coxeter group is finite if and only if each irreducible component of its diagram appears in this list. □

Next, we introduce the geometric representation. This is a crucial tool for understanding Coxeter groups as groups generated by linear reflections.

We now turn our attention to study the connection between Coxeter groups and orthogonal transformations, which leads to the idea of the geometric representation.

Let $O(\mathbb{R}^n)$ denote the orthogonal group of $\mathbb{R}^n$, consisting of all linear transformations that preserve the Euclidean inner product.

Definition 2.8 (Reflection)**.** A *reflection* is an orthogonal transformation $s \in O(\mathbb{R}^n)$ whose fixed subspace is a hyperplane $H_s \subset \mathbb{R}^n$. Such a reflection can be written explicitly as

$$s(v) = v - 2(v, \alpha_s)\alpha_s,$$

where α_s is a unit normal vector to H_s.

Theorem 2.9. Let W be a finite subgroup of $O(\mathbb{R}^n)$ generated by reflections. Then W admits a Coxeter presentation.

Proof. Let $S \subset W$ be the set of reflections generating W. For each pair $s, t \in S$, let m_{st} denote the order of st in W. We claim that W has the presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S\rangle.$$

To verify this, we must show that all relations in W follow from the given ones. Since W is finite, the subgroup generated by any two reflections s, t is a dihedral group of order $2m_{st}$, and the relation $(st)^{m_{st}} = 1$ holds. By the classification of finite reflection groups, these relations suffice to present W. □

Conversely, any finite Coxeter group can be embedded in some orthogonal group, by means of its geometric representation.

Now, we formally define the geometric representation:

Definition 2.10 (Geometric Representation). Let (W, S) be a Coxeter system. The *geometric representation* of W is a linear representation $\rho : W \to \mathrm{GL}(V)$, where V is a real vector space with basis $\{\alpha_s\}_{s \in S}$ indexed by the generators. V is equipped with a symmetric bilinear form $(\cdot, \cdot)$ defined by its values on the basis vectors:

$$(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right) \quad \text{for all } s, t \in S.$$

Here, if $m_{st} = \infty$, we interpret π/m_{st} as 0, so $(\alpha_s, \alpha_t) = -1$. Note that since $m_{ss} = 1$, $(\alpha_s, \alpha_s) = -\cos(\pi) = 1$. The action of a generator $s \in S$ on a vector $v \in V$ is defined as a reflection:

$$\rho(s)(v) = v - 2(v, \alpha_s)\alpha_s.$$

This map ρ extends to a homomorphism from W to $\mathrm{GL}(V)$ because the reflection operators satisfy the Coxeter relations $(st)^{m_{st}} = \mathrm{id}$ on V. (This relies on the fact that the composition of two reflections s_α, s_β in planes with angle θ is a rotation by 2θ; here the angle between hyperplanes orthogonal to α_s, α_t relates to m_{st}).

The most important fact to know about the geometric representation is that it is faithful:

Theorem 2.11. The geometric representation $\rho : W \to \mathrm{GL}(V)$ is faithful (injective) for any Coxeter system (W, S).

Proof. We must show that ρ is injective. Suppose $w \in W$ acts trivially on V. Express w as a reduced word $w = s_1 \cdots s_k$ with $s_i \in S$. Since $w(\alpha_{s_i}) = \alpha_{s_i}$ for all i, we have

$$\alpha_{s_i} = s_1 \cdots s_k(\alpha_{s_i}) = s_1 \cdots s_{i-1}(-\alpha_{s_i}) + \text{terms involving } \alpha_{s_j}, j \neq i.$$

By linear independence of $\{\alpha_s\}_{s \in S}$, this forces $k = 0$, meaning $w = 1$. Thus, $\ker \rho$ is trivial. □

While the geometric representation provides a concrete way to study Coxeter groups, the theory can be developed in a more general framework using the notion of a realization.

Definition 2.12 (Realization). Let (W, S) be a Coxeter system and k a commutative integral domain (typically $\mathbb{R}$, $\mathbb{C}$, or a field of characteristic not 2). A *realization* of (W, S) over k is a triple $\mathfrak{h} = (\mathfrak{h}, \{\alpha_s\}_{s \in S}, \{\alpha_s^\vee\}_{s \in S})$ where:

1. $\mathfrak{h}$ is a free k-module of finite rank.
2. $\{\alpha_s\}_{s \in S}$ is a family of elements in the dual module $\mathfrak{h}^* = \mathrm{Hom}_k(\mathfrak{h}, k)$ (the simple roots).
3. $\{\alpha_s^\vee\}_{s \in S}$ is a family of elements in $\mathfrak{h}$ (the simple coroots).
4. These elements satisfy the condition $\langle \alpha_s, \alpha_t^\vee \rangle := \alpha_s(\alpha_t^\vee) \in k$ for all $s, t \in S$, and critically, $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ for all $s \in S$.
5. The assignment $s \mapsto \sigma_s$, where $\sigma_s(v) := v - \langle \alpha_s, v \rangle \alpha_s^\vee$ for $v \in \mathfrak{h}$, defines a linear action of W on $\mathfrak{h}$ (i.e., the σ_s satisfy the braid relations corresponding to (W, S)).

Further technical conditions, such as *balancedness* and *Demazure surjectivity*, are often imposed for the theory to develop smoothly, particularly for diagrammatic approaches. [14] We will not do that here.

Many other important realizations arise naturally from the root data of reductive algebraic groups, but we will not expand on that viewpoint here.

2.1.3 From Matrix Groups to Bruhat Order

The combinatorics of Coxeter groups is a very popular area of research, see [4]. The most basic concept in this study is the length function, which we define as follows:

Definition 2.13 (Length). Let (W, S) be a Coxeter system. Any $w \in W$ can be written as a product

$$w = s_1 \cdots s_k$$

with $s_i \in S$, and the *length* $\ell(w)$ is defined as the minimal such k for which this is possible.

We have already highlighted several reasons why Coxeter groups are significant. Here is another: they arise naturally in the study of classical matrix groups, such as the general linear group $\mathrm{GL}_n(\mathbb{C})$. While matrix groups themselves are not Coxeter groups, through the theory of Weyl groups and BN-pairs, we can connect the two. Let's briefly outline this connection, which serves as a primary motivation for the definition of the Bruhat order.

We first introduce the definition of a BN-pair:

Definition 2.14 (BN-pair)**.** A pair (B, N) of subgroups of a group G is called a *BN-pair* if the following hold:

1. $B \cup N$ generates G, and $B \cap N$ is normal in N.
2. $W := N/(B \cap N)$ is generated by some set S of involutions.
3. For all $s \in S$ and $w \in W$, $BsB \cdot BwB \subseteq BswB \cup BwB$.
4. For all $s \in S$, $BsB \cdot BsB \neq B$.

The group W is called the *Weyl group* and the number $|S|$ is the *rank* of the BN-pair $(G; B, N)$.

Proposition 2.15. *S is uniquely determined and that the pair (W, S) is a Coxeter system.*

Proof. The uniqueness of S follows from the fact that it consists of the minimal nontrivial elements in W (with respect to the Bruhat order induced by the BN-pair). Specifically, by the BN-pair axioms, each $s \in S$ corresponds to a double coset BsB that is minimal among those not equal to B. The third BN-pair axiom ensures that multiplication of double cosets respects the Coxeter relations, meaning that (W, S) satisfies the defining properties of a Coxeter system. The fourth axiom guarantees that each $s \in S$ is indeed an involution and cannot be further decomposed. Thus, the pair (W, S) is uniquely determined and forms a Coxeter system. □

A magical fact about groups with BN-pairs is the existence of a decomposition of a particular form:

Theorem 2.16 (Bruhat Decomposition)**.** *Let G be a group with a BN-pair (B, N). Then*

$$G = \biguplus_{w \in W} BwB$$

is a disjoint decomposition of G into double cosets indexed by the Weyl group $W = N/(B \cap N)$.

Proof. We proceed by induction on the length $\ell(w)$ of elements in W.

Base case: The identity $e \in W$ corresponds to the double coset $B = BeB$.

Inductive step: Suppose every element of length $\leq k$ lies in some BwB. For w' of length $k+1$, write $w' = sw$ where $\ell(w) = k$ and $s \in S$. By the BN-pair axioms,

$$Bw'B = BswB \subseteq BsB \cdot BwB.$$

The right-hand side is either $BswB$ or BwB (by Axiom 3), but since $\ell(w') > \ell(w)$, the former must hold. Thus, $Bw'B$ is distinct from all shorter double cosets.

By induction, we are done. □

Let's consider a motivating example:

Example 2.17. Let $G = GL_n(\mathbb{C})$ and B be the upper-triangular matrices. The quotient G/B is the *flag variety*, a smooth projective variety that decomposes into Bruhat cells:

$$G/B = \bigsqcup_{w \in S_n} C_w, \quad \text{where } C_w = BwB/B.$$

This decomposition gives G/B with the structure of a CW-complex. Then, the natural partial order to put on this structure is the *Bruhat order* $\leq$ on S_n, which is defined by the closure relation:

$$\overline{C_u} \subseteq \overline{C_w} \iff u \leq w.$$

Remarkably, this order depends only on the Coxeter group structure of S_n and not on the matrix realization.

The Bruhat order can be defined purely in terms of a Coxeter system (W, S), without any reference to matrix groups.

Definition 2.18 (Bruhat Order and Graph)**.** Let $T := \{wsw^{-1} \mid w \in W, s \in S\}$ denote the set of reflections in W.

- For $x, y \in W$, write $x \to y$ if $y = xt$ for some $t \in T$ and $\ell(x) < \ell(y)$.
- The *Bruhat graph* is the directed graph with vertex set W and edges given by $\to$.
- The *Bruhat order* $\leq$ is the transitive closure of $\to$: $x \leq y$ if there exists a path

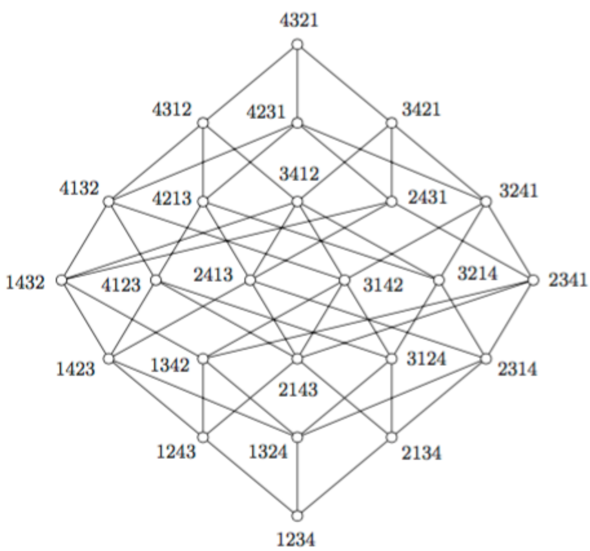
$$x = x_0 \to x_1 \to \cdots \to x_k = y.$$

Proposition 2.19. The Bruhat order $\leq$ is a partial order on W.

Proof. Reflexivity is trivial; take $k = 0$. For antisymmetry, suppose $x \leq y$ and $y \leq x$. Any path $x \to \cdots \to y$ increases length, so $\ell(x) \leq \ell(y)$. Similarly, $\ell(y) \leq \ell(x)$. Thus, $\ell(x) = \ell(y)$, and the only possible path is $x = y$. Finally, transitivity is immediate from the definition of transitive closure. □

We can use the Bruhat order to create Hasse diagrams, which are visually appealing but get complicated very quickly. Here is an example of one:

Example 2.20. The Hasse diagram of the Bruhat order on S_4 is given by:



2.1.4 Root Systems in the Geometric Representation

Now we move onto discuss roots in the geometric representation. But before this, let's review some basic definitions about root systems.

Let V be a finite-dimensional real vector space equipped with a positive-definite inner product $(\cdot, \cdot)$. A root system is a highly symmetric configuration of vectors that arises naturally in the classification of semisimple Lie algebras, Weyl groups, and algebraic groups.

Definition 2.21 (Root system)**.** A finite set $\Phi \subset V \setminus \{0\}$ is a *root system* if it satisfies:

1. Φ spans V.
2. For every $\alpha \in \Phi$, the reflection s_α defined by

$$s_\alpha(\lambda) := \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}\alpha \quad \text{for all } \lambda \in V$$

   preserves Φ; i.e., $s_\alpha(\Phi) = \Phi$.
3. For all $\alpha, \beta \in \Phi$, the number $\langle \beta, \alpha^\vee \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.
4. If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$.

The rank of Φ is $\dim V$.

Definition 2.22 (Positive (and Negative) Roots)**.** Fix a hyperplane in V not containing any root (e.g., by choosing a linear form $h \in V^*$ such that $h(\alpha) \neq 0$ for all $\alpha \in \Phi$). Define:

$$\Phi^+ := \{\alpha \in \Phi : h(\alpha) > 0\}, \quad \Phi^- := -\Phi^+.$$

Then $\Phi = \Phi^+ \sqcup \Phi^-$ and exactly one of $\{\alpha, -\alpha\}$ lies in Φ^+. The elements of Φ^+ are called *positive roots* and the elements of Φ^- are called *negative roots*.

Definition 2.23 (Simple Roots)**.** A root $\alpha \in \Phi^+$ is *simple* if it cannot be written as a sum of two elements of Φ^+. The set of all simple roots is called the *base* Δ of Φ. It forms a basis of V such that every root $\beta \in \Phi$ can be written uniquely as:

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with all $c_\alpha \in \mathbb{Z}$, either all ≥ 0 or all ≤ 0.

Now, we present an alternate definition of the Weyl group.

Definition 2.24 (Weyl Group)**.** The *Weyl group* W of a root system Φ is the subgroup of $\mathrm{O}(V)$ generated by all reflections s_α with $\alpha \in \Phi$.

Definition 2.25 (Cartan Matrix)**.** Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be an ordered base. The corresponding *Cartan matrix* is the $r \times r$ integer matrix $A = (a_{ij})$ defined by:

$$a_{ij} := \langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}.$$

This matrix encodes both angular and length data of the base roots.

Remark 2.26. The only possible angles θ between distinct roots in an irreducible root system are 90°, 120°, 135°, and 150°, corresponding to Cartan integers $a_{ij}a_{ji} = 0, 1, 2, 3$. The classification of root systems thus reduces to a finite list, associated to the Dynkin diagrams of types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2.

We are interested in discussing roots in the geometric representation. Denote by

$$T = \bigcup_{x \in W} xSx^{-1}$$

the set of all reflection. For any $t \in T$ choose $x \in W$ and $s \in S$ with $t = xsx^{-1}$ and $xs > x$ in Bruhat order and set

$$\alpha_t := x(\alpha_s) \in V_{\mathrm{geom}}. \tag{†}$$

The inequality $xs > x$ guarantees that α_t is *positive*.

Proposition 2.27. The positive roots α_t for $W = S_n$ are

$$\{\, e_i - e_j \mid 1 \leq i < j \leq n\}.$$

Proof. Identify S_n with permutation matrices acting on $V_{\text{geom}} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. For the simple transposition $s_i = (i, i+1)$ one has $\alpha_{s_i} = e_i - e_{i+1}$. Any reflection in S_n is a transposition $t = (a, b)$ with $a < b$. Conjugating s_a by a permutation that carries $\{a, a+1\}$ to $\{a, b\}$ gives

$$\alpha_{(a,b)} \;=\; e_a - e_b.$$

Hence $\Phi^+ = \{\, e_i - e_j \mid 1 \le i < j \le n\}$, the standard positive root system of type A_{n-1}. □

Proposition 2.28. α_t for $t \in T$ is well-defined.

Proof. We show the following:

(i) s fixes a hyperplane in V_{geom} and sends α_s to $-\alpha_s$. Make a similar conclusion for xsx^{-1} and deduce that α_t is well defined up to a scalar.

(ii) By exploiting a suitable form on V_{geom}, show that $(\alpha_t, \alpha_t) = 1$ and thus deduce that α_t is well-defined up to ± 1.

(iii) Show that under the assumption that $xs > x$, we have

$$x(\alpha_s) \in \oplus_{s \in S} \mathbb{R}_{\ge 0} \alpha_s.$$

Deduce that α_t is well-defined.

For a simple reflection s,

$$s(\alpha_s) = -\alpha_s, \qquad s(v) = v \text{ for all } v \in H_s := \{v \mid (v, \alpha_s) = 0\}.$$

Hence s fixes the hyperplane H_s pointwise and acts by -1 on the line $\mathbb{R}\alpha_s$. Conjugating by $x \in W$ gives

$$t = xsx^{-1} \;:\; x(\alpha_s) \mapsto -x(\alpha_s), \quad x(H_s) \text{ fixed}.$$

Thus the (-1)–eigenspace of t is the *one–dimensional* line $\mathbb{R} \cdot x(\alpha_s)$. If t also equals yry^{-1} with $r \in S$, then $y(\alpha_r)$ lies in the same eigenspace, so

$$y(\alpha_r) = \lambda\, x(\alpha_s) \qquad (\lambda \in \mathbb{R}^\times),$$

showing that α_t is determined up to a non–zero scalar.

The form $(\,,\,)$ is W–invariant and satisfies $(\alpha_s, \alpha_s) = 1$. Hence $(x(\alpha_s), x(\alpha_s)) = 1 = (y(\alpha_r), y(\alpha_r))$. Taking norms in $y(\alpha_r) = \lambda\, x(\alpha_s)$ yields $\lambda^2 = 1$, so $\lambda = \pm 1$. Therefore every valid construction of α_t differs at most by an overall sign.

By [22], Proposition 5.7: For $\alpha \in \Pi$ and $w \in W$, one has $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha)$ is a positive root. Applying the proposition to our factorisation with $w = x$ and $\alpha = \alpha_s$ gives $x(\alpha_s) \in \Phi^+$. Because a positive root is a non-negative linear combination of the simple roots,

$$x(\alpha_s) \;\in\; \bigoplus_{r \in S} \mathbb{R}_{\ge 0}\, \alpha_r.$$

Its negative lies in $-\Phi^+$, so the sign ambiguity from part (ii) is resolved: we must keep the positive vector $x(\alpha_s)$. If a second valid presentation $t = yry^{-1}$ also satisfies $yr > y$, the same argument makes $y(\alpha_r)$ positive; but the equality $y(\alpha_r) = \pm x(\alpha_s)$ then forces the plus sign. Thus every allowable choice of factorisation yields exactly the same vector.

□

2.1.5 An Alternative View: Borel Subgroups

Having explored the connection between Coxeter groups and orthogonal transformations through the geometric representation, we now shift our perspective to an alternative viewpoint arising from the theory of algebraic groups. This section, following the exposition in [23], briefly explores how the connection between positive roots and Borel subgroups.

Let G be a connected reductive algebraic group over an algebraically closed field F. In this section, we explore how the choice of a Borel subgroup $B \subset G$ determines a distinguished set of positive roots in the root system of G, and vice versa.

Definition 2.29 (Algebraic Group)**.** An *algebraic group* is a group that is also an algebraic variety, such that the group operations (multiplication and inversion) are morphisms of varieties.

In the world of algebraic groups, a particular class of them are important:

Definition 2.30 (Unipotent)**.** An algebraic group is called *unipotent* if it acts by unipotent operators in any rational representation. A matrix is unipotent if all its eigenvalues are equal to 1.

Building upon this, we can define a particularly important subgroup within any algebraic group.

Definition 2.31 (Unipotent Radical)**.** Let G be an algebraic group. Then G has a unique maximal normal unipotent subgroup, called the *unipotent radical* of G, denoted $R_u(G)$.

This leads us to the most important class of algebraic groups for our discussion:

Definition 2.32 (Reductive Algebraic Group)**.** An algebraic group G is *reductive* if its unipotent radical is trivial: $R_u(G) = \{1\}$.

To further understand the structure of reductive algebraic groups, we introduce the concept of a torus.

Definition 2.33 (Torus)**.** A *torus* is an algebraic group isomorphic to a finite product of copies of the multiplicative group $\mathbb{G}_m = F^\times$. A *maximal torus* $T \subset G$ is a torus that is maximal with respect to inclusion among all tori in G.

Now, we can define the important subgroup that governs the choice of positive roots.

Definition 2.34 (Borel Subgroup)**.** A *Borel subgroup* $B \subset G$ is a maximal connected solvable subgroup. All Borel subgroups of a connected reductive group are conjugate, and their quotient G/B is a projective algebraic variety.

To connect Borel subgroups with root systems, we need the concept of weights associated with the maximal torus.

Definition 2.35 (Weight)**.** Let $T \subset G$ be a maximal torus. The group of characters of T, denoted $X(T)$, is the lattice of group homomorphisms $\chi : T \to \mathbb{G}_m$. The elements of $X(T)$ are called *weights*.

These weights then give rise to specific subgroups within G:

Definition 2.36 (Root Subgroup)**.** A *root subgroup* $U' \subset G$ with respect to T is a subgroup isomorphic to $\mathbb{G}_a$ (the additive group), such that there exists a character $\alpha \in X(T)$ satisfying:

$$tut^{-1} = \alpha(t)u \quad \text{for all } t \in T,\ u \in U'.$$

The character α is then called a *root*, and we denote the corresponding subgroup by U_α.

The collection of all such roots forms the root system of the algebraic group.

Definition 2.37 (Root System, Again)**.** The set of all roots $\Delta \subset X(T)$ is called the *root system* of G with respect to T. This set forms a reduced root system.

Finally, the Weyl group, which we encountered earlier in the context of Coxeter groups, also has a definition within the framework of algebraic groups.

Definition 2.38 (Weyl Group, Again)**.** The *Weyl group* $W = N_G(T)/T$ is defined as the quotient of the normalizer of T in G by T itself. The Weyl group acts on the root system Δ.

Now, we state the main point of this section, without proof:

Theorem 2.39. Let G be a connected reductive group, and $T \subset G$ a maximal torus.

1. Each root $\alpha \in \Delta$ determines a unique root subgroup $U_\alpha \subset G$.
2. The group G is generated by T and all root subgroups U_α.

3. The pair $(U_\alpha, U_{-\alpha})$ generates a subgroup of G isomorphic to SL_2 or PGL_2.
4. The set Δ is a reduced root system, and the group $W = N_G(T)/T$ is its Weyl group.
5. Any choice of a Borel subgroup $B \supset T$ determines a subset of roots $\Delta^+ \subset \Delta$ (called the *positive roots*) such that the corresponding root subgroups U_α for $\alpha \in \Delta^+$ multiply to form the unipotent radical $U \subset B$, and $B = TU$.
6. This gives a bijection between Borel subgroups containing T and systems of positive roots in Δ.

Remark 2.40. The set of simple roots $\Pi \subset \Delta^+$ forms a basis of Δ, and all positive roots are non-negative integer linear combinations of simple roots. The choice of B thus determines not only Δ^+ but also the corresponding Dynkin diagram.

Example 2.41. In $G = \mathrm{GL}_n$, take T to be the subgroup of diagonal matrices. The roots are $\alpha_{ij} = \varepsilon_i - \varepsilon_j$, where ε_k is the character sending $\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_k$. The upper-triangular subgroup B_n defines the positive roots as α_{ij} with $i < j$.

Remark 2.42. The flag variety G/B, whose points correspond to Borel subgroups of G, can be understood as the variety of all possible choices of positive root systems. As we saw earlier, geometric structure of G/B reflects the combinatorics of the Weyl group and the Bruhat decomposition.

2.2 Hecke Algebras

2.2.1 Hecke Algebras in Nature

This section is taken out of [9].

While out primary interest in Hecke algebra is for their role in Soergel bimodules, it is important to note that Hecke algebras play a central role in many areas of math, not just in Kazhdan-Lusztig theory. Here are two major examples in number theory:

- *Finite Groups of Lie Type*: For G over $\mathbb{F}_q$ with Weyl group W, $\mathcal{H}_q(W) \subset \mathbb{C}[G(\mathbb{F}_q)]$ captures key aspects of the representation theory of $G(\mathbb{F}_q)$, e.g., $\mathrm{GL}_n(\mathbb{F}_q)$.
- *p-adic groups*: For a nonarchimedean local field F (e.g., $\mathbb{Q}_p$), the affine Weyl group W_{aff} yields $\mathcal{H}_q(W_{\mathrm{aff}})$ as a convolution algebra on $G(F)$. This is important in p-adic representation theory and the theory of automorphic forms.

Beyond number theory, Hecke algebras also appear in other areas of representation theory, including:

- quantum groups and statistical mechanics (e.g., Temperley–Lieb algebras, Jimbo's deformation),
- knot theory (e.g., Jones' definition of the Jones polynomial),
- modular representation theory (e.g., the work of Dipper–James),

and much more.

We won't be studying these perspectives, but they're still worth being aware of.

2.2.2 From Generic Hecke Algebras to the Hecke Algebra

The generic Hecke algebra provides a uniform framework generalizing both group algebras and Hecke algebras. Contrary to most texts, we present the general Hecke algebra before presenting the Hecke algebra.

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{Z}_{\geq 0}$. Fix a commutative ring A and parameters $\{a_s, b_s\}_{s \in S}$ satisfying $a_s = a_t$ and $b_s = b_t$ whenever s, t are conjugate in W.

Definition 2.43. The *generic Hecke algebra* $\mathcal{H}$ is the free A-module with basis $\{T_w \mid w \in W\}$, equipped with the associative multiplication determined by:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ a_s T_w + b_s T_{sw} & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $s \in S$ and $w \in W$, with T_1 acting as the identity.

To analyze the structure of $\mathcal{H}$, we introduce endomorphisms $X_s \in \mathrm{End}(\mathcal{H})$ defined by:

$$X_s(T_w) = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ a_s T_w + b_s T_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\mathcal{E} \subseteq \mathrm{End}(\mathcal{H})$ denote the subalgebra generated by $\{X_s\}_{s \in S}$.

An important property of these endomorphisms is their compatibility with the multiplicative structure of the generic Hecke algebra.

Lemma 2.44. The endomorphisms $\{X_s\}_{s \in S}$ commute with the multiplication relations defining $\mathcal{H}$.

Proof. For any $s, t \in S$ and $w \in W$, we verify:

- If $\ell(sw) > \ell(w)$, then $X_s(T_t T_w) = X_s(T_{tw})$ (if $\ell(tw) > \ell(w)$) or $X_s(a_t T_w + b_t T_{tw})$. Comparing with $T_t X_s(T_w)$ yields equality in both cases.
- The case $\ell(sw) < \ell(w)$ follows similarly by direct computation.

□

Now that we have established some properties of the endomorphisms X_s, we introduce a homomorphism that connects the subalgebra $\mathcal{E}$ of endomorphisms to the generic Hecke algebra $\mathcal{H}$ itself. Define an A-algebra homomorphism $\varphi : \mathcal{E} \to \mathcal{H}$ by $\varphi(X_s) = T_s$.

Theorem 2.45. The map φ is an isomorphism of A-algebras.

Proof.

Surjectivity: Since $\{T_s\}_{s \in S}$ generates $\mathcal{H}$, φ is surjective.

Injectivity: Suppose $\varphi(X) = 0$. We induct on $\ell(w)$. For $w = 1$, $X(T_1) = 0$ implies $X = 0$. Assume $X(T_u) = 0$ for all u with $\ell(u) < n$. For $\ell(w) = n$, write $w = sw'$ with $\ell(w') = n - 1$. Then:

$$X(T_w) = X(T_s T_{w'}) = X_s X(T_{w'}) = 0$$

by the induction hypothesis. Thus, $X = 0$. □

Having explored the generic Hecke algebra, we now turn our attention to a specific and important instance of this construction: the Hecke algebra associated with a Coxeter system. Fix a Coxeter system (W, S) and introduce an indeterminate v over the ring of integers $\mathbb{Z}$. The Hecke algebra $\mathcal{H}$ is defined as follows:

Definition 2.46. The *Hecke algebra* $\mathcal{H}$ is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{\delta_s \mid s \in S\}$, subject to:

- *Quadratic relation:* $\delta_s^2 = (v^{-1} - v)\delta_s + 1$ for all $s \in S$.
- *Braid relations:* $\underbrace{\delta_s \delta_t \delta_s \cdots}_{m_{st}} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{m_{st}}$ for all $s, t \in S$.

The Hecke algebra is often described as a deformation of the group algebra of the Coxeter group W. The following theorem explains why.

Theorem 2.47. The specialization $v \mapsto 1$ induces an isomorphism

$$\mathcal{H} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbb{Z} \cong \mathbb{Z}[W].$$

Proof. At $v = 1$, the quadratic relation becomes $\delta_s^2 = 1$, and the braid relations reduce to the defining relations of W. Thus, the map $\delta_s \mapsto s$ extends to an algebra isomorphism. $\square$

2.2.3 The Two Bases: Standard and Kazhdan-Lusztig

The Hecke algebra admits two particularly important bases: the standard basis and the Kazhdan-Lusztig basis. We will first introduce the standard basis, which is directly defined in terms of the generators of the algebra.

Definition 2.48. The *standard basis* of $\mathcal{H}$ is $\{\delta_w \mid w \in W\}$, where $\delta_w = \delta_{s_1} \cdots \delta_{s_k}$ for any reduced expression $w = s_1 \cdots s_k$.

Proposition 2.49. The standard basis is a basis of the Hecke algebra.

Proof.

Spanning. We prove the general case: For the generic Hecke algebra $\mathcal{H}_A(a_s, b_s)$ the set $\{T_w \mid w \in W\}$ is an A-basis. By definition $\mathcal{H}_A(a_s, b_s)$ is the free A-module on the symbols T_w, so they span.

Linear independence. Now we specialize the parameters to $a_s = 0,\ b_s = 1$. With these values the multiplication relations become

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w), \\ T_{sw} & \ell(sw) < \ell(w), \end{cases}$$

which are exactly the relations of the group algebra $A[W]$. The specialisation map

$$\mathcal{H}_A(a_s, b_s) \longrightarrow A[W], \qquad T_w \longmapsto w,$$

is A-linear and sends distinct T_w to distinct group elements. If we had a non-trivial A-linear relation $\sum_{w \in W} c_w T_w = 0$, applying the map would give $\sum_{w \in W} c_w w = 0$ in $A[W]$, contradicting the linear independence of $\{w \mid w \in W\}$ there. Hence all coefficients c_w must be zero, proving independence. $\square$

Alongside the standard basis, the Hecke algebra possesses another basis, the Kazhdan-Lusztig basis. This basis arises from an involution defined on the Hecke algebra:

Definition 2.50. The *Kazhdan-Lusztig involution* is the $\mathbb{Z}$-linear automorphism $\overline{(-)} : \mathcal{H} \to \mathcal{H}$ determined by:

- $\overline{v} = v^{-1}$,
- $\overline{\delta_s} = \delta_s^{-1} = \delta_s + (v - v^{-1})$,
- $\overline{ab} = \overline{a} \cdot \overline{b}$ for all $a, b \in \mathcal{H}$.

This involution is the centerpiece of the characterization of the Kazhdan-Lusztig basis:

Definition 2.51. The *Kazhdan-Lusztig basis* $\{b_w \mid w \in W\}$ is the unique set of self-dual elements ($\overline{b_w} = b_w$) satisfying a triangularity condition:

$$b_w = \delta_w + \sum_{y < w} h_{y,w}(v) \delta_y,$$

where $h_{y,w}(v) \in v\mathbb{Z}[v]$ and $<$ denotes the Bruhat order. The polynomials $h_{y,w}(v)$ are the *Kazhdan-Lusztig polynomials.*

Theorem 2.52. The Kazhdan-Lusztig basis exists and is unique.

Proof. We construct the Kazhdan–Lusztig basis $\{b_w\}_{w \in W}$ inductively on the Bruhat order. The construction is carried out entirely within the Hecke algebra $\mathcal{H}$, equipped with the standard basis $\{\delta_w\}$, where each δ_w is defined as the product of the generators δ_s along a fixed reduced expression for w. The goal is to define elements $b_w \in \mathcal{H}$ satisfying the following conditions:

1. $\overline{b_w} = b_w$, where the bar involution is determined by $\overline{v} = v^{-1}$ and $\overline{\delta_s} = \delta_s^{-1}$ extended anti-linearly.
2. $b_w = \delta_w + \sum_{y<w} h_{y,w}(v)\delta_y$ with $h_{y,w}(v) \in v\mathbb{Z}[v]$.

We begin by setting $b_e := \delta_e$, where e is the identity element in W. This element is evidently fixed by the bar involution and has no lower terms.

Suppose now that b_y has been defined for all $y < x$, and we wish to construct b_x. Choose a simple reflection $s \in S$ such that $xs < x$. Set $y := xs$, so that $y < x$ and b_y is available by the inductive hypothesis. Consider the product $b_y\delta_s$. Since $\ell(y) < \ell(x)$, and $\ell(x) = \ell(y) + 1$, this product can be expressed as:

$$b_y\delta_s = b_x + \sum_{z<x} \mu(z, y; s) b_z,$$

where the coefficients $\mu(z, y; s) \in \mathbb{Z}$ depend only on the known elements b_z with $z < x$. We define:

$$b_x := b_y\delta_s - \sum_{z<x} \mu(z, y; s) b_z.$$

This formula uniquely determines b_x. By construction, b_x is bar-invariant. Since b_y and δ_s are bar-invariant, so is their product. Moreover, all correction terms b_z are bar-invariant by the inductive hypothesis, and the coefficients $\mu(z, y; s) \in \mathbb{Z}$ are fixed under the involution. Therefore $\overline{b_x} = b_x$.

It remains to verify that b_x expands in the standard basis with the correct leading term and coefficients. Write

$$b_x = \delta_x + \sum_{z<x} h_{z,x}(v)\delta_z.$$

The triangularity and coefficient conditions follow by examining the structure of the product $b_y\delta_s$ in the standard basis. In the case $\ell(ys) > \ell(y)$, this product equals b_{ys} plus a linear combination of b_z for $z < y$, and thus lies in $\delta_{ys} + \sum_{z<ys} v\mathbb{Z}[v]\delta_z$. The subtraction of the correction terms ensures that the undesired coefficients are canceled, leaving $h_{z,x}(v) \in v\mathbb{Z}[v]$ as desired.

Uniqueness of b_x follows from the fact that any two elements satisfying the conditions of self-duality and triangularity must differ by a bar-invariant element supported strictly below x, with coefficients in both $v\mathbb{Z}[v]$ and $v^{-1}\mathbb{Z}[v^{-1}]$, which forces the difference to vanish.

This completes the inductive construction of the Kazhdan–Lusztig basis.

□

2.2.4 Computing Kazhdan–Lusztig Polynomials

The Kazhdan–Lusztig polynomials $h_{y,w}(v)$ introduced above are uniquely determined by the triangularity and self-duality conditions defining the basis $\{b_w\}$, but Kazhdan and Lusztig also provided a recursive algorithm to compute them explicitly. This recursive method is essential for practical computations, especially when dealing with Coxeter systems of higher rank.

To formulate the recursion, it is convenient to introduce an alternative $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis $\{C'_w\}_{w\in W}$ of $\mathcal{H}$ given by:

$$C'_w = q^{-\frac{\ell(w)}{2}} \sum_{y\leq w} P_{y,w}(q)\, T_y,$$

where $P_{y,w}(q)$ denotes the Kazhdan–Lusztig polynomial, normalized so that $P_{w,w}(q) = 1$ and $P_{y,w}(q) = 0$ unless $y \leq w$ in the Bruhat order. Once again, the defining property of this basis is invariance under the bar involution:

$$\overline{C'_w} = C'_w.$$

This condition uniquely determines the polynomials $P_{y,w}(q)$, which satisfy the degree constraint:

$$\deg P_{y,w}(q) \leq \frac{1}{2}(\ell(w) - \ell(y) - 1) \quad \text{if } y < w.$$

In their work, Kazhdan and Lusztig also introduced auxiliary polynomials $R_{x,y}(q)$ appearing in the expansion of inverse basis elements:

$$T_{y^{-1}}^{-1} = \sum_{x} D(R_{x,y})\, q^{-\ell(x)} T_x,$$

where D is the bar involution on $\mathcal{H}$, extended to polynomials by $D(q^{1/2}) = q^{-1/2}$.

The $R_{x,y}(q)$ satisfy the following recursion:

$$R_{x,y}(q) = \begin{cases} 0, & \text{if } x \not\leq y, \\ 1, & \text{if } x = y, \\ R_{sx,sy}(q), & \text{if } \ell(sx) < \ell(x) \text{ and } \ell(sy) < \ell(y), \\ R_{xs,ys}(q), & \text{if } \ell(xs) < \ell(x) \text{ and } \ell(ys) < \ell(y), \\ (q-1)R_{sx,y}(q) + qR_{sx,sy}(q), & \text{if } \ell(sx) > \ell(x) \text{ and } \ell(sy) < \ell(y). \end{cases}$$

These are computed entirely in terms of the Bruhat order and the Coxeter length function $\ell : W \to \mathbb{Z}_{\geq 0}$.

Once the $R_{x,y}(q)$ are known, one can compute the $P_{x,w}(q)$ recursively from the relation:

Proposition 2.53. The Kazhdan-Lusztig polynomials are computed as

$$\begin{aligned} &q^{\frac{1}{2}(\ell(w)-\ell(x))}\overline{P_{x,w}(q)} - q^{\frac{1}{2}(\ell(x)-\ell(w))}P_{x,w}(q) \\ = &\sum_{x<y\leq w} (-1)^{\ell(x)+\ell(y)} q^{\frac{1}{2}(-\ell(x)+2\ell(y)-\ell(w))}\overline{R_{x,y}(q)}P_{y,w}(q), \end{aligned}$$

which determines $P_{x,w}(q)$ by descending induction on w and ascending induction on x. This recursion ensures both the self-duality condition and the required triangularity with respect to the standard basis $\{T_w\}$.

While this method is impractical to carry out by hand beyond rank 3 or 4, it is well suited for computer implementation. Kazhdan–Lusztig polynomials for affine Weyl groups or high-rank finite Coxeter groups have been tabulated computationally (via Sage's `Coxeter3` package), although the large number of them eventually exceeds the memory of a computer.

Although the recursive formula determines Kazhdan–Lusztig polynomials $P_{y,w}(q)$, explicit computations reveal deeper features of their combinatorics. We collect several fundamental facts and examples.

Proposition 2.54. Let $y, w \in W$ with $y \leq w$ in the Bruhat order. Then:

1. The constant term of $P_{y,w}(q)$ is always 1.
2. If $\ell(w) - \ell(y) \in 0, 1, 2$, then $P_{y,w}(q) = 1$.
3. If $w = w_0$ is the longest element in a finite Coxeter group W, then $P_{y,w_0}(q) = 1$ for all y.
4. If W has rank at most 2 (e.g., type A_1, A_2, or $I_2(m)$), then $P_{y,w}(q) = 1$ if $y \leq w$, and 0 otherwise.

Example 2.55. For the Coxeter group of type A_3 with simple reflections $S = a, b, c$ and commutation relation $ac = ca$, we have:

$$\begin{aligned} P_{b,bacb}(q) &= 1 + q, \\ P_{ac,acbca}(q) &= 1 + q. \end{aligned}$$

These are the first examples of non-constant Kazhdan–Lusztig polynomials in type A.

Remark 2.56. The simplicity of Kazhdan–Lusztig polynomials in low-rank Coxeter groups does not generalize. There is also an analogue of Kazhdan-Lusztig polynomials for many other contexts, such as for the representation of real Lie groups. In high-rank groups, especially exceptional types, the polynomials become increasingly complex. For instance, in the split real form of type E_8, one of the most complicated Lusztig–Vogan polynomials (a variant of Kazhdan–Lusztig polynomials relevant to the representation theory of real Lie groups) is given by:

$$\begin{aligned} P(q) = {} & 152q^{22} + 3472q^{21} + 38791q^{20} + 293021q^{19} + 1370892q^{18} \\ & + 4067059q^{17} + 7964012q^{16} + 11159003q^{15} + 11808808q^{14} \\ & + 9859915q^{13} + 6778956q^{12} + 3964369q^{11} + 2015441q^{10} \\ & + 906567q^{9} + 363611q^{8} + 129820q^{7} + 41239q^{6} + 11426q^{5} \\ & + 2677q^{4} + 492q^{3} + 61q^{2} + 3q. \end{aligned}$$

While extensive research has focused on developing faster algorithms for computing Kazhdan–Lusztig polynomials, we will not discuss those advancements here.

3 Soergel Bimodules

3.1 Constructing Soergel Bimodules

3.1.1 The Polynomial Ring and its Structure

Given a realization $\mathfrak{h}$, the fundamental algebraic object is the ring of polynomial functions on $\mathfrak{h}$.

Definition 3.1 (Polynomial Ring)**.** Let $\mathfrak{h}$ be a realization. The associated polynomial ring is $R := \mathrm{Sym}(\mathfrak{h}^*)$, the symmetric algebra[1] on the dual space $\mathfrak{h}^*$. If $\mathfrak{h}^*$ has basis $\{\beta_1, \dots, \beta_n\}$, then $R = k[\beta_1, \dots, \beta_n]$.

We impose a grading on R by declaring that all elements of $\mathfrak{h}^*$ have degree 2: $\deg(\alpha) = 2$ for all $\alpha \in \mathfrak{h}^* \setminus \{0\}$. Consequently, $R = \bigoplus_{k \geq 0} R^{2k}$, where $R^{2k} = \mathrm{Sym}^k(\mathfrak{h}^*)$. This grading convention is motivated by connections to the cohomology rings of flag varieties in geometric settings, where $H^2(G/B; k) \cong \mathfrak{h}^*$.

The action of W on $\mathfrak{h}$ induces a dual action on $\mathfrak{h}^*$ via $(w \cdot \alpha)(v) = \alpha(w^{-1}v)$ for $w \in W, \alpha \in \mathfrak{h}^*, v \in \mathfrak{h}$. This action extends uniquely to a degree-preserving action of W on R by k-algebra automorphisms. Explicitly, $(w \cdot f)(v) = f(w^{-1}v)$ for $f \in R, v \in \mathfrak{h}$.

Subgroups of W give rise to invariant subrings of R. For any subset $I \subseteq S$, let $W_I = \langle I \rangle \subseteq W$ be the standard parabolic subgroup generated by I.

Definition 3.2. The ring of W_I-invariants is $R^I := \{f \in R \mid w(f) = f \text{ for all } w \in W_I\}$. For $I = \{s\}$, we write $R^s := R^{\{s\}}$.

The structure of these invariant rings, particularly when W_I is finite, is governed by the Chevalley-Shephard-Todd theorem.

Theorem 3.3 (Chevalley-Shephard-Todd Theorem)**.** Let $\mathfrak{h}$ be a finite-dimensional vector space over a field k of characteristic zero, and let $G \subseteq GL(\mathfrak{h})$ be a finite group. The invariant ring $R^G = \mathrm{Sym}(\mathfrak{h}^*)^G$ is isomorphic to a polynomial ring if and only if G is generated by pseudo-reflections[2]. Furthermore, if R^G is polynomial, then R is a free graded module over R^G of rank $|G|$.

Proof. See [10, 25] or standard texts on invariant theory. □

Remark 3.4. Many variants of this theorem exist. For example, the theorem can be restated as follows: Suppose $I \subset S$ such that W_I is finite. Then R^I is a polynomial ring. Moreover, R is a graded free module of finite rank over R^I.

The Chevalley-Shephard-Todd theorem provides the essential algebraic guarantee that the invariant rings R^I for finite parabolic subgroups W_I (which are generated by reflections, hence pseudo-reflections) are well-structured polynomial rings, and crucially, that R is a free module over them. This freeness property is fundamental for the construction of Soergel bimodules, which has $B_s := R \otimes_{R^s} R(1)$ as a building block - this involves tensoring R over R^s. Since $W_{\{s\}} = \{e, s\}$ is always finite of order 2, the theorem guarantees that R^s is polynomial and R is a free R^s-module. This ensures that the tensor product $R \otimes_{R^s} R$ is well-behaved and its structure (e.g., rank) can be readily determined from the structure of R as an R^s-module. Without this freeness, the definition and properties of B_s would be significantly more complicated, potentially obstructing the entire categorification program.

The case $I = \{s\}$ is crucial for the construction. Since $W_{\{s\}} = \{e, s\}$ is finite, the theorem applies. We have the following structural result:

Lemma 3.5. For any $s \in S$, the polynomial ring R is a free graded R^s-module of rank two. More explicitly, R decomposes as a direct sum of s-invariants and s-antiinvariants:

$$R = R^s \oplus R^s \cdot \alpha_s$$

[1] The symmetric algebra $\mathrm{Sym}(\mathfrak{h}^*) := T(\mathfrak{h}^*)/\langle \alpha \otimes \beta - \beta \otimes \alpha \rangle_{\alpha, \beta \in \mathfrak{h}^*}$ where $T(\mathfrak{h}^*) := \bigoplus_{n=0}^{\infty} \mathfrak{h}^{*\otimes n}$ is the tensor algebra on $\mathfrak{h}^*$.

[2] An element w is a *pseudo-reflection* if $\mathrm{rk}(\mathrm{id} - w) = 1$, or equivalently, w fixes a hyperplane pointwise.

where $R^s \cdot \alpha_s = \{f\alpha_s \mid f \in R^s\}$. Since $\deg(\alpha_s) = 2$, this decomposition yields an isomorphism of graded R^s-modules:

$$R \cong R^s \oplus R^s(-2)$$

Proof. Any $f \in R$ can be written as:

$$f = \frac{f + s(f)}{2} + \frac{f - s(f)}{2}.$$

The first term is s-invariant. Since $f - s(f)$ vanishes on the reflecting hyperplane $\ker(\alpha_s)$, it must be divisible by α_s. That is, we can write:

$$f - s(f) = g\alpha_s \quad \text{for some } g \in R.$$

Now consider the action of s on both sides:

$$s(f - s(f)) = s(f) - f = -g\alpha_s,$$

and also:

$$s(f - s(f)) = s(g\alpha_s) = s(g)s(\alpha_s) = s(g)(-\alpha_s).$$

Comparing both expressions:

$$s(g)(-\alpha_s) = -g\alpha_s \quad \Rightarrow \quad s(g) = g.$$

Thus, $g \in R^s$. Therefore:

$$\frac{f - s(f)}{2} = \frac{g}{2}\alpha_s \in R^s \cdot \alpha_s,$$

assuming $\mathrm{char}(k) \neq 2$. The sum is direct: suppose $f\alpha_s \in R^s \cdot \alpha_s$ is s-invariant. Then:

$$f\alpha_s = s(f\alpha_s) = s(f)s(\alpha_s) = f(-\alpha_s),$$

which implies:

$$2f\alpha_s = 0.$$

Since $\alpha_s \neq 0$ and R is a domain, this forces $f = 0$. Therefore, $\{1, \alpha_s\}$ forms a basis for R as a free module over R^s. Finally, the graded isomorphism follows from the fact that $\deg(\alpha_s) = 2$.

□

This decomposition $R \cong R^s \oplus R^s(-2)$ is the cornerstone for understanding the structure of the basic bimodules B_s.

3.1.2 Demazure Operators

The decomposition $R = R^s \oplus R^s(-2)$ can be made more explicit, and its structure further explored, using Demazure operators. These operators are the major tool in relating the left and right module structures in Soergel bimodules and are fundamental tools for calculations.

Definition 3.6 ([12])**.** For $s \in S$, the *Demazure operator* $\partial_s : R \to R(-2)$ is the k-linear map defined by

$$\partial_s(f) := \frac{f - s(f)}{\alpha_s}$$

for $f \in R$.

The operator ∂_s is well-defined for the following reasons:

1. As shown in the proof of Lemma 3.5, for any $f \in R$, the polynomial $f - s(f)$ vanishes on the hyperplane $\ker(\alpha_s)$. In a polynomial ring over a field, this implies $f - s(f)$ is divisible by α_s, provided $\alpha_s \neq 0$.
2. The operator lowers degree by $\deg(\alpha_s) = 2$, hence the target $R(-2)$.
3. Crucially, the result $\partial_s(f)$ is s-invariant, meaning $\partial_s(f) \in R^s$.

Proof of s-invariance.

$$s(\partial_s(f)) = s\left(\frac{f - s(f)}{\alpha_s}\right) = \frac{s(f) - s^2(f)}{s(\alpha_s)} = \frac{s(f) - f}{-\alpha_s} = \frac{f - s(f)}{\alpha_s} = \partial_s(f)$$

using $s^2 = \mathrm{id}$ and $s(\alpha_s) = -\alpha_s$. □

Thus, ∂_s can be viewed as a map $\partial_s : R \to R^s(-2)$.

Demazure operators satisfy several fundamental algebraic properties, summarized in Table 1.

Table 1: Properties of Demazure Operators

Property	**Formula**	**Notes**
R^s-Linearity	$\partial_s(gf) = g\partial_s(f)$ for $g \in R^s$	Left R^s-linearity
	$\partial_s(fg) = \partial_s(f)g$ if $g \in R^s$	Right R^s-linearity (less common)
Interaction with s	$\partial_s s = -\partial_s$, $s\partial_s = \partial_s$	Relates operator to reflection
Nilpotence	$\partial_s^2 = 0$	Fundamental algebraic property
Twisted Leibniz Rule	$\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$	Rule for products
Braid Relations	$\underbrace{\partial_s\partial_t\cdots}_{m_{st}} = \underbrace{\partial_t\partial_s\cdots}_{m_{st}}$ $(m_{st} < \infty)$	Connection to W, defines ∂_w

Proof.

- R^s-Linearity: $\partial_s(gf) = (gf - s(gf))/\alpha_s = (gf - s(g)s(f))/\alpha_s = (gf - gs(f))/\alpha_s = g(f - s(f))/\alpha_s = g\partial_s(f)$ since $g \in R^s$. Right linearity is similar.
- Interaction with s: $\partial_s(s(f)) = (s(f) - s^2(f))/\alpha_s = (s(f) - f)/\alpha_s = -\partial_s(f)$. Also $s(\partial_s(f)) = \partial_s(f)$ as shown before.
- Nilpotence: $\partial_s(\partial_s(f)) = (\partial_s(f) - s(\partial_s(f)))/\alpha_s = (\partial_s(f) - \partial_s(f))/\alpha_s = 0$.
- Twisted Leibniz Rule: A direct calculation using the definition of ∂_s and properties of s.
- Braid Relations: This is a deeper result, often proven geometrically or by careful algebraic manipulation. See [12] or related literature.

□

The fact that the Demazure operators satisfy the same braid relations as the simple reflections $s \in S$ is very important. It indicates that the algebraic structure generated by these operators reflects the combinatorial structure of the Coxeter group W itself. This allows one to define an operator ∂_w for any $w \in W$ by choosing a reduced expression $\underline{w} = (s_1, \ldots, s_n)$ for w and setting $\partial_w := \partial_{s_1} \cdots \partial_{s_n}$. The braid relations guarantee that ∂_w is independent of the choice of reduced expression. This connection is essential for linking the algebraic constructions involving R and its bimodules back to the underlying combinatorics of W and the Hecke algebra. Without the braid relations for ∂_s, the theory would likely remain tied to specific sequences of reflections, lacking a connection to the group elements w.

The role of the Demazure operator is to provide the projection onto the s-antiinvariant component $R^s(-2)$ in the decomposition $R = R^s \oplus R^s(-2)$. Assuming $\mathrm{char}(k) \neq 2$, any $f \in R$ can be written as:

$$f = \underbrace{\frac{f + s(f)}{2}}_{\in R^s} + \underbrace{\frac{f - s(f)}{2}}_{\in R^s \cdot \alpha_s} = \frac{f + s(f)}{2} + \alpha_s \frac{\partial_s(f)}{2}$$

Alternatively, as noted in [13], one can write (again assuming $\mathrm{char}(k) \neq 2$):

$$f = \underbrace{\partial_s\left(f\frac{\alpha_s}{2}\right)}_{\in R^s} + \underbrace{\frac{\alpha_s}{2}\partial_s(f)}_{\in R^s \cdot \alpha_s}$$

The map $f \mapsto \partial_s(f)$ isolates the $R^s(-2)$ component, up to multiplication by $\alpha_s/2$.

3.1.3 Basic Bimodules

Using the polynomial ring R, its invariant subrings R^s, and the Demazure operators ∂_s, we can now begin to construct one side of our story (Soergel bimodules) through various bimodule[3] structures.

We work within the category of graded R-bimodules.

Definition 3.7 ($\mathrm{Hom}^\bullet_{R\text{-gbim}}(M,N)$)**.** Let R-gbim denote the category whose objects are $\mathbb{Z}$-graded (R,R)-bimodules $M = \bigoplus_{k\in\mathbb{Z}} M^k$ such that $R^iM^jR^l \subseteq M^{i+j+l}$, and whose morphisms are R-bimodule homomorphisms $\phi : M \to N$ that preserve the grading (i.e., $\phi(M^k) \subseteq N^k$). We denote by $\mathrm{Hom}^\bullet_{R\text{-gbim}}(M,N)$ the graded k-module $\bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{R\text{-gbim}}(M, N(d))$, where $\mathrm{Hom}_{R\text{-gbim}}(M, N(d))$ consists of homomorphisms of degree d.

This category is monoidal under the tensor product $\otimes_R$. For $M, N \in R$-gbim, the tensor product $M \otimes_R N$ is graded by $(M \otimes_R N)^k = \mathrm{span}\{m \otimes n \mid m \in M^i, n \in N^j, i + j = k\}/(\text{relations})$. The identity object is R itself, considered as a bimodule concentrated in degree 0. The shift functors (n) are monoidal autoequivalences.[4]

We are primarily interested in the full subcategory $R\text{-gbim}_{\mathrm{fg}}$ consisting of bimodules that are finitely generated as both left and right R-modules.

The simplest non-trivial Soergel bimodules are associated with simple reflections.

Definition 3.8 (Basic Bimodule)**.** For each $s \in S$, the *basic (Soergel) bimodule* B_s is defined as

$$B_s := R \otimes_{R^s} R(1)$$

This is an object in $R\text{-gbim}_{\mathrm{fg}}$. The tensor product is taken over the ring of s-invariants R^s, meaning $rf \otimes r' = r \otimes fr'$ for $f \in R^s$. The grading shift (1) indicates that the element $1 \otimes 1$ has degree $\deg(1 \otimes 1) = \deg(1) + \deg(1) - 1 = 0 + 0 - 1 = -1$.

The structure of B_s as an R-module (forgetting one side of the bimodule structure) is straightforward to determine using the rank-two freeness lemma.

Lemma 3.9. As a graded left R-module (or equivalently, as a graded right R-module), B_s is free of rank two. Specifically, there is an isomorphism of graded left R-modules:

$$B_s \cong R(1) \oplus R(-1)$$

Consequently, the graded rank is $\underline{\mathrm{rk}}_R B_s = v + v^{-1}$.

Proof. We use the isomorphism of graded right R^s-modules $R \cong R^s \oplus R^s(-2)$ from Lemma 3.5. Since tensoring over R^s is right exact and commutes with direct sums, and $R^s \otimes_{R^s} R(1) \cong R(1)$, we have:

$$\begin{aligned} B_s = R \otimes_{R^s} R(1) &\cong (R^s \oplus R^s(-2)) \otimes_{R^s} R(1) \\ &\cong (R^s \otimes_{R^s} R(1)) \oplus (R^s(-2) \otimes_{R^s} R(1)) \\ &\cong R(1) \oplus (R(-2) \otimes_{R^s} R^s(1)) \quad (\text{as } R^s \text{ is central in } R^s) \\ &\cong R(1) \oplus R(-2)(1) \\ &\cong R(1) \oplus R(-1) \end{aligned}$$

The isomorphism holds for left R-modules by viewing R as a left R-module and $R(1)$ as a left R^s-module (which is isomorphic to $R(1)$ as R^s acts centrally). The graded rank polynomial corresponding to $R(1) \oplus R(-1)$ is $v^1 + v^{-1}$. □

[3] A *bimodule* over two algebras A and B is a module M that is both a left A-module and a right B-module, satisfying the compatibility condition: $a \cdot (m \cdot b) = (a \cdot m) \cdot b$ for all $a \in A$, $b \in B$, $m \in M$.

[4] A monoidal functor is a *monoidal autoequivalence* if it is an equivalence of categories and its inverse is also monoidal.

3.1.4 Polynomial Forcing Relations

The interaction between the left and right R-actions on B_s is crucial and is governed by the R^s-module structure used in the tensor product. This interaction can be made explicit using a basis and the Demazure operator. Assuming $\text{char}(k) \neq 2$, we can define basis elements:

- $c_{\text{id}} := 1 \otimes 1 \in (B_s)^{-1}$
- $c_s := \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \in (B_s)^1$

These elements $\{c_{\text{id}}, c_s\}$ form a basis for B_s as both a left and right free R-module.

Proposition 3.10. The basis elements $\{c_{\text{id}}, c_s\}$ satisfy the following relations for any $f \in R$:

$$f \cdot c_{\text{id}} = c_{\text{id}} \cdot s(f) + c_s \cdot \partial_s(f) \tag{1}$$
$$f \cdot c_s = c_s \cdot s(f) + \alpha_s c_{\text{id}} \cdot \partial_s(f) \tag{2}$$

(Note: Relation (2) is a more generalized version of $f \cdot c_s = c_s \cdot f$ from [13], which holds only if $f \in R^s$. The corrected version reflects the interaction more accurately, aligning with diagrammatic presentations found in sources like [14].)

Proof. These relations arise from the definition $B_s = R \otimes_{R^s} R(1)$ and the properties of ∂_s. For (1): $f \cdot (1 \otimes 1) = f \otimes 1$. We decompose $f = \frac{f+s(f)}{2} + \alpha_s \frac{\partial_s(f)}{2}$. Since $\frac{f+s(f)}{2} \in R^s$, it can pass through the tensor: $f \otimes 1 = \frac{f+s(f)}{2} \otimes 1 + \alpha_s \frac{\partial_s(f)}{2} \otimes 1 = 1 \otimes \frac{f+s(f)}{2} + \alpha_s \frac{\partial_s(f)}{2} \otimes 1$. We need to relate this to $c_{\text{id}} \cdot s(f) + c_s \cdot \partial_s(f) = (1 \otimes 1)s(f) + \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)\partial_s(f) = 1 \otimes s(f) + \frac{1}{2}(\alpha_s \partial_s(f) \otimes 1 + 1 \otimes \alpha_s \partial_s(f))$. This can be verified by just manipulating the tensor product around some more. □

The relations essentially encode how the left action $f \cdot -$ is expressed in terms of the right action $- \cdot g$ and the Demazure operator ∂_s. These relations are fundamental for understanding the bimodule structure.

3.1.5 Bott-Samelson Bimodules

Iterating the tensor product of basic bimodules yields the Bott-Samelson bimodules.

Definition 3.11 (Bott Samuleson Bimodule)**.** Let $\underline{w} = (s_1, s_2, \ldots, s_n)$ be any sequence (expression) of simple reflections $s_i \in S$. The corresponding *Bott-Samelson bimodule* is defined as:

$$BS(\underline{w}) := B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_n}$$

For the empty expression $\underline{w} = \emptyset$ (corresponding to the identity element $e \in W$), we define $BS(\emptyset) := R$. This gives $BS(\underline{w}) \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R(n)$.

Concatenation of expressions corresponds to the tensor product: $BS(\underline{u}) \otimes_R BS(\underline{v}) = BS(\underline{uv})$.

Lemma 3.12. Any Bott-Samelson bimodule $BS(\underline{w})$ is graded free of finite rank as a left (and right) R-module. The graded rank is $\underline{\text{rk}}_R BS(\underline{w}) = (v + v^{-1})^n$, where n is the length of the expression $\underline{w}$.

Proof. This follows by induction on the length n of $\underline{w}$, using the fact that B_s is free of rank $v + v^{-1}$ (Lemma 3.9) and that the tensor product of free modules over a polynomial ring is free. □

Remark 3.13. It is crucial to remember that the isomorphism class of the Bott-Samelson bimodule $BS(\underline{w})$ depends on the specific sequence $\underline{w}$, not just on the element $w = s_1 s_2 \cdots s_n \in W$. For example, if $w = st = ts$ (i.e., $m_{st} = 2$), then $BS(s,t) = B_s \otimes_R B_t$ is generally not isomorphic to $BS(t,s) = B_t \otimes_R B_s$, even though they correspond to the same group element w. However, if $\underline{w}$ and $\underline{w}'$ are two different reduced expressions for the same element w, the corresponding Bott-Samelson bimodules $BS(\underline{w})$ and $BS(\underline{w}')$ share a unique indecomposable direct summand (up to isomorphism and shift), which will be denoted B_w.

3.1.6 The Category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$

Bott-Samelson bimodules serve as building blocks, but they depend on the choice of expression. The intrinsic objects associated with elements of W are their indecomposable direct summands. The category encompassing these objects is the category of Soergel bimodules, $\mathbb{S}\mathrm{Bim}$.

Definition 3.14 ($\mathbb{S}\mathrm{Bim}$)**.** The *category of Soergel bimodules*, denoted $\mathbb{S}\mathrm{Bim}$, is the smallest strictly full subcategory[5] of R-$\mathrm{gbim}_{\mathrm{fg}}$ that satisfies the following properties:

1. It contains the regular bimodule $R = BS(\emptyset)$.
2. It contains all basic bimodules $B_s = R \otimes_{R^s} R(1)$ for $s \in S$.
3. It is closed under taking finite direct sums ($\oplus$).
4. It is closed under grading shifts (n) for all $n \in \mathbb{Z}$.
5. It is closed under taking direct summands (i.e., it is Karoubian or idempotent complete).

Equivalently, $\mathbb{S}\mathrm{Bim}$ is the Karoubi envelope[6] of the additive monoidal category generated by the objects $\{B_s(n) \mid s \in S, n \in \mathbb{Z}\}$ under $\oplus$ and $\otimes_R$. The Karoubi envelope formally adds objects corresponding to images of idempotent endomorphisms.

The category $\mathbb{S}\mathrm{Bim}$ inherits several important structural properties:

- **Additive:** $\mathbb{S}\mathrm{Bim}$ is an additive category, meaning it has finite direct sums and a zero object (the zero bimodule).
- $\mathbb{Z}[v^{\pm 1}]$**-linear:** The grading shifts endow $\mathbb{S}\mathrm{Bim}$ with the structure of a $\mathbb{Z}[v^{\pm 1}]$-linear category[7], where the action of v^n corresponds to the shift (n).
- **Monoidal:** $\mathbb{S}\mathrm{Bim}$ is a monoidal category with tensor product $\otimes_R$ and unit object R. This follows because the tensor product of Bott-Samelson bimodules is Bott-Samelson, and the property of being a direct summand is preserved under tensor products with free modules like B_s.
- **Krull-Schmidt:** $\mathbb{S}\mathrm{Bim}$ is a Krull-Schmidt category. This means that every object in $\mathbb{S}\mathrm{Bim}$ decomposes into a finite direct sum of indecomposable objects, and this decomposition is unique up to isomorphism and permutation of summands. This property relies on the fact that $\mathbb{S}\mathrm{Bim}$ is additive, Karoubian, and the endomorphism rings of objects are appropriately finite-dimensional after suitable base change (e.g., $R \to k$).
- **Freeness:** Every object $B \in \mathbb{S}\mathrm{Bim}$ is graded free of finite rank as both a left and a right R-module. This follows from the freeness of Bott-Samelson bimodules and the fact that direct summands of graded free modules over polynomial rings (over a field) are themselves graded free (a graded analogue of the Quillen-Suslin theorem[8], which is simpler in the graded setting).

It is important to note that $\mathbb{S}\mathrm{Bim}$ is generally *not* an abelian category, as it is not typically closed under taking kernels or cokernels within the larger category R-gbim.

3.1.7 Indecomposable Objects and Examples

The Krull-Schmidt property guarantees the existence and uniqueness of decomposition into indecomposable objects. A fundamental result, formalized in Theorem 3.2.3, states that the

[5] A *strictly full subcategory* $\mathcal{A} \subset \mathcal{C}$ is a subcategory such that: $\mathcal{A}$ is full (for all $X, Y \in \mathrm{Ob}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$) and isomorphism-closed (if $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{C})$ with $X \cong Y$ in $\mathcal{C}$, then $Y \in \mathrm{Ob}(\mathcal{A})$).

[6] The *Karoubi envelope* of a category $\mathcal{C}$ is the universal category $\mathrm{Kar}(\mathcal{C})$ containing $\mathcal{C}$ in which every idempotent morphism $e : X \to X$ splits; that is, there exist Y, $r : X \to Y$, and $s : Y \to X$ such that $e = s \circ r$ and $r \circ s = \mathrm{id}_Y$.

[7] A $\mathbb{Z}[v^{\pm 1}]$*-linear category* is a category $\mathcal{C}$ such that for all objects $X, Y \in \mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a module over the ring $\mathbb{Z}[v^{\pm 1}]$and the composition of morphisms is $\mathbb{Z}[v^{\pm 1}]$-bilinear.

[8] The *Quillen–Suslin theorem* states that every finitely generated projective module over a polynomial ring $k[x_1, \ldots, x_n]$ (with k a field) is free.

indecomposable objects in $\mathbb{S}$Bim are, up to isomorphism and grading shift, in one-to-one correspondence with the elements of the Coxeter group W. We denote the indecomposable object corresponding to $w \in W$ (normalized appropriately, e.g., generated in degree $-\ell(w)$) by B_w.

The structure of $\mathbb{S}$Bim and the decomposition of Bott-Samelson bimodules become apparent through examples.

Example 3.15. The Coxeter group has two elements, e and s. The corresponding indecomposable Soergel bimodules (up to shift) are $B_e = R$ and $B_s = R \otimes_{R^s} R(1)$. The tensor square of B_s decomposes as:

$$B_s \otimes_R B_s \cong B_s(1) \oplus B_s(-1)$$

This isomorphism can be established by constructing explicit projection and inclusion maps or by analyzing the endomorphism ring $\mathrm{End}_{\mathbb{S}\mathrm{Bim}}(B_s \otimes_R B_s)$. This decomposition directly reflects the quadratic relation in the Hecke algebra, often written as $H_s^2 = (v + v^{-1})H_s + (1 - v^2)$ or in terms of the Kazhdan-Lusztig basis element $C'_s = H_s + vH_e$, for which $C'_sC'_s = (v + v^{-1})C'_s$. The categorical decomposition $B_sB_s \cong B_s(1) \oplus B_s(-1)$ lifts the algebraic relation $[B_s]^2 = (v + v^{-1})[B_s]$ in the Grothendieck group $[\mathbb{S}\mathrm{Bim}]_\oplus$.

Example 3.16. Here $W = \{e, s, t, st, ts, sts = tst\}$, with $|W| = 6$. The indecomposable objects in $\mathbb{S}$Bim (up to shift) are $\{B_w \mid w \in S_3\}$. These are $B_e = R$, B_s, B_t. The products $B_s \otimes_R B_s \cong B_s(1) \oplus B_s(-1)$ and $B_t \otimes_R B_t \cong B_t(1) \oplus B_t(-1)$ yield no new indecomposables. The Bott-Samelson bimodules $BS(s,t) = B_s \otimes_R B_t$ and $BS(t,s) = B_t \otimes_R B_s$ correspond to the elements st and ts. These are indecomposable and are denoted B_{st} and B_{ts} respectively. They are generally not isomorphic, $B_{st} \ncong B_{ts}$. The length 3 Bott-Samelson bimodules decompose according to the braid relation $sts = tst$. Explicit calculations show:

$$BS(s,t,s) = B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$$
$$BS(t,s,t) = B_t \otimes_R B_s \otimes_R B_t \cong B_{tst} \oplus B_t$$

Here B_{sts} (which is isomorphic to B_{tst}) is the new indecomposable Soergel bimodule corresponding to the longest element $w_0 = sts = tst$. The appearance of B_s and B_t as summands reflects the structure constants in the Hecke algebra when multiplying basis elements. For instance, in the Kazhdan-Lusztig basis $\{C'_w\}$, one finds relations like $C'_sC'_tC'_s = C'_{sts} + C'_s$. Further tensor products decompose in terms of these six indecomposables, e.g., $B_{sts} \otimes_R B_s \cong B_{sts}(1) \oplus B_{sts}(-1)$. The set of indecomposable Soergel bimodules $\{B_w \mid w \in S_3\}$ provides a basis for the Grothendieck group $[\mathbb{S}\mathrm{Bim}]_\oplus$, mirroring the structure of the Hecke algebra $\mathcal{H}(A_2)$.

These examples illustrate a crucial point: the decomposition patterns of Bott-Samelson bimodules under the tensor product $\otimes_R$ precisely mirror the multiplicative structure of the Hecke algebra $\mathcal{H}$, particularly the quadratic relation ($s^2 = e$) and the braid relations ($stst... = tsts...$). The emergence of unique indecomposable objects B_w associated with each group element w provides concrete evidence for the categorification claim. The correspondence between categorical operations ($\otimes_R, \oplus$) and algebraic operations (multiplication, addition) is the essence of how $\mathbb{S}$Bim categorifies $\mathcal{H}$.

3.2 Soergel's Categorification Theorem

The examples above hint at a deep, structural connection between the category of Soergel bimodules $\mathbb{S}$Bim and the Hecke algebra $\mathcal{H}$. Soergel's Categorification Theorem formalizes this relationship, establishing $\mathbb{S}$Bim as a categorical realization of $\mathcal{H}$. The theorem, in its original formulation [27, 28], relies on the assumption that the underlying realization $\mathfrak{h}$ is *reflection faithful*. While later developments have relaxed this condition [14, 1], we present the theorem under this assumption first.

3.2.1 More Building Blocks

As introduced previously, the additive and monoidal structure of $\mathbb{S}$Bim is captured algebraically by its split Grothendieck group.

Definition 3.17 (Split Grothendieck Group)**.** The *split Grothendieck group* of $\mathbb{S}$Bim, denoted $[\mathbb{S}\mathrm{Bim}]_\oplus$, is the free abelian group generated by the isomorphism classes $[B]$ of objects $B \in \mathbb{S}\mathrm{Bim}$, subject to the relation $[B] = [B'] + [B'']$ whenever $B \cong B' \oplus B''$. It forms a $\mathbb{Z}[v^{\pm 1}]$-algebra where the ring structure is given by multiplication $[B] \cdot [B'] := [B \otimes_R B]$ with unit $[R]$, and the module structure is given by scalar multiplication $v^n \cdot [B] := [B(n)]$ for $n \in \mathbb{Z}$.

To connect $[\mathbb{S}\mathrm{Bim}]_\oplus$ to a specific basis of $\mathcal{H}$, Soergel introduced standard objects and filtrations.

Definition 3.18 (Standard Bimodule)**.** For $w \in W$, the *standard bimodule* R_w is defined as the graded (R, R)-bimodule which is R as a set, with the standard left R-action $(r \cdot m = rm)$ and the right action twisted by w: $m \cdot r = mw(r)$ for $m \in R_w, r \in R$. The grading is the standard grading of R.

Remark 3.19. Standard bimodules R_w for $w \neq e$ are generally *not* objects in $\mathbb{S}$Bim. They serve as reference objects for defining filtrations. They form their own category $\mathcal{S}td\mathcal{B}im$ where $R_x \otimes_R R_y \cong R_{xy}$ and $\mathrm{Hom}^\bullet_{R\text{-gbim}}(R_x, R_y) \cong \delta_{x,y} R$. The Grothendieck group $[\mathbb{S}\mathrm{Bim}]_\oplus$ is isomorphic to the group ring $\mathbb{Z}[v^{\pm 1}][W]$.

Definition 3.20 (Standard Filtration)**.** A *standard filtration* (or Δ-filtration) of an object $M \in \mathbb{S}\mathrm{Bim}$ is a finite filtration by graded sub-bimodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that each subquotient M_i/M_{i-1} is isomorphic to a direct sum of shifts of standard bimodules:

$$M_i/M_{i-1} \cong \bigoplus_{x \in W, n \in \mathbb{Z}} R_x(n)^{\oplus m_{i,x,n}}$$

for some non-negative integers $m_{i,x,n}$. Soergel proved that every $M \in \mathbb{S}\mathrm{Bim}$ admits such filtrations. Furthermore, there exists a unique Δ-filtration where the elements x appearing in the subquotients M_i/M_{i-1} can be ordered such that x only appears after all $y > x$ (in Bruhat order) have appeared. For such a filtration, the *graded multiplicity* $h_x(M) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ of R_x (defined as $h_x(M) = \sum_{i,n} m_{i,x,n} v^{-n}$) is an invariant of M. A dual notion of ∇-filtration also exists, where the order is reversed.

Example 3.21. Consider $B_s = R \otimes_{R^s} R(1) \cong R(1) \oplus R(-1)$. A standard filtration is $0 \subset M_1 \subset B_s$, where M_1 is an appropriate sub-bimodule isomorphic to $R_s(-1)$. The quotient B_s/M_1 is then isomorphic to $R_e(1) = R(1)$. Thus, the standard subquotients are $R_s(-1)$ and $R_e(1)$. The graded multiplicities are $h_s(B_s) = v^{-(-1)} = v^1$ and $h_e(B_s) = v^{-1}$. (Note: Normalization conventions for h_x can vary, affecting powers of v. This calculation needs careful checking against a specific source's convention).

3.2.2 The Character Map

The standard filtrations allow defining a map from the Grothendieck group to the Hecke algebra.

Definition 3.22 (Character Map)**.** The *(standard) character map*

$$\mathrm{ch} : [\mathbb{S}\mathrm{Bim}]_\oplus \to \mathcal{H}$$

is the $\mathbb{Z}[v^{\pm 1}]$-linear map defined on the classes of indecomposable Soergel bimodules $[B] \in [\mathbb{S}\mathrm{Bim}]_\oplus$ by

$$\mathrm{ch}([B]) = \sum_{x \in W} h_x(B) H_x,$$

where $h_x(B) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ denotes the graded multiplicity of the Bott–Samelson bimodule R_x in a fixed Δ-filtration of B, and $\{H_x\}_{x \in W}$ denotes the standard basis of the Hecke algebra $\mathcal{H}$ over $\mathbb{Z}[v^{\pm 1}]$.

We adopt the normalization convention of [13], in which $H_s = T_s + 1$ for each simple reflection s, where the generators T_s satisfy the quadratic relation

$$T_s^2 = (v - v^{-1}) T_s + 1.$$

The graded multiplicities $h_x(B)$ depend on a chosen normalization of the Δ-filtration consistent with the standard basis $\{H_x\}$.

Using the multiplicities computed in the example above ($h_s(B_s) = v$, $h_e(B_s) = v^{-1}$), we obtain

$$\mathrm{ch}([B_s]) = h_e(B_s)H_e + h_s(B_s)H_s = v^{-1}H_e + vH_s.$$

This differs from the expression $\mathrm{ch}([B_s]) = vH_e + H_s$ found in [13], reflecting a difference in normalization conventions for the graded multiplicities and basis elements in the Hecke algebra.

3.2.3 Soergel's Categorification Theorem Statement

We can now state the main theorem formalizing the categorification.

Theorem 3.23 (Soergel's Categorification Theorem)**.** Assume that the realization $\mathfrak{h}$ is reflection faithful over a field k of characteristic 0. Then:

1. **Indecomposable bimodules B_w:** For each $w \in W$, there exists an indecomposable Soergel bimodule $B_w \in \mathbb{S}\mathrm{Bim}$, unique up to isomorphism and grading shift. This B_w is characterized as the unique indecomposable direct summand of any Bott–Samelson bimodule $BS(\underline{w})$ (for a reduced expression $\underline{w}$ of w) such that the standard bimodule R_w appears in its Δ-filtration with graded multiplicity $h_w(B_w)$ having a nonzero constant term. This is usually normalized so that $h_w(B_w) = 1 + \cdots$, which may be achieved by shifting B_w to $B_w(\ell(w))$. All other indecomposable summands of $BS(\underline{w})$ are isomorphic to grading shifts of B_x for $x < w$ in the Bruhat order. The set

$$\{B_w(n) \mid w \in W,\ n \in \mathbb{Z}\}$$

forms a complete set of representatives for the isomorphism classes of indecomposable objects in $\mathbb{S}\mathrm{Bim}$.

2. **Categorification isomorphism:** There exists a unique isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras

$$c : \mathcal{H} \xrightarrow{\sim} [\mathbb{S}\mathrm{Bim}]_{\oplus}$$

determined by sending the standard basis element H_s (for each simple reflection $s \in S$) to the class $[B_s]$ in the Grothendieck group. This map respects multiplication via the monoidal structure on $\mathbb{S}\mathrm{Bim}$.

3. **Character map:** The character map

$$\mathrm{ch} : [\mathbb{S}\mathrm{Bim}]_{\oplus} \to \mathcal{H}$$

defined via graded multiplicities in Δ-filtrations (Definition 3.22) is a $\mathbb{Z}[v^{\pm 1}]$-linear isomorphism and is the inverse of c:

$$\mathrm{ch} = c^{-1}.$$

4. **Hom formula:** For any $B, B' \in \mathbb{S}\mathrm{Bim}$, the graded k-vector space

$$\mathrm{Hom}^{\bullet}_{\mathbb{S}\mathrm{Bim}}(B, B') := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(B, B'(n))$$

is finite-dimensional. Moreover, $\mathrm{Hom}^{\bullet}_{\mathbb{S}\mathrm{Bim}}(B, B')$ is a finitely generated graded free R-bimodule, and its graded rank over R satisfies:

$$\underline{\mathrm{rk}}_R \mathrm{Hom}^{\bullet}_{\mathbb{S}\mathrm{Bim}}(B, B') = (\mathrm{ch}([B]), \mathrm{ch}([B']))_{\mathcal{H}}\,,$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the standard sesquilinear form on the Hecke algebra $\mathcal{H}$ satisfying

$$(H_x, H_y) = \delta_{x,y}$$

after appropriate normalization (e.g., taking $(v^{-\ell(x)}H_x, v^{-\ell(y)}H_y)$ to be orthonormal).

The Hom formula (part 4) is particularly powerful. It establishes a direct link between the internal structure of the category $\mathbb{S}$Bim, specifically the size and grading of its morphism spaces, and the algebraic structure of the Hecke algebra $\mathcal{H}$, namely its canonical inner product. Since the graded rank $\underline{\mathrm{rk}}_R$ encodes dimensions of Hom spaces between objects (which are non-negative integers when viewed as dimensions of vector spaces over k after base change $R \to k$), this formula implies that the structure constants or inner products in the Hecke algebra can be computed by counting dimensions of certain homomorphism spaces in $\mathbb{S}$Bim. This provides a categorical explanation for combinatorial properties, especially positivity phenomena, observed in the Hecke algebra.

3.2.4 Remarks on Realizations

The construction and main theorems regarding Soergel bimodules depend on the choice of a realization $\mathfrak{h}$ for the Coxeter system (W, S). We revisit this concept and discuss the technical conditions often required.

Recall from Definition 2.12 that a realization consists of a free k-module $\mathfrak{h}$ equipped with families of roots $\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^*$ and coroots $\{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$ satisfying $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ and inducing the correct W-action $s(v) = v - \langle \alpha_s, v\rangle \alpha_s^\vee$.

While the geometric realization over $\mathbb{R}$ is standard, other realizations are important, particularly those arising from the root data of semisimple Lie algebras or Kac-Moody algebras, where $\mathfrak{h}$ is the Cartan subalgebra (or its dual).

For the theory to function smoothly, especially for diagrammatic approaches and certain proofs, additional technical conditions on the realization are often imposed:

- **Balancedness:** Introduced by Elias-Williamson [14], this condition relates the pairings $\langle \alpha_s, \alpha_t^\vee \rangle$ and $\langle \alpha_t, \alpha_s^\vee \rangle$ to the integer m_{st} via certain "quantum numbers". It simplifies diagrammatic calculations and ensures a consistent notion of positive roots within the diagrammatics.
- **Demazure Surjectivity:** This condition requires that each root map $\alpha_s : \mathfrak{h} \to k$ is surjective (or an equivalent condition on coroots). It ensures that the Demazure operator $\partial_s = (id - s)/\alpha_s$ is well-defined (as division by α_s makes sense in an appropriate localization) and that the ring extension $R^s \subset R$ is Frobenius.[9][14]

Additionally, a crucial condition in Soergel's original development was reflection faithfulness.

Definition 3.24 (Reflection Faithful)**.** Let $\mathfrak{h}$ be a realization of (W, S) over a field k. We say $\mathfrak{h}$ is *reflection faithful* if:

1. The W-action on $\mathfrak{h}$ is faithful (only $e \in W$ acts as the identity).
2. An element $w \in W$ acts as a pseudo-reflection on $\mathfrak{h}$ if and only if w is a reflection in W (i.e., w is conjugate in W to some $s \in S$).

Reflection faithfulness creates a tight geometric correspondence between the abstract reflections within the group W and the elements acting as actual reflections on the representation space $\mathfrak{h}$. Soergel's original proofs of the Categorification Theorem (Theorem 3.23) and the Hom formula relied on this assumption.

However, this condition is not always met by natural or standard realizations. A major limitation is that the geometric realization V_{geom} is often *not* reflection faithful for infinite Coxeter groups, including affine Weyl groups, which are crucial in Lie theory and related areas. This limitation was a significant motivator for developing alternative approaches.

Soergel proved that reflection faithful realizations always exist over $\mathbb{R}$ for any Coxeter system. However, constructing them explicitly can be non-trivial, and they might not be the most natural realization for certain applications.

[9] A ring extension A/B is called Frobenius if there exists a linear map $E : A \to B$ satisfying the bimodule condition $E(bac) = bE(a)c$ for all $b, c \in B$ and $a \in A$, and there exist elements $\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n \in A$ such that for all $a \in A$, we have $\sum_{i=1}^n E(ax_i)y_i = a = \sum_{i=1}^n x_i E(y_i a)$.

Modern developments, particularly the diagrammatic category of Elias-Williamson [14] and related algebraic constructions by Abe [1], have successfully established categorification results under weaker assumptions, often requiring only balancedness and Demazure surjectivity, thereby bypassing the need for reflection faithfulness. These frameworks are more robust and directly applicable to situations like affine Weyl groups acting on Cartan subalgebras.

4 Braden-MacPherson Sheaves

We significantly depart from the exposition in [13], instead following the original treatment in [7]. While [13] is largely driven by computations and overlooks the intersection cohomology aspect, we fully embrace this perspective and focus less on computational details. For readers specifically interested in the parts of the Braden-MacPherson framework most relevant to Soergel bimodules, [13] provides a clearer exposition than the present one.

4.1 Preliminaries: Varieties with Torus Actions

The theory of Braden-MacPherson sheaves operates within a specific geometric context: complex algebraic varieties equipped with actions of algebraic tori satisfying certain regularity conditions. We start by establishing this context, defining the key objects and assumptions that underpin the entire framework.

4.1.1 Algebraic Tori, Actions, and Orbits

We begin by defining the actors of our story: algebraic tori and their actions on varieties. Throughout, we work over the ground field $\mathbb{C}$ of complex numbers, and all algebraic varieties are assumed to be separated and of finite type.

Definition 4.1 (Algebraic Torus)**.** An *algebraic torus* of dimension d is an affine algebraic group T isomorphic to the d-fold product of the multiplicative group $\mathbb{G}_m = \mathbb{C}^*$, i.e., $T \cong (\mathbb{C}^*)^d$ for some integer $d \geq 1$.

Remark 4.2. Algebraic tori over $\mathbb{C}$ are commutative affine algebraic groups. As Lie groups, $T \cong (S^1)^d \times (\mathbb{R}^+)^d$, containing the compact torus $K = (S^1)^d$ as a maximal compact subgroup. While the theory of torus actions in symplectic geometry often focuses on the action of the compact torus K and associated moment maps, the Braden-MacPherson theory uses the algebraic structure of T itself, particularly its characters.

Definition 4.3 (Character Lattice)**.** The *character lattice* of an algebraic torus $T \cong (\mathbb{C}^*)^d$ is the group of algebraic group homomorphisms $X^*(T) = \mathrm{Hom}_{alg}(T, \mathbb{G}_m)$. It is a free abelian group of rank d. The associated *cocharacter lattice* is $X_*(T) = \mathrm{Hom}_{alg}(\mathbb{G}_m, T)$, also a free abelian group of rank d, dual to $X^*(T)$ under the natural pairing $\langle \chi, \lambda \rangle \in \mathbb{Z}$ defined by $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for $\chi \in X^*(T)$, $\lambda \in X_*(T)$, and $t \in \mathbb{G}_m$.

Characters $\chi \in X^*(T)$ can be thought of as Laurent monomials in coordinates $t_1, \ldots, t_d$ if $T = (\mathbb{C}^*)^d$. The character lattice encodes the algebraic structure of the torus action. We denote the Lie algebra of T by $\mathfrak{t} \cong \mathbb{C}^d$, and its dual space by $\mathfrak{t}^*$. The character lattice embeds into $\mathfrak{t}^*$ (via differentiation), and we often work with the rational vector space $\mathfrak{t}^*_{\mathbb{Q}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 4.4 (T-Variety)**.** A *T-variety* is a complex algebraic variety X equipped with a morphism $a : T \times X \to X$ satisfying the axioms of a (left) group action.

Linear actions on vector spaces provide simple examples. If T acts linearly on a finite-dimensional vector space V, then V decomposes into a direct sum of weight spaces $V = \bigoplus_{\chi \in X^*(T)} V_\chi$, where $V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \in T\}$. The characters χ appearing in this decomposition are called the weights of the action.

Definition 4.5 (Orbit and Stabilizer)**.** For a T-variety X and a point $x \in X$, the *orbit* through x is the set $T \cdot x = \{t \cdot x \mid t \in T\}$. The *stabilizer* of x is the subgroup $T_x = \{t \in T \mid t \cdot x = x\}$. The orbit $T \cdot x$ is isomorphic as a variety to the quotient T/T_x. Its dimension is $d - \dim T_x$.

Definition 4.6 (One-dimensional Orbit)**.** A *one-dimensional T-orbit* is an orbit $T \cdot x$ such that $\dim(T \cdot x) = 1$. In the contexts considered by Braden-MacPherson, the Zariski closure $\overline{T \cdot x}$ of such an orbit is assumed to be isomorphic to the projective line $\mathbb{P}^1$.

4.1.2 Fixed Points and Contracting Subgroups

Fixed points play a central role, acting as anchors for the combinatorial structure.

Definition 4.7 (Fixed-Point Set)**.** For a T-variety X, the *fixed-point set* is $X^T = \{x \in X \mid t \cdot x = x \text{ for all } t \in T\}$.

The local behavior near fixed points is often analyzed using one-parameter subgroups.

Definition 4.8 (One-Parameter Subgroup)**.** A *one-parameter subgroup* of T is an algebraic group homomorphism $\lambda : \mathbb{G}_m \to T$. These correspond bijectively to cocharacters $\lambda \in X_*(T)$.

Definition 4.9. Let $x \in X^T$. A one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is *contracting at x* if there exists a Zariski-open neighborhood $U \subseteq X$ of x such that

$$\lim_{t\to 0} \lambda(t) \cdot y = x$$

for all $y \in U$. The limit is taken in an analytic topology on X, which coincides with the Zariski topology for such limits involving algebraic actions.

The existence of a contracting 1-PS at a fixed point x implies that x is an attractive fixed point for the action of the subtorus $\lambda(\mathbb{G}_m)$. This provides a powerful tool for analyzing the local structure of X near x, as it induces a flow towards x within a neighborhood. This concept is fundamental to the Białynicki-Birula decomposition.

4.1.3 Białynicki-Birula Decomposition

The existence of contracting subgroups is intimately related to the Białynicki-Birula (BB) decomposition, which provides a canonical way to partition a variety based on the flow induced by a $\mathbb{G}_m$-action.

Let X be a smooth projective variety over $\mathbb{C}$ with a $\mathbb{G}_m$-action induced by a 1-PS $\lambda : \mathbb{G}_m \to T$. Assume the fixed point set $X^\lambda = X^{\lambda(\mathbb{G}_m)}$ is finite. For each fixed point $x_i \in X^\lambda$, the *Białynicki-Birula cell* (or stratum) is defined as:

$$X_i^+ = \{x \in X \mid \lim_{t\to 0} \lambda(t) \cdot x = x_i\}.$$

Theorem 4.10 ([3])**.** Let X be a smooth projective variety over $\mathbb{C}$ with a $\mathbb{G}_m$-action (via λ) such that X^λ is finite. Then:

1. X decomposes into a disjoint union of the locally closed cells $X = \bigsqcup_{x_i \in X^\lambda} X_i^+$.
2. Each cell X_i^+ is isomorphic to an affine space $\mathbb{C}^{n_i}$.
3. The map $p_i : X_i^+ \to \{x_i\}$ defined by $p_i(x) = \lim_{t\to 0} \lambda(t) \cdot x$ makes X_i^+ a locally trivial algebraic fiber bundle over $\{x_i\}$ with fiber $\mathbb{C}^{n_i}$.
4. The dimension n_i is the number of positive weights of the $\mathbb{G}_m$-action on the tangent space $T_{x_i}X$.

This decomposition provides a cellular structure on X determined by the torus action. The closure relation among these cells induces a partial order on the fixed point set: $x_i \preceq x_j$ if $X_i^+ \subseteq \overline{X_j^+}$. While the original theorem requires smoothness and projectivity, the existence of contracting subgroups (Assumption 2 in the BM framework) and a T-invariant Whitney stratification by affine spaces (Assumption 4) ensures a similar decomposition into affine strata exists for the varieties considered by Braden and MacPherson, even if they are singular. The partial order defined later for the moment graph directly reflects the closure relationships between these affine strata, analogous to the BB cell closures.

4.1.4 Whitney Stratifications by Affine Spaces

Intersection cohomology is sensitive to the singular structure of a variety. Stratifications provide a way to decompose a singular space into smooth pieces (strata) in a controlled manner.

Definition 4.11 (Stratification)**.** A *stratification* of a topological space X is a locally finite partition $X = \bigsqcup_\alpha S_\alpha$ into disjoint, connected, locally closed smooth manifolds S_α, called *strata*.

For intersection cohomology, Whitney stratifications are particularly important because they impose regularity conditions on how strata meet.

Definition 4.12 (Whitney Stratification)**.** Let S_α and S_β be two strata in a stratification of a subset $X \subseteq \mathbb{C}^N$. The pair (S_α, S_β) satisfies:

(A) *Whitney's Condition (A)* if for any point $y \in S_\beta$ and any sequence $\{x_m\} \subset S_\alpha$ converging to y, if the tangent spaces $T_{x_m}S_\alpha$ converge to a plane L (in the Grassmannian), then $T_y S_\beta \subseteq L$.

(B) *Whitney's Condition (B)* if for any point $y \in S_\beta$, any sequence $\{x_m\} \subset S_\alpha$ converging to y, and any sequence $\{y_m\} \subset S_\beta$ converging to y, if the tangent spaces $T_{x_m}S_\alpha$ converge to a plane L and the secant lines $\overline{x_m y_m}$ converge to a line ℓ, then $\ell \subseteq L$.

A stratification is a *Whitney stratification* if every pair of strata satisfies Whitney's Condition (A) or (B), which are equivalent.

Whitney's conditions ensure that the tangent spaces of nearby strata align in a controlled way as they approach a lower-dimensional stratum, guaranteeing a certain uniformity of the singularity structure along each stratum. Existence theorems guarantee that complex algebraic and analytic varieties admit Whitney stratifications. Algorithms exist to compute these stratifications.

The Braden-MacPherson theory requires a very specific type of stratification:

Definition 4.13. A *T-invariant Whitney stratification by affine spaces* of a T-variety X is a partition such that:

1. Each stratum C_x is T-stable (i.e., $t \cdot C_x = C_x$ for all $t \in T$).
2. Each stratum C_x contains exactly one T-fixed point, namely x.
3. Each stratum C_x is isomorphic as a variety to an affine space $\mathbb{C}^{n(x)}$ for some $n(x) \geq 0$.
4. The collection $\{C_x\}_{x \in X^T}$ forms a Whitney stratification of X.

Remark 4.14. The condition that each stratum C_x is isomorphic to an affine space is very strong. Since T acts on the affine space C_x and x is the unique fixed point, standard results imply that the action is linear in suitable coordinates and x corresponds to the origin. This affine structure, combined with the Whitney conditions ensuring uniform singularity behavior, is what enables the reduction of intersection cohomology computations to the combinatorial data encoded in the moment graph. The unique fixed point property is a direct consequence of the strata being affine.

4.1.5 The Braden-MacPherson Assumptions

We can now state the four fundamental assumptions on the pair (X, T) required by the Braden-MacPherson theory. These assumptions collectively ensure that the variety X has a structure that allows for a combinatorial study via its torus action, specifically allowing the computation of its intersection cohomology from the moment graph.

Assumptions Let X be an irreducible complex algebraic variety equipped with an action of an algebraic torus $T \cong (\mathbb{C}^*)^d$. The pair (X, T) is assumed to satisfy the following four conditions:

(1) ***T*-action with isolated fixed points.** The fixed point set X^T is finite.

(2) **Local contraction.** For every fixed point $x \in X^T$, there exists a contracting one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ at x.

(3) **Finiteness of one-dimensional orbits.** The T-action has only finitely many one-dimensional orbits, and the Zariski closure of each such orbit is isomorphic to $\mathbb{P}^1$.

(4) **Whitney stratification.** There exists a T-invariant Whitney stratification by affine spaces, $X = \bigsqcup_{x \in X^T} C_x$.

Remark 4.15 (Motivation for Assumptions)**.** These four conditions work together to bridge the geometry of X with the combinatorics of the moment graph:

- **Assumption (1)** reduces the problem to analyzing a finite set of points, which will become the vertices of the moment graph.
- **Assumption (2)** guarantees that the local geometry near each fixed point is governed by the torus action, providing the basis for a Białynicki-Birula-like decomposition. The contracting flow defines the directions inherent in the partial order of the moment graph.
- **Assumption (3)** ensures that the graph connecting these fixed points (the moment graph) has only finitely many edges, corresponding to the $\mathbb{P}^1$ closures of these 1D orbits.
- **Assumption (4)** is the strongest structural condition. The Whitney property ensures that singularities behave uniformly along strata, allowing intersection cohomology (which measures singularities) to be understood locally. The affine space property simplifies the structure of each stratum immensely, ensuring a unique fixed point and allowing the local equivariant geometry (and thus local intersection cohomology) to be captured by the tangent space action at the fixed point. This ultimately enables the computation of global intersection cohomology from data localized at the fixed points (stalks[10] of the BM sheaf) and their immediate connections (edges/weights of the moment graph).

Table 2: Summary of Braden-MacPherson Assumptions

Assumption	**Idea**	**Role**
Isolated Fixed Pts	T acts on X, X^T is finite.	Reduces global analysis to a finite set of points. Simplifies combinatorics (finite vertex set for graph).
Local Contraction	$\forall x \in X^T$, $\exists \lambda : \mathbb{G}_m \to T$ contracting at x.	Ensures local geometry near x is controlled by T-action (BB decomposition). Provides local coordinates/flow, defines partial order.
Finite 1D Orbits	Finitely many 1D T-orbits $\mathcal{O}$, $\overline{\mathcal{O}} \cong \mathbb{P}^1$.	Ensures the moment graph has finitely many edges. Simplifies combinatorics.
Whitney Stratification	$X = \bigsqcup_{x \in X^T} C_x$, C_x T-stable, $\cong \mathbb{C}^{n(x)}$, Whitney (A,B).	Guarantees local structure/singularities are uniform along strata, determined by fixed points. Enables IC computation via stalks on the graph.

It is now natural to wonder which varieties satisfy the desired assumptions.

Let G be a complex semisimple algebraic group, B a Borel subgroup, and $T \subset B$ a maximal torus. The flag variety $X = G/B$ is a smooth projective variety. For $w \in W = N_G(T)/T$, the Weyl group, the *Schubert cell* is $C_w = BwB/B \subset G/B$, and the *Schubert variety* is its closure $X_w = \overline{C_w}$. Schubert varieties are, in general, singular.

The maximal torus T acts on G/B (by left multiplication, for instance), and this action stabilizes each Schubert variety X_w.

Proposition 4.16. The pair (X_w, T) satisfies the Braden-MacPherson assumptions. More generally, Schubert varieties in a flag manifold and affine flag varieties[11] satisfy the Braden-MacPherson assumptions.

[10]The *stalk* of a sheaf $\mathcal{F}$ at a point p is the direct limit of the sections of $\mathcal{F}$ over open neighborhoods of p:

$$\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U)$$

[11]Let G be a semisimple complex algebraic group, $G(\mathbb{C}((t))$ the corresponding loop group, I an Iwahori subgroup, $P \supseteq I$ a parahoric subgroup. Then $M = G/P$ is a *affine flag variety*.

For the (X_w, T) case:

- The T-fixed points in G/B are indexed by the Weyl group W, specifically $X_w^T = \{x \in W \mid x \leq w\}$, where $\leq$ is the Bruhat order. This set is finite.
- The remaining assumptions (local contraction, finite 1D orbits $\cong \mathbb{P}^1$, Whitney stratification by affine spaces via the Bruhat decomposition) also hold.

4.2 Construction of Braden-MacPherson Sheaves

Given a pair (X, T) satisfying the Braden-MacPherson assumptions (Assumption 4.1.5), we construct a finite, directed, labeled graph $\mathcal{G}(X, T)$, called the moment graph. This combinatorial object encapsulates the essential T-equivariant geometry arising from the zero- and one-dimensional orbits, providing the foundation upon which the Braden-MacPherson sheaf is built. Its structure is reminiscent of graphs used in GKM (Goresky-Kottwitz-MacPherson) theory for equivariant cohomology computations.

4.2.1 The Moment Graph $\mathcal{G}(X, T)$

We will present two (equivalent) definitions of the moment graph. Our first one will be in terms of the geometry of the space X under the torus action, with a focus on edge weights and closures of orbits. The second definition describes a more abstract moment graph, emphasizing the vector space structure, the direction of edges, and the partial order. Both definitions aim to capture the combinatorial and geometric features of moment graphs, but they do so from slightly different perspectives.

Here is the first definition of the moment graph.

Definition 4.17 (Moment Graph #1)**.** Let $\mathfrak{t}$ be a complex vector space (typically the Lie algebra of the torus T). A $\mathfrak{t}$-moment graph Γ is a finite graph endowed with two additional structures:

1. For each edge L, a one-dimensional subspace V_L of the dual vector space $\mathfrak{t}^*$, called the direction of L.
2. A partial order $\leq$ on the set of vertices $\mathcal{V}$ such that if an edge L connects vertices x and y, then either $x \leq y$ or $y \leq x$ (and $x \neq y$).

The set of edges is denoted by $\mathcal{E}$. For a vertex $x \in \mathcal{V}$, U_x denotes the set of edges connecting x to a vertex y with $x \leq y$ ("up"), and D_x denotes the set of edges connecting x to a vertex y with $y \leq x$ ("down").

The components of the moment graph are derived directly from the low-dimensional orbits of the T-action on X.

Definition 4.18 (Vertices and Edges)**.** Let (X, T) satisfy Assumptions 4.1.5.

- The *vertex set* of the moment graph $\mathcal{G}(X, T)$ is $V := X^T$. This set is finite by Assumption (1).
- The *edge set* E consists of unordered pairs $\{x, y\}$ of distinct vertices such that there exists a one-dimensional T-orbit $\mathcal{O}$ whose closure is $\overline{\mathcal{O}} = (T \cdot z) \cup \{x, y\}$ for some $z \in X$. By Assumption (3), there are finitely many such orbits, and each closure $\overline{\mathcal{O}}$ is isomorphic to $\mathbb{P}^1$ and connects exactly two fixed points $x, y \in V$. The edges correspond to these one-dimensional orbits.

Each edge carries a weight, an algebraic invariant derived from the torus action.

Definition 4.19 (Weight of an Edge)**.** Let $e = \{x, y\} \in E$ be an edge corresponding to the curve $\overline{\mathcal{O}} \cong \mathbb{P}^1$. The torus T acts on $\overline{\mathcal{O}}$. At the fixed point x, the tangent space $T_x\overline{\mathcal{O}}$ is a one-dimensional vector space isomorphic to $\mathbb{C}$. The induced T-action on $T_x\overline{\mathcal{O}}$ is linear and non-trivial (since $\mathcal{O}$ is not a fixed point). Being a one-dimensional representation of T, this action is determined by a unique non-zero character $\alpha_{x,e} \in X^*(T)$. Similarly, the action on $T_y\overline{\mathcal{O}}$ determines a character $\alpha_{y,e} \in X^*(T)$. These characters are related by $\alpha_{y,e} = -\alpha_{x,e}$. We choose one of these, say $\alpha_e = \alpha_{x,e}$, and call it the *weight* of the edge e. We view α_e as a non-zero element in the rational vector space $\mathfrak{t}^*_{\mathbb{Q}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The weight is also referred to as the direction V_L in some sources, a one-dimensional subspace of $\mathfrak{t}^*$.

Remark 4.20. The weight α_e encodes the character by which T scales the tangent direction along the edge $\overline{\mathcal{O}}$ emanating from x. Choosing the opposite orientation (thinking of the edge emanating from y) negates the character. For the purpose of defining the sheaf structure later, only the one-dimensional subspace $\langle \alpha_e \rangle_{\mathbb{Q}} \subset \mathfrak{t}^*_{\mathbb{Q}}$ spanned by the weight is intrinsically defined by the unoriented edge e. However, fixing a choice of α_e (up to sign) is necessary for defining the edge rings A_L. Geometrically, α_e can also be understood via the stabilizer subgroup T_z for any $z \in \mathcal{O}$. Since $\dim \mathcal{O} = 1$, $\dim T_z = d - 1$. The Lie algebra $\mathrm{Lie}(T_z)$ is a hyperplane in $\mathfrak{t}$. The weight vector α_e spans the one-dimensional annihilator of this hyperplane in $\mathfrak{t}^*$. This weight captures the essential infinitesimal T-equivariant geometry connecting the fixed points x and y.

The moment graph is endowed with a partial order reflecting the flow structure induced by the torus action, closely related to the Białynicki-Birula decomposition.

Definition 4.21 (Partial Order)**.** Fix a *generic* one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ such that the limit $\lim_{t\to 0} \lambda(t) \cdot z$ exists for all $z \in X$ and converges to a point in X^T. (The existence of contracting 1-PS, Assumption (2), ensures such limits exist locally; genericity ensures a consistent global flow structure, often related to choosing λ in a particular Weyl chamber in representation-theoretic contexts). Define a relation $\preceq$ on the vertex set $V = X^T$ by declaring:

$$x \preceq y \iff y = \lim_{t\to 0} \lambda(t) \cdot x$$

Equivalently, using the flow interpretation, $x \preceq y$ if y lies in the closure of the $\lambda(\mathbb{G}_m)$-orbit flowing from x towards $t = 0$. An alternative, equivalent definition uses the strata from Assumption (4):

$$x \preceq y \iff C_y \subseteq \overline{C_x}$$

(Note: the convention might be reversed, $C_x \subseteq \overline{C_y}$, depending on whether λ is contracting or expanding; we follow the convention linked to contracting 1-PS). This relation $\preceq$ defines a partial order on V. For any edge $e = \{x, y\} \in E$, the points x and y are comparable under $\preceq$, meaning either $x \prec y$ or $y \prec x$.

We now combine these elements to give our second definition of the moment graph.

Definition 4.22 (Moment Graph #2)**.** The *moment graph* $\mathcal{G}(X, T)$ is the quadruple consisting of the finite vertex set $V = X^T$, the finite edge set E (viewed as pairs $\{x, y\}$ corresponding to closures of 1D orbits), the weight map $\alpha : E \to \mathfrak{t}^*_{\mathbb{Q}} \setminus \{0\}$ assigning a weight α_e to each edge $e \in E$ (well-defined up to sign), and the partial order $\preceq$ on V.

The moment graph is thus a finite, directed graph whose edges are labeled by non-zero rational characters (up to sign). Its significance lies in its ability to encode the necessary geometric information for computing intersection cohomology.

Remark 4.23. The Braden-MacPherson assumptions, particularly the Whitney stratification by affine spaces (Assumption 4), ensure that the local structure of X near each fixed point x and along each 1D orbit $\mathcal{O}$ is sufficiently well-behaved. The intersection cohomology sheaf $\mathrm{IC}^\bullet(X)$ is locally constant along the strata $C_x \setminus \{x\}$ and its behavior at the fixed point x is determined by how the strata C_y for y connected to x by an edge come together at x. This interaction is governed by the T-action on the normal directions at x, which are precisely captured by the weights α_e of the edges e incident to x. The partial order $\preceq$ reflects the inclusion relationships of strata closures. Therefore, the moment graph $\mathcal{G}(X, T)$ packages exactly the combinatorial and equivariant data needed to reconstruct the stalks and global sections of the (equivariant) intersection cohomology sheaf.

Remark 4.24 (Relation to Symplectic Geometry)**.** The name "moment graph" is motivated by symplectic geometry. If X is projective, one can choose a T-equivariant embedding into projective space $X \hookrightarrow \mathbb{P}^N$. With the Fubini-Study metric, the T-action is Hamiltonian, leading to a moment map $\mu : X \to \mathfrak{t}^*_K$, where $\mathfrak{t}_K$ is the Lie algebra of the maximal compact torus $T_K \subset T$. The image $\mu(\overline{L})$ of the closure of a 1D orbit is a line segment connecting $\mu(x)$ and $\mu(y)$. The direction vector $\mu(y) - \mu(x)$ (or $\mu(x) - \mu(y)$) spans the real line underlying the complex line $V_L \subset \mathfrak{t}^* \cong \mathfrak{t}_K^* \otimes_{\mathbb{R}} \mathbb{C}$. Similar graphs appear in the work of Guillemin and Zara on equivariant cohomology and Morse theory on graphs [20, 19, 21].

For those particularly interested in the connection to Soergel bimodules, most moment graphs (in that context) are Hasse diagrams of the Bruhat order.

Example 4.25 (SL_3/B Moment Graph)**.** Let $X = G/B$ where $G = SL_3(\mathbb{C})$ and B is the Borel subgroup of upper triangular matrices. Let T be the maximal torus of diagonal matrices. The fixed points X^T are indexed by the Weyl group $W = S_3 = \{\mathrm{id}, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$. The Bruhat order provides the partial order $\leq$. Edges connect $w, v \in W$ if $v = wr$ for some reflection r and $w < v$. The moment graph is the Hasse diagram of the Bruhat order. The directions V_L correspond to roots of SL_3.

4.2.2 Sheaves on Moment Graphs

Having constructed the combinatorial moment graph $\Gamma = \mathcal{G}(X, T)$, we now define algebraic objects living on this graph: Γ-sheaves. These sheaves, particularly the canonical one defined in the next chapter, are the central objects in the Braden-MacPherson theory, designed to capture the (equivariant) intersection cohomology of the original variety X.

The sheaves are built from modules over specific graded rings associated with the vertices and edges of the moment graph. The base ring reflects the global torus action, while the edge rings incorporate local geometric information.

Definition 4.26. Let $\mathfrak{t} = \mathrm{Lie}(T)$ be the Lie algebra of the torus T, and let $\mathfrak{t}^*$ be its dual space. The primary *coefficient ring* is the symmetric algebra of $\mathfrak{t}^*$: We equip A with a grading by placing elements of $\mathfrak{t}^*$ in degree 2. If $\{\beta_1, \ldots, \beta_d\}$ is a basis for $\mathfrak{t}^*$, then $A \cong \mathbb{C}[\beta_1, \ldots, \beta_d]$ is a graded polynomial ring.

Remark 4.27. This is very similar to the setup in the previous section.

Remark 4.28 (Connection to Equivariant Cohomology)**.** The ring $A = \mathrm{Sym}(\mathfrak{t}^*)$ is canonically isomorphic to the T-equivariant cohomology ring of a point, $H_T^*(\mathrm{pt}; \mathbb{C})$. The grading corresponds to the cohomological degree. Thus, modules over A naturally model objects carrying T-equivariant information.

Definition 4.29. For each edge $L \in E$ of the moment graph Γ, let $\alpha_L \in \mathfrak{t}_{\mathbb{Q}}^* \setminus \{0\}$ be its associated weight. By clearing denominators, we can assume $\alpha_L \in \mathfrak{t}^*$. The *edge ring* associated to L is the quotient ring:

$$A_L := A/(\alpha_L)$$

where (α_L) is the principal ideal generated by the linear form α_L. Since α_L is homogeneous of degree 2, A_L inherits a grading from A. The natural projection map $q_L : A \to A_L$ makes A_L a graded A-algebra.

The edge ring A_L incorporates the local constraint imposed by the torus action along the edge L. In GKM theory, compatibility conditions between data at adjacent fixed points often involve congruence modulo the edge weight α_L. Using A_L-modules as building blocks encodes this local compatibility directly into the sheaf structure.

A Γ-sheaf packages modules associated with vertices and edges, linked by restriction maps respecting the graph structure.

Definition 4.30. Let $\Gamma = (V, E, \alpha, \preceq)$ be a moment graph. A Γ-*sheaf* $\mathcal{M}$ consists of the following data:

1. For each vertex $x \in V$, a graded A-module $\mathcal{M}_x$.
2. For each edge $L \in E$, a graded A_L-module $\mathcal{M}_L$ (which is also a graded A-module via the projection $A \to A_L$).
3. For each pair (x, L) where $x \in V$ is an endpoint of $L \in E$, a degree-preserving A-linear homomorphism $\rho_{x,L} : \mathcal{M}_x \to \mathcal{M}_L$.

All modules $\mathcal{M}_x$ and $\mathcal{M}_L$ are assumed to be *finitely generated* over their respective rings (A or A_L).

Remark 4.31 (Intuition)**.** One should think of $\mathcal{M}_x$ as the "stalk" of the sheaf at the vertex x, and $\mathcal{M}_L$ as the "stalk" along the edge L. The maps $\rho_{x,L}$ are "restriction maps" that enforce compatibility between the data at a vertex and the data along an incident edge. This definition discretizes the usual notion of a sheaf on a topological space, adapting it to the combinatorial structure of the graph Γ. The use of A-modules and A_L-modules incorporates the T-equivariant structure via the coefficient rings derived from the torus characters and edge weights.

While defined combinatorially, Γ-sheaves can also be viewed as traditional sheaves on a suitable topological space associated with the graph.

Definition 4.32. Let $\Gamma = (V, E, \alpha, \preceq)$ be a moment graph. Define the set $S(\Gamma) := V \cup E$. We endow $S(\Gamma)$ with a topology by declaring a subset $O \subseteq S(\Gamma)$ to be *open* if it satisfies the condition:

$$\text{if } x \in O \cap V, \text{ then } L \in O \text{ for all edges } L \in E \text{ incident to } x.$$

This defines a topology where the minimal open neighborhood of a vertex x consists of x and all edges incident to it. $S(\Gamma)$ becomes a finite T_0 topological space, sometimes called an Alexandrov space.

Definition 4.33. Let $\mathcal{M}$ be a Γ-sheaf and let $Z \subseteq S(\Gamma)$ be any subset. The module of *sections of* $\mathcal{M}$ *over* Z is defined as:

$$\mathcal{M}(Z) := \left\{ ((s_x)_{x \in Z \cap V}, (s_L)_{L \in Z \cap E}) \in \bigoplus_{a \in Z} \mathcal{M}_a \;\middle|\; \rho_{x,L}(s_x) = s_L \text{ whenever } x \subset L \text{ and } \{x, L\} \subset Z \right\}$$

where $\mathcal{M}_a$ denotes $\mathcal{M}_x$ if $a = x \in V$ and $\mathcal{M}_L$ if $a = L \in E$. Elements of $\mathcal{M}(Z)$ are compatible families of elements from the vertex and edge modules within Z. The *global sections* are $\Gamma(\mathcal{M}) := \mathcal{M}(S(\Gamma))$. $\mathcal{M}(Z)$ forms a graded module over the ring of sections $A(Z)$ of the structure sheaf $\mathcal{A}$ (defined below).

Definition 4.34. The *structure sheaf* (or sheaf of coefficient rings) on Γ is the Γ-sheaf $\mathcal{A}$ defined by:

- $\mathcal{A}_x = A = \mathrm{Sym}(\mathfrak{t}^*)$ for all $x \in V$.
- $\mathcal{A}_L = A_L = A/(\alpha_L)$ for all $L \in E$.
- $\rho_{x,L} : \mathcal{A}_x \to \mathcal{A}_L$ is the natural quotient map $q_L : A \to A_L$.

For any Γ-sheaf $\mathcal{M}$, the modules $\mathcal{M}_x$ are $A = \mathcal{A}_x$-modules and $\mathcal{M}_L$ are $A_L = \mathcal{A}_L$-modules, and the maps $\rho_{x,L}$ are compatible with these structures in a natural sense, making $\mathcal{M}$ a "sheaf of $\mathcal{A}$-modules" in the combinatorial setting.

The combinatorial definition of a Γ-sheaf is equivalent to the standard definition of a sheaf on the topological space $S(\Gamma)$.

Proposition 4.35 ([7, Proposition 1.1])**.** The assignment $\mathcal{M} \mapsto (O \mapsto \mathcal{M}(O))$ establishes a bijection between:

- Γ-sheaves in the sense of Definition 4.30.
- Sheaves of graded $\mathcal{A}$-modules on the topological space $S(\Gamma)$ (Definition 4.32), where $\mathcal{A}$ is viewed as a sheaf of rings on $S(\Gamma)$.

Proof. Given a Γ-sheaf $\mathcal{M}$, the map $O \mapsto \mathcal{M}(O)$ defines a presheaf on $S(\Gamma)$. The compatibility condition $\rho_{x,L}(s_x) = s_L$ in the definition of $\mathcal{M}(O)$ ensures the gluing axiom holds for the specific topology on $S(\Gamma)$, making it a sheaf. Conversely, given a sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules on $S(\Gamma)$, we can recover the Γ-sheaf data by setting $\mathcal{M}_x = \mathcal{F}_x$ (stalk at x) and $\mathcal{M}_L = \mathcal{F}_L$ (stalk at L), with $\rho_{x,L}$ being the restriction map induced by the topology. More precisely, one can define $\mathcal{M}_x = \mathcal{F}(x^\circ)$ where x° is the minimal open set containing x, and

$\mathcal{M}_L = \mathcal{F}(L^\circ)$ (or simply $\mathcal{F}_L$ if edges are considered closed points in some sense). The equivalence ensures consistency. □

Remark 4.36. This equivalence is powerful. It allows us to primarily use the concrete, combinatorial definition of Γ-sheaves for constructions and computations, involving only finitely many modules ($\mathcal{M}_x, \mathcal{M}_L$) and explicit maps ($\rho_{x,L}$). Simultaneously, it grants access to the full machinery of sheaf theory on topological spaces, such as sheaf cohomology and derived functors, via the interpretation as a sheaf on $S(\Gamma)$.

4.2.3 The Braden-MacPherson Algorithm

The core of the Braden-MacPherson theory is the construction of a specific, canonical Γ-sheaf $\mathcal{M}$ associated to the moment graph Γ of X. From the geometric side: among all possible Γ-sheaves, there exists a unique one, the *canonical sheaf* $\mathcal{M}$, whose global sections compute the equivariant intersection cohomology of the original variety X. Remarkably, this construction can be carried out purely combinatorially, relying solely on the structure of Γ. To achieve this, [7] developed a recursive algorithm, now widely known as the Braden-MacPherson algorithm.

The core of the Braden-MacPherson theory is the construction of a specific, canonical Γ-sheaf $\mathcal{M}$ associated to the moment graph Γ of X. This construction is purely combinatorial, relying only on Γ (vertices, edges, directions V_L, and partial order $\leq$).

The construction proceeds by induction, moving downwards through the partial order $\leq$ on $\mathcal{V}$. We first recall the notion of a projective cover.

Definition 4.37 (Projective Cover)**.** Let R be a ring (typically $A = \mathrm{Sym}(\mathfrak{t}^*)$ here) and M be an R-module. A *projective cover* of M is a projective R-module P together with a surjective homomorphism $\pi : P \to M$ such that the induced map $\overline{\pi} : P/(\mathfrak{m}P) \to M/(\mathfrak{m}M)$ is an isomorphism, where $\mathfrak{m}$ is the Jacobson radical of R. For our graded ring $A = \mathrm{Sym}(\mathfrak{t}^*)$, where the unique maximal homogeneous ideal is $(\mathfrak{t}^*)A$, this means $\overline{\pi} : P \otimes_A \mathbb{C} \to M \otimes_A \mathbb{C}$ is an isomorphism. Such a P is typically constructed as $P = (M \otimes_A \mathbb{C}) \otimes_{\mathbb{C}} A$.

Braden-MacPherson Algorithm, ([8]):

Let Γ be the moment graph of X.

1. **Initialization:** Since X is irreducible, there is a unique maximal vertex $x_0 \in \mathcal{V}$ under the partial order $\leq$. Define $M_x[x_0] = A$.
2. **Inductive Step:** Assume $M_x[y]$, M_L, and $\rho_{y,L}$ have been defined for all vertices $y > x$ and all edges L connecting such vertices. We want to define M_x.
   (a) **Define modules on incoming edges:** For each edge $L \in U_x$ (connecting x to y with $x < y$), define $M_L = M_x[y]/V_L M_x[y]$. Let $\rho_{y,L} : M_x[y] \to M_L$ be the quotient map.
   (b) **Define boundary module:** Consider the subgraph $\tilde{\Gamma}_{>x} = \Gamma_{>x} \cup U_x$ (vertices $> x$, edges between them, and edges from x up to them). Define the "boundary module" $M_{\partial x}$ as the image of the restriction map ϕ:
   $$\phi : \mathcal{M}(\tilde{\Gamma}_{>x}) \longrightarrow \mathcal{M}(U_x) = \bigoplus_{L \in U_x} M_L$$
   An element of $M_{\partial x}$ is thus a collection $(s_L)_{L \in U_x}$ with $s_L \in M_L$ that can be extended compatibly to all of $\tilde{\Gamma}_{>x}$.
   (c) **Define vertex module via projective cover:** Define M_x to be the projective cover of $M_{\partial x}$. Let $\pi_x : M_x \twoheadrightarrow M_{\partial x}$ be the covering map.
   (d) **Define restriction maps:** For $L \in U_x$, define $\rho_{x,L} : M_x \to M_L$ as the composition $M_x \xrightarrow{\pi_x} M_{\partial x} \hookrightarrow \mathcal{M}(U_x) \xrightarrow{\mathrm{proj}_L} M_L$.

(e) **Consistency for outgoing edges:** For edges $L \in D_x$ (connecting x to y with $y < x$), the modules M_L and maps $\rho_{x,L}$ will be defined when the construction reaches y. The overall consistency (e.g., that M_L is also isomorphic to $M_x/V_L M_x$ if $L \subset C_x$) is implicitly guaranteed by the main theorem identifying $\mathcal{M}$ with the intersection cohomology sheaf.

This process defines the Γ-sheaf $\mathcal{M} = (\{M_x\}, \{M_L\}, \{\rho_{x,L}\})$ for the entire graph Γ.

The construction, particularly the use of projective covers, might seem non-canonical. However, the resulting sheaf $\mathcal{M}$ possesses strong uniqueness properties.

Proposition 4.38 (Rigidity, Prop 1.2 [8])**.** Let $M_x \to M_{\partial x}$ and $N_x \to M_{\partial x}$ be two projective covers as constructed in step (2c). Then there exists a unique isomorphism $f : M_x \to N_x$ of graded A-modules such that the diagram commutes (i.e., f respects the maps to $M_{\partial x}$).

Proof. The proof given in [8] relies on the geometric interpretation via intersection cohomology (Theorem 1.8) and properties of compactly supported IH (Theorem 3.8, Lemma 4.2). It is not purely combinatorial. □

Corollary 4.39 (Automorphisms, Cor 1.3 [8])**.** The group of automorphisms of the canonical sheaf $\mathcal{M}$ as a graded $\mathcal{A}$-module, $\mathrm{Aut}_{\mathcal{A}}(\mathcal{M})$, is isomorphic to $\mathbb{C}^*$ acting by scalar multiplication. This follows from Prop 4.38 by induction starting from $\mathrm{Aut}_A(M_{x_0}) = \mathrm{Aut}_A(A) \cong \mathbb{C}^*$.

This rigidity ensures that although choices might be made in constructing projective covers, the resulting sheaf $\mathcal{M}$ is well-defined up to a unique isomorphism, making subsequent identifications with geometric objects canonical.

An alternative characterization of $\mathcal{M}$ is given using the notion of "purity".

Definition 4.40 (Pure $\mathcal{A}$-module [8] §1.4)**.** An $\mathcal{A}$-module (or Γ-sheaf) $\mathcal{N}$ is called *pure* if for all $x \in \mathcal{V}$:

1. $\mathcal{N}(x)$ is a free A-module.
2. $\mathcal{N}(L) = \mathcal{N}(x)/V_L\mathcal{N}(x)$ whenever $L \in D_x$.
3. The image of the restriction map $\mathcal{N}(x^\circ) \to \mathcal{N}(U_x)$ is equal to the image of the restriction map $\mathcal{N}(\tilde{\Gamma}_{>x}) \to \mathcal{N}(U_x)$, where $x^\circ = \{x\} \cup U_x \cup D_x$.

Theorem 4.41 (Characterization via Purity, Thm 1.4 [8])**.** The canonical sheaf $\mathcal{M}$ constructed above is the unique (up to isomorphism and shift) indecomposable pure $\mathcal{A}$-module on Γ satisfying $\mathcal{M}(x_0) = A$. Any pure $\mathcal{A}$-module on Γ decomposes as a direct sum of shifted copies of the canonical sheaves associated with the subgraphs $\Gamma_{\leq x}$ for $x \in \mathcal{V}$.

4.3 Intersection Cohomology and the Main Theorems

See [17], [18], and [6] for a more thorough iscussion

4.3.1 Brief Introduction to Intersection Cohomology

Intersection cohomology, introduced by Goresky and MacPherson, is a cohomology theory for singular spaces that generalizes Poincaré duality.

Definition 4.42 (Intersection Cohomology)**.** Let X be an irreducible complex algebraic variety of dimension n. Let $\mathcal{L}$ be a local system on a dense smooth open subset $U \subseteq X$. The *intersection cohomology complex* with coefficients in $\mathcal{L}$ is an object $\mathrm{IC}^\bullet(X; \mathcal{L})$ in the derived category $D_c^b(X)$ of constructible sheaves. It is characterized by properties related to its stalks and costalks relative to a stratification. When $\mathcal{L}$ is the constant sheaf $\underline{\mathbb{C}}_U$, we write $\mathrm{IC}^\bullet(X)$. The *intersection cohomology groups* are the hypercohomology groups:

$$IH^k(X; \mathcal{L}) := \mathbb{H}^k(X, \mathrm{IC}^\bullet(X; \mathcal{L}))$$

We primarily consider the constant coefficient case $IH^k(X) := IH^k(X; \underline{\mathbb{C}})$.

When a torus T acts on X, we can define the equivariant version.

Definition 4.43 (Equivariant Intersection Cohomology)**.** Let T be an algebraic torus acting on X. Let $ET \to BT$ be the universal principal T-bundle, where ET is a contractible space with a free T-action and $BT = ET/T$ is the classifying space. The *Borel construction* or *homotopy quotient* is $X_T := (X \times ET)/T$. The T-equivariant intersection cohomology of X is defined as the ordinary intersection cohomology of the homotopy quotient: The equivariant intersection cohomology $IH_T^*(X) = \bigoplus_k IH_T^k(X)$ forms a graded module over the equivariant cohomology of a point, $H_T^*(\mathrm{pt}) \cong A = \mathrm{Sym}(\mathfrak{t}^*)$.

We also need the local versions.

Definition 4.44 (Local Intersection Cohomology)**.** For a point $x \in X$, the *local intersection cohomology* at x, denoted $IH^*(X)_x$, is defined via the stalk of the intersection cohomology complex $\mathrm{IC}^\bullet(X)$ at x. Specifically, $IH^k(X)_x = \mathcal{H}^k(\mathrm{IC}^\bullet(X)_x)$. Similarly, the *local T-equivariant intersection cohomology* $IH_T^*(X)_x$ is defined using the stalks of an equivariant version of the IC complex, or equivalently via the restriction map $IH_T^*(X) \to IH_T^*(\{x\}) \cong H_T^*(\mathrm{pt}) \otimes IH^*(X)_x$. If $x \in X^T$, then $IH_T^*(X)_x$ is a graded A-module.

4.3.2 Braden-MacPherson Sheaves for Intersection Cohomology

The theorems in [7] provide a way to compute these potentially complicated topological invariants $IH_T^*(X)$ and $IH_T^*(X)_x$ using the purely algebraic and combinatorial construction of the canonical sheaf $\mathcal{M}$ on the moment graph Γ, provided (X, T) satisfies Assumptions 4.1.5. Let's see how to do this:

Theorem 4.45 (Global Sections vs Global IH, Thm 1.5 [8])**.** Let Γ be the moment graph of X and $\mathcal{M}$ be the canonical Γ-sheaf. There is a canonical isomorphism of graded A-modules:

$$\mathrm{IH}_T^*(X) \cong \mathcal{M}(\Gamma)$$

Consequently, the ordinary intersection cohomology is given by:

$$\mathrm{IH}^*(X) \cong \mathcal{M}(\Gamma) \otimes_A \mathbb{C} = \overline{\mathcal{M}(\Gamma)}$$

Furthermore, $\mathrm{IH}_T^*(X)$ is a free A-module.

Theorem 4.46 (Stalks vs Local IH, Thm 1.6 [8])**.** For each fixed point $x \in X^T = \mathcal{V}$, there is a canonical isomorphism of graded A-modules between the stalk of $\mathcal{M}$ at x and the local equivariant intersection cohomology at x:

$$\mathrm{IH}_T^*(X)_x \cong M_x$$

The local ordinary intersection cohomology is given by:

$$\mathrm{IH}^*(X)_x \cong M_x \otimes_A \mathbb{C} = \overline{M_x}$$

Theorem 4.47 (Module Structures, Thm 1.7 [8])**.** The identifications in Theorems 4.45 and 4.46 respect the module structures over the equivariant and ordinary cohomology rings $H_T^*(X)$ and $H^*(X)$. These cohomology rings themselves can be identified with the global sections of the sheaf of rings $\mathcal{A}$, $H_T^*(X) \cong \mathcal{A}(\Gamma)$ and $H^*(X) \cong \overline{\mathcal{A}(\Gamma)}$ (under suitable conditions, cf. [16]). The $H_T^*(X)$-module structure on $\mathrm{IH}_T^*(X)$ corresponds precisely to the $\mathcal{A}(\Gamma)$-module structure on $\mathcal{M}(\Gamma)$.

These theorems are remarkable. They provide a purely combinatorial algorithm, starting from the moment graph Γ, to compute the equivariant intersection cohomology of X both globally and locally.

5 Fiebig's Correspondence

Now that we have seen both Soergel bimodules and Braden-MacPherson sheaves, it's finally time to combine them. To do this, we turn to Fiebig's correspondence, which states the following:

Proposition 5.1 (Fiebig's Correspondence)**.** There exists a bijectiion between between the following two categories:

additive categories of Soergel bimodules	$\leftrightarrow$	Braden-MacPherson Sheaves BM($\mathcal{G}$)

Specifically, where the indecomposable Soergel bimodules are sent to the indecomposable normalized Braden-MacPherson sheaves.

Soon, we will state this more formally. Throughout this section, we follow the exposition of the original paper, [15].

5.1 The Set Up

5.1.1 Assumptions

Before we dive straight into the math, let's briefly mention some of the assumptions that are required for this correspondence to work.

- **Coxeter System:** (W, S) is a Coxeter system, $\mathcal{T} \subset W$ the set of reflections, k is a field with $\text{char}(k) \neq 2$
- **Representation:** V is a finite-dimensional, reflection faithful representation of W over k
- **Hyperplanes and Forms:** For $t \in \mathcal{T}$, $V^t = \ker(t - \text{id})$ is a hyperplane. $\alpha_t \in V^*$ is a non-zero linear form with $\ker(\alpha_t) = V^t$. These forms are unique up to scalars and distinct for distinct reflections.
- **Symmetric Algebra:** $\mathcal{S} = S_k(V^*)$ is the symmetric algebra on V^*, graded by $\deg(V^*) = 2$. We identify $\mathcal{S}$ with polynomial functions on V when k is infinite. All modules and maps are graded unless stated otherwise. $M\{l\}$ denotes a grading shift: $(M\{l\})_n = M_{n+l}$
- **W-Orbit and Order:** $\Lambda \subset V$ is a W-orbit. It is equipped with a partial order $\leq$ such that $x, y \in \Lambda$ are comparable if $y = tx$ for some $t \in \mathcal{T}$. For the main theorem, Λ will be a *regular* orbit.

5.1.2 Category $\mathcal{V}(\Lambda)$: The Sheaf-Theoretic Side

Next, we describe the category $\mathcal{V}(\Lambda)$, which is built using an auxiliary algebra and linked to sheaves on a moment graph.

Definition 5.2 (Structure Algebra $\mathcal{Z}(\Lambda)$)**.** The *structure algebra* $\mathcal{Z} = \mathcal{Z}(\Lambda)$ is the $\mathcal{S}$-subalgebra of $\prod_{x\in\Lambda} \mathcal{S}$ defined by compatibility across reflections:

$$\mathcal{Z}(\Lambda) := \left\{ (z_x)_{x\in\Lambda} \in \prod_{x\in\Lambda} \mathcal{S} \;\middle|\; z_x \equiv z_{tx} \pmod{\alpha_t} \text{ for all } x \in \Lambda, t \in \mathcal{T} \text{ with } tx \in \Lambda \right\}.$$

$\mathcal{Z}$ is commutative, associative, $\mathbb{Z}$-graded, and an $\mathcal{S}$-algebra via the diagonal embedding $f \mapsto (f, f, \dots)$.

Building on the structure algebra $\mathcal{Z}(\Lambda)$, we now define the category $\mathcal{V}(\Lambda, \leq)$.

Definition 5.3 (Category $\mathcal{V}(\Lambda, \leq)$)**.** Let $\mathcal{Z}$-mod^f be the category of graded $\mathcal{Z}$-modules that are finitely generated and torsion-free[12] over $\mathcal{S}$, and whose action factors through $\mathcal{Z}^\Omega$

[12] A module is *torsion-free* if for any non-zero element m in the module and any non-zero scalar $r \in \mathcal{Z}$, the equation $rm = 0$ implies $m = 0$.

for some finite $\Omega \subset \Lambda$. For $M \in \mathcal{Z}\text{-mod}^f$ and an upwardly closed $\Omega \subset \Lambda$, let M^Ω be the canonical quotient supported on Ω. M admits a *Verma flag* if M^Ω is a graded free $\mathcal{S}$-module for all upwardly closed Ω. The category $\mathcal{V} = \mathcal{V}(\Lambda, \leq)$ is the full subcategory of $\mathcal{Z}\text{-mod}^f$ of objects admitting a Verma flag.

Remark 5.4 (Moment Graph Interpretation)**.** The orbit Λ with its partial order defines a *moment graph* $\mathcal{G}(\Lambda)$ whose vertices are Λ, and edges $x \leftrightarrow tx$ are labelled by $k \cdot \alpha_t \in \mathbb{P}V^*$. There is a localization functor $\mathcal{L}$ from $\mathcal{Z}$-modules to sheaves $\mathcal{SH}(\mathcal{G}(\Lambda))$, and $\mathcal{V}(\Lambda)$ fully embeds into $\mathcal{SH}(\mathcal{G}(\Lambda))$. Objects in the image $\mathcal{L}(\mathcal{V})$ are characterized by being generated by global sections, being flabby[13], and having graded free "local stalks" $\mathcal{H}^{[x]}$. This links $\mathcal{V}$ to Braden-MacPherson's framework.

5.1.3 Category $\mathcal{F}_\nabla$: The Bimodule Side

Now that we have defined the category $\mathcal{V}(\Lambda)$, we now turn our attention to the bimodule side of the story. On this side, the category is constructed from $\mathcal{S}$-bimodules, assuming k is infinite.

Definition 5.5 (Twisted Diagonals $\mathrm{Gr}(x)$ and Bimodules $S(x)$)**.** For $x \in W$, $\mathrm{Gr}(x) = \{(x^{-1}v, v) \in V \times V \mid v \in V\}$. Let $S(x)$ be the regular functions on $\mathrm{Gr}(x)$. The projection $\mathrm{pr}_2 : \mathrm{Gr}(x) \to V$ identifies $S(x)$ with $\mathcal{S}$ as a (right) $\mathcal{S}$-module. The $\mathcal{S}$-bimodule structure is given by $(f \otimes_k g) \cdot h = (f^x \cdot g)h$, where $f^x(v) = f(x^{-1}v)$.

Next, we present a lemma that explains the codimension of intersections of twisted diagonals, which is important for understanding the filtration structure in the category $\mathcal{F}_\nabla$.

Lemma 5.6 (Intersection Codimension [15, Lemma 4.1])**.**

$$\mathrm{codim}_{\mathrm{Gr}(x)}(\mathrm{Gr}(x) \cap \mathrm{Gr}(y)) = 1 \iff y = tx$$

for some $t \in \mathcal{T}$.

With this lemma in hand, we are now ready to define the category $\mathcal{F}_\nabla$, which is constructed from graded $\mathcal{S}$-bimodules and involves a filtration structure indexed by the length function $l : W \to \mathbb{N}_0$. For $M \in \mathcal{S}\text{-mod-}\mathcal{S}$, let $M_{\leq i}$ be the submodule supported on $\bigcup_{l(x) \leq i} \mathrm{Gr}(x)$.

Definition 5.7 (Category $\mathcal{F}_\nabla$)**.** The category $\mathcal{F}_\nabla$ is the full subcategory of graded $\mathcal{S}$-bimodules M such that:

1. M is supported on $\mathrm{Gr}(A) = \bigcup_{x \in A} \mathrm{Gr}(x)$ for some finite $A \subset W$.
2. For each $i \geq 0$, the filtration quotient $M_{\leq i+1}/M_{\leq i}$ is isomorphic to a finite direct sum of shifted basic bimodules:

$$M_{\leq i+1}/M_{\leq i} \cong \bigoplus_{x:\, l(x)=i+1} \bigoplus_j S(x)\{k_{x,j}\} \quad (k_{x,j} \in \mathbb{Z}).$$

   (Note: Fiebig's text states $l(x) = i$; we follow the interpretation consistent with filtration steps, where the $(i+1)$-th layer involves elements of length $i + 1$.)

5.1.4 The Equivalence Theorem (Theorem 4.3)

We now present the main result and sketch the proof strategy:

Central Setup: Let $\Lambda \subset V$ be a *regular* W-orbit[14]. Fix $v_0 \in \Lambda$ to identify $W \cong \Lambda$ via $w \mapsto w \cdot v_0$. Equip Λ with the partial order induced by the Bruhat order on W. Let $\mathcal{V} = \mathcal{V}(\Lambda)$ and $\mathcal{G} = \mathcal{G}(\Lambda)$ be the associated category and moment graph.

Theorem 5.8 ([15, Theorem 4.3])**.** Under the central setup above, there exists an equivalence of additive, graded categories:

$$\boxed{\mathcal{F}_\nabla \cong \mathcal{V}}$$

[13] A *flabby sheaf* on a topological space X is a sheaf $\mathcal{F}$ such that for every open subset $U \subset X$ and every inclusion of open subsets $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

[14] A *regular W-orbit* is an orbit of an element v under the group W such that the stabilizer $\mathrm{Stab}_W(v)$ is trivial, i.e., $\mathrm{Stab}_W(v) = \{e\}$.

The proof constructs functors $G : \mathcal{F}_\nabla \to \mathcal{V}$ and $F : \mathcal{V} \to \mathcal{F}_\nabla$ and shows they are quasi-inverse[15].

5.2 Proof

5.2.1 Step 1. Construction of the Functor $G : \mathcal{F}_\nabla \to \mathcal{V}$

This functor translates the bimodule structure into a sheaf on the moment graph, whose global sections yield an object in $\mathcal{V}$.

Local Data from Bimodules. Let $M \in \mathcal{F}_\nabla$, supported on $\mathrm{Gr}(A)$ for finite $A \subset W$.

Definition 5.9. For each vertex $x \in W \cong \Lambda$, define the *vertex module* (stalk) by restricting M to the diagonal $\mathrm{Gr}(x)$:

$$M_x := M|_{\mathrm{Gr}(x)}.$$

Via the isomorphism $\mathrm{pr}_2 : \mathrm{Gr}(x) \xrightarrow{\cong} V$, M_x is identified with a graded right $\mathcal{S}$-module (isomorphic to a direct sum of copies of $\mathcal{S}\{k\}$ by Definition 5.7).

Definition 5.10. For each edge E in $\mathcal{G}$ connecting x and $y = tx$ ($t \in \mathcal{T}$), the corresponding hyperplanes intersect: $\mathrm{Gr}(x) \cap \mathrm{Gr}(y) \cong V^t$. Define the *edge module* by restricting M to this intersection:

$$M^{x \cap tx} := M|_{\mathrm{Gr}(x) \cap \mathrm{Gr}(y)}.$$

Since $\mathrm{Gr}(x) \cap \mathrm{Gr}(y)$ is defined by $\alpha_t = 0$ (acting from the right via pr_2), the right action of α_t on $M^{x \cap tx}$ is zero: $M^{x \cap tx} \cdot \alpha_t = 0$.

Gluing via Restrictions: Defining the Sheaf $\mathcal{M}$.

There are natural restriction maps of right $\mathcal{S}$-modules induced by the inclusions of varieties:

$$\rho_{x, x \cap tx} : M_x \to M^{x \cap tx} \quad \text{and} \quad \rho_{tx, x \cap tx} : M^{tx} \to M^{x \cap tx}.$$

These data define a sheaf $\mathcal{M}$ (denoted $\mathcal{H}$ in [15, Section 4, Proof of Thm 4.3]) on the moment graph $\mathcal{G} = \mathcal{G}(\Lambda \cong W)$:

- $\mathcal{M}^x := M_x$ (module at vertex x).
- $\mathcal{M}^E := M^{x \cap tx}$ (module at edge $E : x \leftrightarrow tx$).
- The structure maps $\mathcal{M}^x \to \mathcal{M}^E$ are the restriction maps $\rho_{x, x \cap tx}$.

The condition $M^{x \cap tx} \cdot \alpha_t = 0$ matches the sheaf requirement $l(E) \cdot \mathcal{M}^E = 0$, where $l(E) = k \cdot \alpha_t$ is the label of edge E.

Local-to-Global Principle. A cornerstone result relates the global bimodule M to its local pieces M_x.

Proposition 5.11 ([15, Proposition 4.4])**.** Let $M \in \mathcal{F}_\nabla$ supported on $\mathrm{Gr}(A)$. The canonical map $M \hookrightarrow \bigoplus_{x \in A} M_x$ identifies M with the subspace of tuples $(m_x)_{x \in A}$ satisfying the matching conditions on all intersections corresponding to edges in $\mathcal{G}$:

$$M \cong \left\{ (m_x)_{x \in A} \in \bigoplus_{x \in A} M_x \;\middle|\; \rho_{x, x \cap tx}(m_x) = \rho_{tx, x \cap tx}(m_{tx}) \text{ for all edges } E : x \leftrightarrow tx \right\}.$$

Proof Idea). The statement is checked by localizing at height one graded prime ideals $\mathfrak{p} \subset \mathcal{S}$. Lemma 4.5 shows that $M_\mathfrak{p}$ decomposes into pieces corresponding to $S(x)_\mathfrak{p}$ and, if $\mathfrak{p} = (\alpha_t)$, extensions involving $S(x, tx)_\mathfrak{p}$. The proposition holds for these basic components, implying it holds for M. ☐

[15] Two categories $\mathcal{C}$ and $\mathcal{D}$ are *quasi-inverse equivalences* of categories if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is naturally isomorphic to the identity functor on $\mathcal{D}$ and $G \circ F$ is naturally isomorphic to the identity functor on $\mathcal{C}$.

This proposition means exactly that M is isomorphic to the global sections of the sheaf $\mathcal{M}$:

$$\Gamma(\mathcal{G}, \mathcal{M}) \cong M.$$

Furthermore, the matching conditions $z_x \equiv z_{tx} \pmod{\alpha_t}$ defining the structure algebra $\mathcal{Z}$ (Definition 5.2) allow $\mathcal{Z}$ to act naturally on $\Gamma(\mathcal{G}, \mathcal{M})$ by pointwise multiplication: $(z \cdot (m_x))_y = z_y \cdot m_y$. This endows $\Gamma(\mathcal{G}, \mathcal{M})$ with a $\mathcal{Z}$-module structure.

Landing in $\mathcal{V}$: The Verma Flag Condition. We define $G(M) := \Gamma(\mathcal{G}, \mathcal{M})$, now viewed as a $\mathcal{Z}$-module. We must show $G(M) \in \mathcal{V}$. This requires verifying the Verma flag condition.

Let $\Omega \subset W \cong \Lambda$ be upwardly closed with respect to. Bruhat order. Consider the quotient $\mathcal{Z}$-module $G(M)^\Omega = \operatorname{im}(G(M) \to \bigoplus_{x\in\Omega} G(M)^x_Q)$. Let M^Ω be the restriction of the bimodule M to $\operatorname{Gr}(\Omega) = \bigcup_{x\in\Omega} \operatorname{Gr}(x)$. A result from [26] implies $M^\Omega \in \mathcal{F}_\nabla$. Applying the functor G to M^Ω yields $G(M^\Omega)$. The construction ensures that $G(M^\Omega) \cong G(M)^\Omega$ as $\mathcal{Z}$-modules. Since $M^\Omega \in \mathcal{F}_\nabla$, its underlying right $\mathcal{S}$-module structure (obtained by forgetting the left action) must be graded free. This follows because M^Ω has a filtration whose quotients are sums of $S(x)\{k\}$, and $S(x)$ is graded free as a right $\mathcal{S}$-module (isomorphic to $\mathcal{S}$). The right $\mathcal{S}$-module underlying $G(M^\Omega)$ is precisely $G(M)^\Omega$. Therefore, $G(M)^\Omega$ is a graded free $\mathcal{S}$-module.

This holds for all upwardly closed Ω, so $G(M)$ admits a Verma flag. The finite generation and torsion-free conditions are also met. Thus, $G(M) \in \mathcal{V}$. This completes the construction $G : \mathcal{F}_\nabla \to \mathcal{V}$.

(Note: Flabbiness is implicitly satisfied because the construction lands in $\mathcal{V}$, whose objects correspond to flabby sheaves under localization.)

5.2.2 Step 2. Construction of the Functor $F : \mathcal{V} \to \mathcal{F}_\nabla$

This functor takes an object $N \in \mathcal{V}$ and endows it with an $\mathcal{S}$-bimodule structure using the structure algebra $\mathcal{Z}$, such that it becomes an object in $\mathcal{F}_\nabla$.

Starting with an Object $N \in \mathcal{V}$. Let N be an object in $\mathcal{V} = \mathcal{V}(\Lambda)$. By definition, N is a graded $\mathcal{Z}$-module admitting a Verma flag.

The Canonical Algebra Homomorphism $\sigma \otimes \tau$. Since $\Lambda \cong W$ is regular, we define two key maps into $\mathcal{Z}$, following [15, Lemma 2.4]:

- $\sigma : \mathcal{S} \to \mathcal{Z}$: This map encodes the left action. It is the algebra homomorphism extending $\sigma : V^* \to \mathcal{Z}$ defined by

  $$\sigma(\lambda)_w = w \cdot \lambda \quad (= \lambda \circ w^{-1})$$

  for $\lambda \in V^*, w \in W \cong \Lambda$. One verifies this definition satisfies the compatibility $w\lambda \equiv tw\lambda \pmod{\alpha_t}$ needed for $\sigma(\lambda)$ to be in $\mathcal{Z}$.
- $\tau : \mathcal{S} \to \mathcal{Z}$: This is the standard $\mathcal{S}$-algebra structure map, $\tau(g) = g \cdot 1_{\mathcal{Z}} = (g, g, \dots)$, encoding the right action.

These combine via the tensor product to give an algebra homomorphism:

$$\sigma \otimes \tau : \mathcal{S} \otimes_k \mathcal{S} \to \mathcal{Z}, \quad f \otimes_k g \mapsto \sigma(f)\tau(g).$$

This map essentially connects the abstract coordinate rings of $V \times V$ (namely $\mathcal{S} \otimes_k \mathcal{S}$) to the structure algebra $\mathcal{Z}$, which Fiebig notes can be viewed as the regular functions on the union of diagonals $\bigcup_{x\in W} \operatorname{Gr}(x)$ (at least for finite W).

Defining the Bimodule $F(N)$ via Restriction. Define $F(N)$ to be the same underlying graded k-vector space as N, but equipped with an $\mathcal{S}$-bimodule structure obtained by *restriction of scalars* along $\sigma \otimes \tau$:

$$(f \otimes_k g) \cdot n := (\sigma \otimes \tau)(f \otimes_k g) \cdot n = (\sigma(f)\tau(g)) \cdot n \quad \text{for } n \in N, f, g \in \mathcal{S}.$$

This defines the functor $F : \mathcal{V} \to \mathcal{S}\text{-mod-}\mathcal{S}$.

Compatibility Check and Landing in $\mathcal{F}_\nabla$. We must show $F(N) \in \mathcal{F}_\nabla$.

- **Support:** Since $N \in \mathcal{V} \subset \mathcal{Z}\text{-mod}^f$, its $\mathcal{Z}$-action factors through $\mathcal{Z}^\Omega$ for some finite $\Omega \subset W \cong \Lambda$. This implies $F(N)$ is supported on $\mathrm{Gr}(\Omega)$.
- **Filtration Quotients:** This is the crucial step. We need to show the filtration $F(N)_{\leq i+1}/F(N)_{\leq i}$ consists of sums of $S(x)\{k\}$.

  A key commutative diagram relates the action i_x defining $S(x)$ and the evaluation map $ev_x : \mathcal{Z} \to \mathcal{S}$:

$$\begin{array}{ccc} \mathcal{S} \otimes_k \mathcal{S} & \xrightarrow{\sigma\otimes\tau} & \mathcal{Z} \\ \downarrow i_x & & \downarrow ev_x \\ \mathcal{S} & \xrightarrow{\text{id}} & \mathcal{S} \end{array}$$

  This diagram means that the bimodule action defined by F, when restricted to the x-component N^x (using the decomposition $N_Q = \bigoplus N_Q^x$), matches the action defining the standard bimodule $S(x)$. Since $N \in \mathcal{V}$, it has a Verma flag. This implies its filtration quotients according to the Bruhat order, say $N^{[x]} = \ker(N^{\geq x} \to N^{>x})$, are graded free $\mathcal{S}$-modules. More usefully, linearizing the cofiltration [15, Section 2.5] shows that the quotient $N_{\{l(\cdot)=i+1\}} := N_{\{l(\cdot)\leq i+1\}}/N_{\{l(\cdot)\leq i\}}$ is isomorphic to a direct sum $\bigoplus_{x:l(x)=i+1} \bigoplus_j M(x)\{k_{x,j}\}$, where $M(x) \in \mathcal{V}$ is the basic object supported only at x (isomorphic to $\mathcal{S}$ as an $\mathcal{S}$-module). The commutative diagram implies $F(M(x)) \cong S(x)$ as $\mathcal{S}$-bimodules. Since F is additive and respects grading shifts, $F(N_{\{l(\cdot)=i+1\}}) \cong \bigoplus_{x:l(x)=i+1} \bigoplus_j S(x)\{k_{x,j}\}$. The map F also preserves the filtration indexed by length: $F(N)_{\leq i} = F(N_{\{l(\cdot)\leq i\}})$. Therefore, the filtration quotient for $F(N)$ is $F(N)_{\leq i+1}/F(N)_{\leq i} \cong F(N_{\{l(\cdot)=i+1\}})$, which has the required form $\bigoplus S(x)\{k\}$ with $l(x) = i + 1$.

This confirms $F(N)$ satisfies the conditions of Definition 5.7, so $F(N) \in \mathcal{F}_\nabla$. This completes the definition $F : \mathcal{V} \to \mathcal{F}_\nabla$.

5.2.3 Step 3. Verifying the Quasi–Inverse Relationship

Finally, we confirm that F and G invert each other up to natural isomorphism.

Checking $G \circ F \cong \mathrm{id}_\mathcal{V}$. Start with $N \in \mathcal{V}$. $F(N)$ is N with the S-bimodule structure via $\sigma \otimes \tau$. Applying G means constructing the sheaf $\mathcal{M}$ from $F(N)$ and taking global sections: $G(F(N)) = \Gamma(\mathcal{G}, \mathcal{M})$. The vertex module $\mathcal{M}^x$ is $F(N)|_{\mathrm{Gr}(x)}$. Due to the commutative diagram relating i_x and ev_x, this restriction process effectively recovers the original $\mathcal{Z}$-module structure on the components. Proposition 4.4 ensures $\Gamma(\mathcal{G}, \mathcal{M}) \cong F(N)$ as bimodules. But viewed as $\mathcal{Z}$-modules, the action on $\Gamma(\mathcal{G}, \mathcal{M})$ is the original action on N. Thus $G(F(N)) \cong N$.

Checking $F \circ G \cong \mathrm{id}_{\mathcal{F}_\nabla}$. Start with $M \in \mathcal{F}_\nabla$. $G(M) = \Gamma(\mathcal{G}, \mathcal{M}) \in \mathcal{V}$, where $\mathcal{M}$ is the sheaf built from M. By Proposition 4.4, $G(M) \cong M$ as underlying spaces, equipped with a $\mathcal{Z}$-action. Applying F restricts this $\mathcal{Z}$-action back to an $\mathcal{S}$-bimodule action via $\sigma \otimes \tau$. We need to show this recovered bimodule action is the original action on M. This follows again from the commutative diagram and how the $\mathcal{Z}$-action relates to the original $S \otimes_k S$ action on M. Essentially, the map $\sigma \otimes \tau$ correctly encodes the relationship between the pointwise $\mathcal{Z}$-action on global sections and the original bimodule structure defined via f^x and g. Thus $F(G(M)) \cong M$.

Compatibility of Structures (Filtrations, Grading, Exactness).

- **Filtrations:** The proof explicitly relies on the fact that F maps the Verma flag filtration (by upwardly closed sets / Bruhat order) in $\mathcal{V}$ to the length filtration in $\mathcal{F}_\nabla$, and G maps the length filtration back to the Verma flag structure. This compatibility is essential.
- **Grading:** Both categories and functors are inherently graded. The constructions preserve the grading degrees, including shifts $M\{k\}$.
- **Exactness:** While not explicitly detailed in the proof of Theorem 4.3 itself, both $\mathcal{V}$ and $\mathcal{F}_\nabla$ carry natural exact structures (Quillen structure on $\mathcal{V}$ [15, Proposition 2.10], structure from filtration on $\mathcal{F}_\nabla$). The functors F and G, being constructed from

restriction/induction-like processes preserving the filtration structures, are exact functors between these categories (or can be shown to be). This is crucial for applications, e.g., relating projective objects [15, Theorem 6.3].

These steps confirm that F and G are mutually quasi-inverse equivalences of categories, and we are done.

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