Topics in Symplectic Geometry: Lagrangian Floer Theory

Rutgers Symplectic Summer School

August 2025

Abstract

From August 18 to August 22, Rutgers University ran a summer school on symplectic geometry that aimed to provide graduate students and advanced undergraduate students tutorials in various advanced topics in symplectic geometry and introductions to recent developments. This year we focus on, but are not restricted to, theories and applications of Lagrangian Floer theories, including Fukaya categories, Floer theory in low-dimensional topology, contact geometry, and Hamiltonian dynamics etc.

This is an unofficial set of notes scribed by Gary Hu, who is responsible for all mistakes. None of these notes have been endorsed by the original lecturers. All of the course content is owned by their respective institutions and their researchers, while mistakes should be attributed solely to me. If you do find any errors, please report them to: gh7@williams.edu

Contents

MINICOURSES

There were three minicourses, each three hours long:

1. Rational Symplectic Field Theory by Zhengyi Zhou (Chinese Academy of Sciences)
2. Atiyah-Floer Conjecture by Aliakbar Daemi (Washington University in St. Louis)
3. Equivariant Lagrangian Floer Theory and Application to Symplectic Khovanov Homology by Cheuk Yu Mak (University of Sheffield)

1 Cheuk Yu Mak: Equivariant Lagrangian Floer Theory and Applications to Symplectic Khovanov Homology

There were three lectures:

- Lecture 1: We will introduce equivariant Lagrangian Floer cohomology. There are many different versions available in the literature. We will discuss the version given by Seidel and Smith. We will also explain the localization theorem.
- Lecture 2: Khovanov homology is a link invariant. Seidel and Smith introduced the symplectic analogue, called the symplectic Khovanov homology, which doesn't involve resolutions of a link. We will see what symplectic Khovanov homology is and understand why it is a link invariant. We will also see an annular version which gives a link invariant of a solid torus.
- Lecture 3: We will explain how to apply equivariant Lagrangian Floer theory to get applications of symplectic (annular) Khovanov homology.

1.1 Lecture 1

1.1.1 Background: Borel Equivariant Cohomology

Let G be a compact Lie group (e.g., $\mathbb{Z}/2\mathbb{Z}, S^1, U(n)$) and let M be a finite G-CW complex. A theory of G-equivariant cohomology, denoted $H^*_G(-)$, is expected to satisfy several properties, such as:

- Functoriality: It should be a contravariant functor from the category of G-spaces to the category of graded rings.
- Normalization for free actions: If G acts freely on M, the theory should recover the ordinary cohomology of the orbit space, i.e., $H^*_G(M) \cong H^*(M/G)$.

When the action of G is not free, the quotient space M/G is often poorly behaved. The Borel construction circumvents this issue by replacing M with a related space on which the action is free. This is accomplished by introducing a contractible space EG on which G acts freely.

Theorem 1.1. *For any Lie group G, such a universal space EG exists.*

We denote the quotient space by $BG := EG/G$, the classifying space of G.

Example 1.2.

- *For $G = \mathbb{Z}/2\mathbb{Z}$, we may take $EG = S^\infty$, on which G acts antipodally. Then $BG = S^\infty/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{RP}^\infty$.*
- *For $G = S^1$, we may also take $EG = S^\infty$ (the unit sphere in $\mathbb{C}^\infty$) with the standard action of S^1 by scalar multiplication. Then $BG = S^\infty/S^1 = \mathbb{CP}^\infty$.*

The diagonal action of G on $M \times EG$ is always free. This leads to our definition of Borel cohomology.

Definition 1.3 (Borel Cohomology)**.** *The **Borel equivariant cohomology** of a G-space M is*

$$H^*_G(M) := H^*\left(\frac{M \times EG}{G}\right).$$

We denote the quotient space $(M \times EG)/G$ by $M \times_G EG$.

This construction has several immediate properties:

- A G-equivariant map $f : M \to N$ naturally induces a map $M \times_G EG \to N \times_G EG$, making the construction functorial.
- If $H \subseteq G$ is a subgroup, EG also serves as an EH, and the natural map $M \times_H EG \to M \times_G EG$ provides a restriction homomorphism.
- The definition is independent of the choice of model for EG, up to canonical isomorphism.

- If G acts freely on M, the projection map $M \times_G EG \to M/G$ is a homotopy equivalence, which induces an isomorphism $H^*(M/G) \cong H^*_G(M)$.

The projection $M \times_G EG \to BG$ endows $H^*_G(M)$ with the structure of an algebra over $H^*(BG)$.

Example 1.4. *Let $G = \mathbb{Z}/2\mathbb{Z}$ and let the coefficient ring be $\mathbb{F}_2$. Then $H^*(BG;\mathbb{F}_2) = H^*(\mathbb{RP}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[q]$, where $\deg(q) = 1$. The ring map $\mathbb{F}_2[q] \to H^*_{\mathbb{Z}/2\mathbb{Z}}(M)$ governs the structure of the equivariant cohomology.*

Multiplication by q has a geometric meaning. The shift map $\tau : (z_0, z_1, z_2, \dots) \mapsto (0, z_0, z_1, \dots)$ on S^∞ induces a self-map of $\mathbb{RP}^\infty$. The induced map on cohomology is precisely multiplication by q.

- *If the $\mathbb{Z}/2\mathbb{Z}$-action on M is trivial, then $M \times_G EG = M \times BG$. By the Künneth formula, $H^*_G(M) \cong H^*(M) \otimes H^*(BG)$. The action of $H^*(BG)$ is free.*
- *If the $\mathbb{Z}/2\mathbb{Z}$-action on M is free, then $H^*_G(M) = H^*(M/G)$ is finite-dimensional. This implies that $H^*_G(M)$ is a torsion $\mathbb{F}_2[q]$-module.*

The map $p : M \times_G EG \to BG$ is a fibration with fiber M.

$$\begin{array}{ccc} M & \longrightarrow & M \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

This gives rise to a Serre spectral sequence with E_2-page $H^*(BG; H^*(M))$ converging to $H^*_G(M)$. A powerful consequence of this structure is the Localization Theorem.

Theorem 1.5 (Localization)**.** *For $G = \mathbb{Z}/2\mathbb{Z}$ with $\mathbb{F}_2$ coefficients, the inclusion of the fixed-point set $M^{\mathbb{Z}/2\mathbb{Z}} = \{x \in M | g(x) = x \forall g \in \mathbb{Z}/2\} \hookrightarrow M$ induces an isomorphism upon inverting q:*

$$H^*_{\mathbb{Z}/2\mathbb{Z}}(M;\mathbb{F}_2) \otimes_{\mathbb{F}_2[q]} \mathbb{F}_2[q,q^{-1}] \xrightarrow{\cong} H^*(M^{\mathbb{Z}/2\mathbb{Z}};\mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[q,q^{-1}].$$

Comparing the ranks of the modules in the theorem gives:

Corollary 1.6 (Smith Inequality)**.**

$$\text{rank } H^*_{\mathbb{Z}/2\mathbb{Z}}(M;\mathbb{F}_2) \geqslant \text{rank } H^*(M^{\mathbb{Z}/2\mathbb{Z}};\mathbb{F}_2).$$

Remark 1.7. *The long exact sequence in equivariant homology for the pair $(M, M^{\mathbb{Z}/2\mathbb{Z}})$,*

$$\dots \to H_{\mathbb{Z}/2\mathbb{Z}}(M, M^{\mathbb{Z}/2\mathbb{Z}}) \to H_{\mathbb{Z}/2\mathbb{Z}}(M) \to H_{\mathbb{Z}/2\mathbb{Z}}(M^{\mathbb{Z}/2\mathbb{Z}}) \to \dots$$

is useful. Furthermore, $H_{\mathbb{Z}/2\mathbb{Z}}(M, M^{\mathbb{Z}/2\mathbb{Z}})$ is built from cells of the form $(\mathbb{Z}/2\mathbb{Z} \times D^n, \mathbb{Z}/2\mathbb{Z} \times S^{n-1})$.

1.1.2 Towards Equivariant Floer Theory

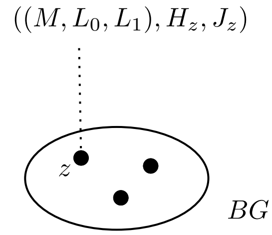
We now sketch how the Borel construction can be applied in symplectic geometry to define an equivariant version of Lagrangian Floer cohomology.

Let a symplectic action of $G = \mathbb{Z}/2\mathbb{Z}$ be given on a symplectic manifold (M, ω), and consider two Lagrangian submanifolds, L_0 and L_1, that are setwise fixed by G. The goal is to perform Floer theory on the homotopy quotient $M \times_G EG$; however, this space is not a symplectic manifold. The strategy, therefore, is to use Morse theory on the base space BG.

We choose a Morse-Smale pair (h, g) on BG, working within the standard filtration by finite-dimensional skeleta $BG_n \subseteq BG_{n+1} \subseteq \dots$, where $BG_n \cong \mathbb{RP}^{2n+1}$. The gradient flow lines between critical points of h will define the differential. For this to be well-defined on the filtration, we require that any flow line originating in a skeleton BG_n remains within BG_n.

Next, we introduce a Hamiltonian $H \in C^\infty([0,1] \times (M \times_G EG))$. For each $z \in BG$, this defines a Hamiltonian on the fiber, $H_z := H|_{\pi^{-1}(z)}$. The construction requires a regularity condition at the critical points of the Morse function: for each $z \in \operatorname{crit}(h)$, the time-1 flow must satisfy the transversality condition $\phi_{X_{H_z}}(L_{0,z}) \pitchfork L_{1,z}$. The Lagrangians in the fiber over z, denoted $L_{i,z}$, are given by the intersection $L_i \cap \pi^{-1}(z)$.

Let G act on a symplectic manifold (M, ω). Let $G = \mathbb{Z}/2$. Consider two Lagrangians L_0, L_1 setwise fixed by G. We want to do Floer theory on the fiber $M \times_G EG$, but this is not a symplectic manifold, so we best we can do is Morse theory on the base BG. Pick (h, g) for BG. Then $BG_n \subseteq BG_{n+1} \subseteq ...$ where $BG_n = \mathbb{RP}^{2n+1}$. We do Morse-Smale on BG_n. Consider $\operatorname{crit}(h)$ as the gradient flowlines. If it starts at $x \in BG_n \subseteq BG_N$, then the flowline stays in BG_n.



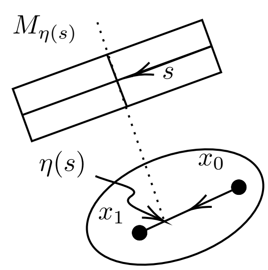
In addition to the Hamiltonian, we consider a family of ω-compatible almost complex structures $(J_z)_{z \in BG}$ parametrized by the base, where each J_z is defined on the fiber $\pi^{-1}(z) \cong M$.

The equivariant Floer complex, $CF_{eq}(L_0, L_1)$, is generated by pairs (z, x), where $z \in \operatorname{crit}(h)$ and x is a generator of the fiberwise Floer complex $CF(L_{0,z}, L_{1,z}, H_z)$. A generator x is an intersection point in the set $\phi^1_{X_{H_z}}(L_{0,z}) \cap L_{1,z}$.

The differential counts pairs (η, u) contributing to the map from a generator (z_0, x_0) to (z_1, x_1). Here, $\eta : \mathbb{R} \to BG$ is a gradient trajectory of (h, g) from z_0 to z_1. The map $u : \mathbb{R} \times [0,1] \to M \times_G EG$ is a finite-energy solution to the parametrized Floer equation

$$(\partial_s u - X_{H_{\eta(s)}}(u) \otimes dt)^{0,1}_{J_{\eta(s)}} = 0.$$

This solution must satisfy the condition $\pi \circ u(s,t) = \eta(s)$, the Lagrangian boundary conditions $u(s,0) \in L_{0,\eta(s)}$ and $u(s,1) \in L_{1,\eta(s)}$, and the asymptotic limits $\lim_{s \to -\infty} u(s, \cdot) = x_0$ and $\lim_{s \to +\infty} u(s, \cdot) = x_1$.



Example 1.8. *We consider the case of $BG = \mathbb{RP}^\infty$ with homogeneous coordinates $[z_0 : z_1 : \dots]$. Let the Morse function h be defined as*

$$h([z_0 : z_1 : \dots]) = \frac{|z_1|^2 + 2|z_2|^2 + 3|z_3|^2 + \dots}{\sum_{k=0}^\infty |z_k|^2}.$$

Let $\tau : \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ be the shift map. Then the pullback of h along τ satisfies

$$\begin{aligned} \tau^* h &= \frac{|z_1|^2 + 2|z_2|^2 + \dots + (|z_0|^2 + |z_1|^2 + \dots)}{\sum |z_k|^2} \\ &= h + 1. \end{aligned}$$

Let g be the standard round metric, and assume the pair (h, g) is chosen to be compatible with τ. For each integer $k \geqslant 0$, there exists a unique critical point $z^{(k)} \in \operatorname{crit}(h)$ whose index (or degree) is k. The resulting

Morse complex has a trivial differential, indicated by

$$C_{Morse}(h) : \dots \xrightarrow{0} \mathbb{F}_2\langle z^{(k)}\rangle \xrightarrow{0} \dots$$

Since the differential is zero, any gradient trajectory η from a critical point $z^{(k)}$ to another point $z^{(l)}$ must be trivial unless $l = k$. The full complex of generators for the equivariant theory, which are pairs $(z^{(k)}, x)$, thus admits a natural filtration:

$$\bigoplus_{k\geqslant 0}(z^{(k)}, x) \supseteq \bigoplus_{k\geqslant 1}(z^{(k)}, x).$$

We now make the simplifying assumption that the Floer data (H_z, J_z) is constant for all critical points, i.e., $H_{z^{(k)}} = H_{z^{(l)}}$ and $J_{z^{(k)}} = J_{z^{(l)}}$ for all k, l. Then the equivariant Floer complex is given by

$$\begin{aligned} CF_{eq}(L_0, L_1) &= \bigoplus_{z^{(k)}} CF(L_0, L_1; H_{z^{(0)}}) \\ &= CF(L_0, L_1; H_{z^{(0)}})[q] \end{aligned}$$

where the equivariant differential has the form $d_{eq} = d_0 + qd_1 + q^2 d_2 + \dots$.

Next time, we will see how this algebraic structure plays an important role in equivariant Lagrangian Floer theory.

1.2 Lecture 2

We begin by recalling the foundational setup for equivariant Lagrangian Floer cohomology. Given a symplectic manifold (M, ω), a pair of Lagrangian submanifolds (L_0, L_1), and a choice of auxiliary data (h, g) consisting of a Morse function h and a Riemannian metric g, we can select a compatible pair (H, J), a Hamiltonian and an almost complex structure, to define the Floer cochain complex $CF^*(L_0, L_1)$.

We now consider the setting where a finite group G acts on M by symplectomorphisms, and the Lagrangians L_0, L_1 are G-invariant. For our purposes, we will specialize to the case $G = \mathbb{Z}/2$. The goal is to construct a G-equivariant version of Floer cohomology. Let $\tau : \mathbb{RP}^\infty \to \mathbb{RP}^\infty$ be the map inducing multiplication by the variable q on cohomology. The $\mathbb{Z}/2$-equivariant Floer cochain complex, denoted $CF^*_{\text{eq}}(L_0, L_1)$, is a free module over the polynomial ring $\mathbb{F}_2[q]$, given by

$$CF^*_{\text{eq}}(L_0, L_1) = CF^*(L_0, L_1) \otimes_{\mathbb{F}_2} \mathbb{F}_2[q].$$

The equivariant differential d_{eq} is an $\mathbb{F}_2[q]$-module endomorphism of this complex, which can be expressed as a power series in q:

$$d_{\text{eq}} = d_0 + qd_1 + q^2 d_2 + \dots,$$

where each $d_i : CF^*(L_0, L_1) \to CF^*(L_0, L_1)$ is a map of degree $+1$. The operator d_0 is the standard, non-equivariant Floer differential.

1.2.1 The Localization Isomorphism

In classical equivariant topology, for a space X with a $\mathbb{Z}/2$-action, there is a fundamental long exact sequence relating the equivariant cohomology of X to that of its fixed-point set $X^{\mathbb{Z}/2}$:

$$\dots \to H^k_{\text{eq}}(X; X^{\mathbb{Z}/2}) \to H^k_{\text{eq}}(X) \to H^k_{\text{eq}}(X^{\mathbb{Z}/2}) \to H^{k+1}_{\text{eq}}(X; X^{\mathbb{Z}/2}) \to \dots$$

At the cochain level, this corresponds to a short exact sequence

$$0 \to C^*_{\text{eq}}(X; X^{\mathbb{Z}/2}) \to C^*_{\text{eq}}(X) \to C^*_{\text{eq}}(X^{\mathbb{Z}/2}) \to 0.$$

One might hope for a direct analogue in Floer theory. However, establishing such a relationship presents two significant technical challenges:

1. The differential d_{eq} does not preserve the decomposition of the chain complex into invariant and non-invariant parts. A generator corresponding to a G-invariant intersection point can have a non-zero differential to a generator that is not invariant, and vice-versa.
2. A pseudo-holomorphic curve contributing to d_{eq} that lies entirely within the fixed locus $M^{\mathbb{Z}/2}$ may be a regular solution in the moduli space over the fixed locus, but fail to be regular in the full equivariant moduli space $M \times_G EG$. This issue of transversality is a central difficulty.

The second problem was resolved by Seidel and Smith. Their work provides a powerful localization theorem that connects the equivariant Floer homology of (M, L_0, L_1) to the ordinary Floer homology of the fixed-point sets, under a specific geometric condition on the normal bundle.

Theorem 1.9 (Seidel-Smith Localization)**.** *Let (M, ω) be a symplectic manifold with a symplectic $\mathbb{Z}/2$-action, and let L_0, L_1 be $\mathbb{Z}/2$-invariant Lagrangian submanifolds. Suppose the normal bundle of the fixed locus $(M^{\mathbb{Z}/2}, L_0^{\mathbb{Z}/2}, L_1^{\mathbb{Z}/2})$ inside (M, L_0, L_1) is stably trivial. Then there is an isomorphism of localized modules:*

$$HF_{eq}(L_0, L_1; \mathbb{F}_2) \otimes_{\mathbb{F}_2[q]} \mathbb{F}_2[q, q^{-1}] \cong HF(L_0^{\mathbb{Z}/2}, L_1^{\mathbb{Z}/2}; \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[q, q^{-1}].$$

The notion of a stably trivial normal bundle is very important and is defined as follows:

Definition 1.10. *The normal bundle triple $(NM^{\mathbb{Z}/2}, NL_0^{\mathbb{Z}/2}, NL_1^{\mathbb{Z}/2})$ is **stably trivial** if there exists an integer $k \geqslant 0$ and an isomorphism of vector bundles over $M^{\mathbb{Z}/2}$,*

$$\phi : NM^{\mathbb{Z}/2} \oplus \mathbb{C}^k \xrightarrow{\cong} M^{\mathbb{Z}/2} \times \mathbb{C}^n,$$

(where the rank of $NM^{\mathbb{Z}/2}$ is $n-k$) such that ϕ restricts to isomorphisms on the Lagrangian normal bundles:

$$\phi(NL_0^{\mathbb{Z}/2} \oplus \mathbb{R}^k) = L_0^{\mathbb{Z}/2} \times \mathbb{R}^n,$$
$$\phi(NL_1^{\mathbb{Z}/2} \oplus (i\mathbb{R})^k) = L_1^{\mathbb{Z}/2} \times (i\mathbb{R})^n.$$

Remark 1.11. *The utility of the stable triviality condition is that it allows one to modify the geometric setup without changing the equivariant Floer cohomology. Given such a trivialization ϕ, one can consider the stabilized manifold $M \times \mathbb{C}^k$ and Lagrangians $L_i \times \mathbb{R}^k$. The equivariant Floer complex remains unchanged, $CF_{eq}(L_0 \times \mathbb{R}^k, L_1 \times i\mathbb{R}^k) \cong CF_{eq}(L_0, L_1)$. One can then use ϕ to construct an equivariant Hamiltonian isotopy to deform the geometry so that the fixed locus becomes $(M^{\mathbb{Z}/2} \times \{0\}, L_0^{\mathbb{Z}/2}, L_1^{\mathbb{Z}/2})$ and its normal bundle is now genuinely trivial. This resolves the transversality issues mentioned earlier. After this modification, there is a well-defined restriction map $\lambda : CF_{eq}(L_0, L_1) \to CF(L_0^{\mathbb{Z}/2}, L_1^{\mathbb{Z}/2})[q]$ which induces the isomorphism in the theorem upon localization.*

1.2.2 Application: Symplectic Khovanov Homology

One application of this machinery is the construction of a symplectic version of Khovanov homology, a powerful link invariant. We provide a basic outline of their construction.

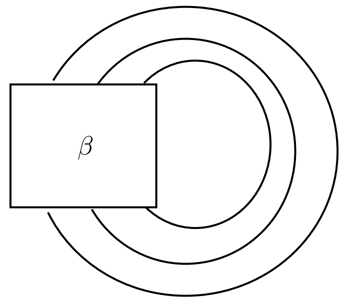
The construction relies on three core ingredients:

1. An exact symplectic manifold (Y, ω).
2. A homomorphism from the braid group $\mathrm{Br}_n(\mathbb{C}) = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$ into the group of Hamiltonian sympathomorphisms of Y, considered up to homotopy:

$$\rho : \mathrm{Br}_n(\mathbb{C}) \to \mathrm{Symp}(Y, \omega)/\mathrm{Ham}.$$

3. An exact Lagrangian submanifold $L \subseteq (Y, \omega)$.

Given a braid $\beta \in \mathrm{Br}_n(\mathbb{C})$, its image under ρ is a Hamiltonian isotopy class, which we denote by ϕ_β. Applying this to the Lagrangian L, we obtain a new Lagrangian $\phi_\beta(L)$. The Floer homology group $HF(L, \phi_\beta(L))$ is then an invariant of the isotopy class. If this construction satisfies certain properties (invariance under Markov moves), the resulting homology theory $HF(L; \beta \circ L)$ depends only on the link obtained by the closure of the braid β:



The specific manifold (Y, ω) is constructed from a family of affine varieties. Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathrm{Conf}_n(\mathbb{C})$ be a configuration of n distinct points in the complex plane. To this, we associate the affine surface in $\mathbb{C}^3$ defined by

$$A_\tau = \{(u, v, z) \in \mathbb{C}^3 \mid uv = (z - \tau_1) \ldots (z - \tau_n)\}.$$

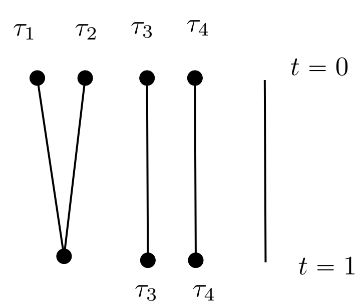
This construction can be globalized. Consider the space $\mathbb{C}^n$ parametrizing monic polynomials of degree n, identified with $\mathrm{Sym}^n(\mathbb{C})$ via the root map. Let $\mathcal{A} \subset \mathbb{C}^3 \times \mathbb{C}^n$ be the total space defined by the equation $uv = z^n - p_1 z^{n-1} - \cdots - p_n$. The projection onto the second factor gives a fibration $\mathcal{A} \to \mathbb{C}^n$. The base space $\mathbb{C}^n \cong \mathrm{Sym}^n(\mathbb{C})$ has a singular locus corresponding to polynomials with repeated roots. The regular part of the base is precisely $\mathrm{Conf}_n(\mathbb{C})$.

The symplectic structure on the fibers A_τ allows for symplectic parallel transport over paths in the regular base $\mathrm{Conf}_n(\mathbb{C})$. This transport defines the monodromy representation

$$\pi_1(\mathrm{Conf}_n(\mathbb{C})) \to \mathrm{Symp}(A_\tau)/\mathrm{Ham},$$

which gives the required braid group action.

When a path $(\tau^t)_{t \in [0,1]}$ in the base approaches the singular locus, for instance when $\tau_1^1 = \tau_2^1$ as $t \to 1$, the fiber A_{τ^t} degenerates. For example, we have:



The local model for this degeneration is $\{uv = (z - \epsilon)(z + \epsilon)\} \to \{uv = z^2\}$ as $\epsilon \to 0$. This process creates a a vanishing Lagrangian sphere S^2 in the smooth fiber $\{uv = z^2 - \epsilon^2\}$ that collapses to the singular point in the degenerate fiber.

To construct a richer theory, Seidel and Smith use a more complicated (and more interesting) space. Let $\mathrm{Sym}_0^{2n}(\mathbb{C}) = \{(\tau_1, \ldots, \tau_{2n}) \in \mathrm{Sym}^{2n}(\mathbb{C}) \mid \sum \tau_i = 0\}$. We consider the map from the space of traceless $2n \times 2n$ matrices to the space of their characteristic polynomial coefficients:

$$\begin{aligned} \pi_{\text{char}} : \mathfrak{sl}(2n) &\to \mathbb{C}^{2n-1} \cong \mathrm{Sym}_0^{2n}(\mathbb{C}) \\ A &\mapsto \text{coefficients of char poly of } A. \end{aligned}$$

The manifold Y is defined as follows:

$$Y = \left\{ \left. \begin{pmatrix} Y_{11} & I & 0 & \cdots & 0 \\ Y_{21} & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Y_{(n-1)1} & 0 & \cdots & 0 & I \\ Y_{n1} & 0 & \cdots & 0 & 0 \end{pmatrix} \right| Y_{11} \in \mathfrak{sl}(2),\ Y_{j1} \in \mathfrak{gl}(2)\ \forall j > 1 \right\} \subseteq \mathfrak{sl}(2n).$$

There is a projection map $\pi : Y \to \mathbb{C}^{2n-1} \cong \mathrm{Sym}_0^{2n}(\mathbb{C})$ induced by π_{char}. The regular locus of the base is $\mathrm{Conf}_0^{2n}(\mathbb{C})$, the space of $2n$ distinct points in $\mathbb{C}$ summing to zero. The monodromy of this fibration provides an action of $\pi_1(\mathrm{Conf}_0^{2n}(\mathbb{C})) = \mathrm{Br}_{2n}(\mathbb{C})$ on the fibers Y_τ for $\tau \in \mathrm{Conf}_0^{2n}(\mathbb{C})$.

Consider a point in the singular locus of the base, e.g., $\tau_* = (0, 0, \tau_3, \ldots, \tau_{2n})$. This means the corresponding matrices have a generalized eigenspace of dimension 2 for the eigenvalue 0. This can happen in two ways:

1. Type 1: The eigenspace is 2-dimensional.
2. Type 2: The eigenspace is 1-dimensional.

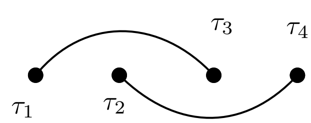
The Type 1 singularities correspond to matrices in the fiber Y_{τ_*} where the bottom-left block Y_{n1} is zero. Such points form the singular locus of the fiber Y_{τ_*}. There is a map from this singular locus to the fiber over a lower-dimensional configuration space:

$$\mathrm{Sing}(Y_{\tau_*}) \to Y_{(\tau_3,\ldots,\tau_{2n})}.$$

Given a Lagrangian $L \subseteq Y_{(\tau_3,\ldots,\tau_{2n})}$, which can be identified with a component of the singular locus $\mathrm{Sing}(Y_{(0,0,\tau_3,\ldots,\tau_{2n})})$, one can consider the set of points in a nearby smooth fiber $Y_{(-\epsilon,\epsilon,\tau_3,\ldots,\tau_{2n})}$ that converge to L as $\epsilon \to 0$. This procedure defines a new Lagrangian $\tilde{L}$ in the smooth fiber, which is topologically an S^2-bundle over L.

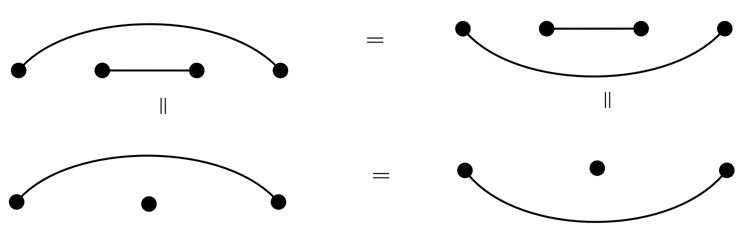
1.2.3 Lagrangians and Khovanov Homology

Lagrangian submanifolds in the fibers Y_τ are constructed from diagrams. A non-crossing matching on $2n$ points defines a Lagrangian submanifold in the fiber $Y_{(\tau_1,\ldots,\tau_{2n})}$:

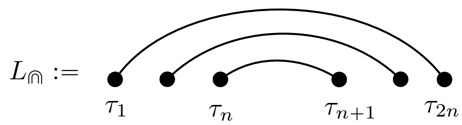


Proposition 1.12. *The construction of Lagrangians from non-crossing matchings has the following properties:*

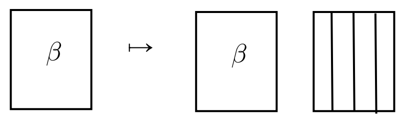
- *The resulting Lagrangian submanifold L is independent of the ordering of the matching paths used in its construction.*
- *Composing diagrams corresponds to applying the braid group action. That is, if a diagram D' is obtained from D by the action of a braid β, then $L_{D'} = \phi_\beta(L_D)$.*
- *The diagrams below represent isotopic Lagrangians:*



To define the link invariant, we fix a reference Lagrangian $L_{\Cap}$ corresponding to the standard "cap" diagram on $2n$ points:



where $\beta \mapsto \beta \times \mathrm{id}$, i.e.



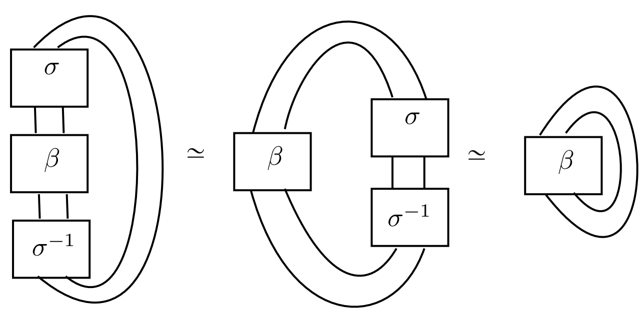
Given a braid $\beta \in \mathrm{Br}_n$, we embed it into Br_{2n} via the map $\beta \mapsto \beta \otimes \mathrm{id}_n$, which acts on the first n strands and leaves the last n strands fixed. The symplectic Khovanov homology of the closure of β, denoted $Kh(\mathrm{cl}(\beta))$, is defined as the Lagrangian Floer homology:

$$Kh(\mathrm{cl}(\beta)) := HF(L_{\Cap}, \phi_\beta(L_{\Cap})).$$

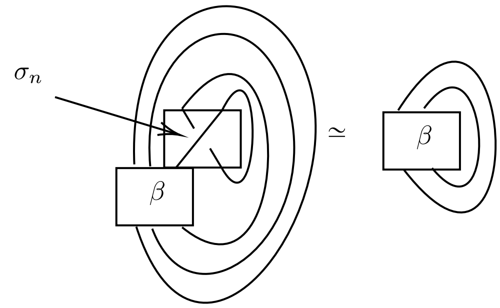
For this definition to yield a well-defined link invariant, it must be invariant under the Markov moves, which relate braids with equivalent closures.

Proposition 1.13. *The constructed homology theory is invariant under the two Markov moves:*

1. $Kh(cl(\beta)) \cong Kh(cl(\sigma\beta\sigma^{-1}))$ *for any* $\sigma \in \mathrm{Br}_n$*:*



2. $Kh(cl(\beta)) \cong Kh(cl(\beta\sigma_n^{\pm 1}))$ *for* $\beta \in \mathrm{Br}_n$*, where the closure is taken in* Br_{n+1}*:*



Proof.

- Let ϕ_σ be the Hamiltonian diffeomorphism corresponding to σ.

$$\begin{aligned} Kh(\mathrm{cl}(\sigma\beta\sigma^{-1})) &= HF(L_{\Cap}, \phi_{\sigma\beta\sigma^{-1}}(L_{\Cap})) \\ &= HF(L_{\Cap}, \phi_\sigma\phi_\beta\phi_{\sigma^{-1}}(L_{\Cap})) \\ &\cong HF(\phi_\sigma^{-1}(L_{\Cap}), \phi_\beta\phi_{\sigma^{-1}}(L_{\Cap})) \\ &= HF(\phi_{\sigma^{-1}}(L_{\Cap}), \phi_\beta(\phi_{\sigma^{-1}}(L_{\Cap}))). \end{aligned}$$

  Because the diagram for σ^{-1} composed with the cap diagram is isotopic to the cap diagram itself (i.e., $\phi_{\sigma^{-1}}(L_{\Cap}) \cong L_{\Cap}$), we can substitute this into the expression:

$$\begin{aligned} \ldots &\cong HF(L_{\Cap}, \phi_\beta(L_{\Cap})) \\ &= Kh(\mathrm{cl}(\beta)). \end{aligned}$$

- We do a very brief sketch since we don't have enough time. We need to analyze the degeneration corresponding to three colliding points, i.e., the fiber over $(0, 0, 0, \tau_4, \ldots, \tau_{2n})$. The singular locus of this fiber is isomorphic to $Y_{(0,\tau_4,\ldots,\tau_{2n})}$, which brings the dimension into the story.

□

1.3 Lecture 3

1.3.1 Hilbert Schemes

Let $\tau = (\tau_1, \ldots, \tau_{2n}) \in \mathrm{Conf}^{2n}(\mathbb{C})$ be a configuration of $2n$ distinct points in the complex plane. We consider the variety Y_τ defined by matrices of a particular block form.

Definition 1.14. *Let Y_τ be the subvariety of a product of matrix spaces defined as:*

$$Y_\tau = \left\{ A = \begin{pmatrix} Y_{11} & I & 0 & \cdots & 0 \\ Y_{21} & 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Y_{(n-1)1} & 0 & \cdots & 0 & I \\ Y_{n1} & 0 & \cdots & 0 & 0 \end{pmatrix} \middle| \; Y_{11} \in \mathfrak{sl}(2),\; Y_{j1} \in \mathfrak{gl}(2)\ \forall j > 1, \det(xI - A) = \prod_{i=1}^{n}(x - \tau_i) \right\}$$

We seek a more geometric description of Y_τ. We can analyze the determinant $\det(xI - A)$ in terms of the block entries Y_{j1}. Let $Y_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$. A block determinant expansion reveals that the characteristic polynomial $\det(xI - A)$ takes the form of a determinant of a 2×2 matrix whose entries are polynomials in x.

$$\begin{aligned}
\det(xI - A) &= \det\big((xI)^n - Y_{11}(xI)^{n-1} + Y_{21}(xI)^{n-2} - \dots\big) \\
&= \det\begin{pmatrix} x^n - a_1x^{n-1} + \dots & -b_1x^{n-1} + b_2x^{n-2} - \dots \\ -c_1x^{n-1} + \dots & x^n - d_1x^{n-1} + d_2x^{n-2} - \dots \end{pmatrix} \\
&= A(x)D(x) - B(x)C(x)
\end{aligned}$$

Here, $A(x), B(x), C(x), D(x)$ are polynomials whose coefficients are determined by the entries of the matrices Y_{j1}. The condition $\det(xI - A) = P_\tau(x) = \prod_{i=1}^{2n}(x - \tau_i)$ gives us a useful algebraic identity:

$$A(x)D(x) - B(x)C(x) = P_\tau(x) \implies A(x)D(x) = B(x)C(x) + P_\tau(x).$$

This algebraic relation suggests a link to a geometric object. We can consider the variety $S_\tau = \operatorname{Spec}\left(\frac{\mathbb{C}[b,c,x]}{(bc+P_\tau(x))}\right)$, which is defined by the equation $bc + P_\tau(x) = 0$ in $\mathbb{C}^3_{b,c,x}$. The algebraic identity shows that for any $A \in Y_\tau$, the roots $z_1, \dots, z_n$ of the polynomial $A(x)$ are related to points in S_τ. Specifically, if $A(z_i) = 0$, then the point $(B(z_i), C(z_i), z_i)$ lies on the variety S_τ. This defines a map $Y_\tau \to \operatorname{Sym}^n(S_\tau)$ by sending a matrix A to the set of points $\{(B(z_i), C(z_i), z_i)\}_{i=1}^n$, where the z_i are the roots of $A(x)$.

Conversely, given n distinct points $(b_i, c_i, z_i) \in S_\tau$ for $i = 1, \dots, n$, we can define a matrix A in Y_τ. First, we construct the polynomial $A(x) = \prod_{i=1}^n (x - z_i)$. Using polynomial interpolation, there exists a unique polynomial $B(x)$ of degree at most $n-1$ such that $B(z_i) = b_i$ for all i. Similarly, there exists a unique polynomial $C(x)$ of degree at most $n-1$ such that $C(z_i) = c_i$ for all i. The identity $b_ic_i + P_\tau(z_i) = 0$ for each i implies that the polynomial $B(x)C(x) + P_\tau(x)$ vanishes at each z_i. This means $A(x)$ must divide $B(x)C(x) + P_\tau(x)$, which then uniquely determines a polynomial $D(x)$ such that $A(x)D(x) - B(x)C(x) = P_\tau(x)$.

To make this correspondence more precise, we need to work using the language of schemes. Let R be the coordinate ring of Y_τ. The roots of the polynomial $A(x)$ over Y_τ correspond to the scheme $\operatorname{Spec}(R[x]/A(x))$, which is a subscheme of $Y_\tau \times \mathbb{C} = \operatorname{Spec}(R[x])$. This leads to the following geometric picture:

$$\begin{array}{ccc}
Z & \subseteq & Y_\tau \times S_\tau \\
& \searrow \quad \swarrow & \\
& Y_\tau & \ni A
\end{array}$$

Here, Z is a closed subscheme of $Y_\tau \times S_\tau$. For a given $A \in Y_\tau$, the fiber of Z over A is a subscheme of S_τ whose ideal is given by $I_A = \{Q(b, c, z) \mid A(x) \text{ divides } Q(B(x), C(x), x)\}$. This ideal defines a subscheme of S_τ of length n.

The collection of all such length-n subschemes is called the Hilbert scheme.

Definition 1.15. *The Hilbert scheme $Hilb^n(S_\tau)$ is the moduli space of all length-n subschemes of S_τ.*

Theorem 1.16. *The Hilbert scheme $Hilb^n(S_\tau)$ is a smooth algebraic variety.*

This theorem allows us to define a map from the variety Y_τ to the Hilbert scheme.

Definition 1.17. *Let $j : Y_\tau \to Hilb^n(S_\tau)$ be the map defined by $A \mapsto I_A$, where I_A is the ideal of the subscheme corresponding to the roots of $A(x)$.*

This map provides an important connection between the algebraic variety Y_τ and the geometric object $\operatorname{Hilb}^n(S_\tau)$.

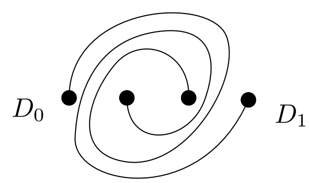
Theorem 1.18 (Manolescu)**.**

1. *The map $j : Y_\tau \to \operatorname{Hilb}^n(S_\tau)$ is an open embedding.*

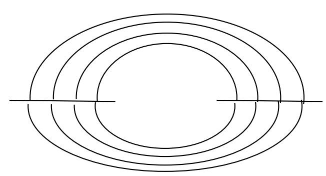
2. *The complement of the image of j, denoted by D_τ, is the closure of the set of collections of points $\{(b_i, c_i, z_i)\}_{i=1}^n \in \mathrm{Conf}^n(S_\tau)$ such that $z_i = z_j$ for some $i \neq j$. This set is a subvariety of $\mathrm{Hilb}^n(S_\tau)$.*

3. *For a given non-crossing matching diagram D, there is a Lagrangian submanifold $L_D^{\text{Seidel-Smith}} \subseteq Y_\tau$. The Hilbert scheme also comes with a natural map to $\mathrm{Sym}^n(S_\tau)$, which is a subset of $\mathrm{Conf}^n(S_\tau)$. Given a Lagrangian sphere $L_{p_i} \subseteq S_\tau$, we can define a Lagrangian submanifold $L_D^M = \mathrm{Sym}(L_p) = \pi(L_{p_1} \times \cdots \times L_{p_n}) \subseteq \mathrm{Sym}^n(S_\tau)$. We have the following chain of inclusions: $\pi(L_{p_1} \times \cdots \times L_{p_n}) \subseteq \mathrm{Conf}^n(S_\tau) \backslash D_\tau \subseteq Y_\tau$. The key result is that L_D^M is Hamiltonian isotopic to $L_D^{\text{Seidel-Smith}}$.*

4. *The rank of the Khovanov homology of a link K can be computed using the Floer homology of the corresponding Lagrangian submanifolds. Specifically, for two diagrams D_0 and D_1 of a link, the Floer homology group $\mathrm{HF}(L_{D_0}, L_{D_1})$ computes the Khovanov homology $\mathrm{Kh}(K)$. The diagrams below illustrate two non-crossing matchings, D_0 and D_1.*

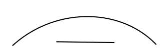


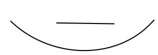
The diagrams in the image depict two non-crossing matching diagrams, D_0 and D_1, which can be used to construct the corresponding Lagrangian submanifolds. The Khovanov homology of the link is then computed by the Floer homology of these two Lagrangian submanifolds. The next image shows the associated link diagram.



This image shows a standard link diagram. The previous image of the matching diagrams, D_0 and D_1, are used as building blocks to define the Lagrangian submanifolds whose intersection Floer homology gives the Khovanov homology of this link.

Corollary 1.19. *Consider the diagrams for D_0 and D_1 from the theorem statement.*

$D_0 =$ 

$D_1 =$ 

The corresponding Lagrangian submanifolds L_{D_0} and L_{D_1} are not Hamiltonian isotopic in the space $\mathrm{Conf}^n(S_\tau) \backslash D_\tau$. However, by the Seidel-Smith lemma and the Manolescu theorem, they are Hamiltonian isotopic in the larger ambient space Y_τ.

1.3.2 Symmetry and Localization

We can use geometric symmetries to learn more about these invariants. Let's consider a 2-periodic link K and its quotient link $\overline{K}$. This symmetry can be lifted to the geometric spaces we have constructed.

Consider a map $\iota : S_\tau \to S_\tau$ defined by $(b, c, z) \mapsto (b, c, -z)$. This involution is well-defined if the polynomial $P_\tau(z)$ satisfies $P_\tau(z) = P_\tau(-z)$. This condition holds if the set of points τ is symmetric with respect to the origin, i.e., $\tau = \{\pm\tau_1, \ldots, \pm\tau_{2n}\} \in \mathrm{Conf}^{4n}(\mathbb{C})$. The involution ι on S_τ induces an involution on the Hilbert scheme $\mathrm{Hilb}^n(S_\tau)$, and furthermore, it lifts to an involution on Y_τ.

We can describe this induced involution on Y_τ more explicitly. If $A \in Y_\tau$ is the matrix corresponding to the set of roots $\{z_i\}$, the image $\iota(A)$ is the matrix corresponding to the roots $\{-z_i\}$. This means that if $A(x) = \prod_{i=1}^{2n}(x - z_i)$, then $\iota(A)$ corresponds to the polynomial $\tilde{A}(x) = \prod_{i=1}^{2n}(x + z_i) = A(-x)$. Similarly, the polynomial $B(x)$ associated with A satisfies $B(z_i) = b_i$. The polynomial $\tilde{B}(x)$ associated with $\iota(A)$ satisfies $\tilde{B}(-z_i) = b_i$. This implies that $B(x) = \tilde{B}(-x)$. This involution acts on the matrices in Y_τ as a sign change on the blocks, as illustrated below.

$$\begin{pmatrix} Y_{11} & I & & \\ \vdots & & \ddots & \\ \vdots & & & I \\ Y_{n1} & & & 0 \end{pmatrix} \xmapsto{\iota} \begin{pmatrix} -Y_{11} & I & & & \\ Y_{21} & & \ddots & & \\ -Y_{31} & & & \ddots & \\ \vdots & & & & I \\ Y_{n1} & & & & \end{pmatrix}$$

The fixed set of this involution is also of interest. A matrix A is fixed by ι if $A = \iota(A)$, which means that the entries must satisfy certain sign relations. This leads to a simplified block matrix structure for the fixed points: the fixed set is defined by:

$$\begin{pmatrix} 0 & I & & & & \\ Y_{21} & & \ddots & & & \\ 0 & & & \ddots & & \\ Y_{41} & & & & \ddots & \\ \vdots & & & & & I \\ Y_{2n\,1} & & & & & \end{pmatrix}$$

This fixed set is isomorphic to a new variety of the same type, but with half the number of parameters: the fixed set is isomorphic to:

$$\begin{pmatrix} Y_{21} & I & & \\ Y_{41} & & \ddots & \\ \vdots & & & I \\ Y_{2n\,1} & & & \end{pmatrix} \in Y_{\tau^2}$$

where $\tau^2 = \{\tau_1^2, \ldots, \tau_{2n}^2\}$.

This structure allows us to apply a localization principle in Floer theory. If we have two Lagrangian submanifolds L and K that are fixed by the $\mathbb{Z}/2$ involution, their Floer homology can be related to the Floer homology of their fixed point sets.

Proposition 1.20 (Localization). *If Lagrangian submanifolds L and K are fixed by a $\mathbb{Z}/2$ involution, their Floer homology satisfies:*

$$\mathrm{rank}(\mathrm{Kh}(K)) = \mathrm{rank}(\mathrm{HF}(L, K)) \geqslant \mathrm{rank}(\mathrm{HF}(L^{\mathbb{Z}/2}, K^{\mathbb{Z}/2})) = \mathrm{rank}(\mathrm{Kh}(\overline{K}))$$

where $L^{\mathbb{Z}/2}$ and $K^{\mathbb{Z}/2}$ denote the fixed point sets of L and K respectively.

2 Zhengyi Zhou: Rational Symplectic Field Theory

There were three lectures:

- Lecture 1: We will explain the algebraic structures arising from rational symplectic field theory (RSFT) and use them to define hierarchy invariants for contact manifolds via RSFT.
- Lecture 2: We will explain properties and applications of the hierarchy invariants, as well as examples.
- Lecture 3: We will explain the functoriality of RSFT and hierarchy invariants under strong symplectic cobordisms, and their applications. Time permitting, we will explain, in all known examples, the non-existence of strong/weak fillings of contact manifolds of dimension at least 5 is obtained via RSFT.

2.1 Lecture 1

2.1.1 L_∞ Algebras

Let V be a $\mathbb{Z}/2$-graded vector space over $\mathbb{Q}$. We denote by SV the symmetric algebra on V,

$$SV = \bigoplus_{k\geqslant 0} S^k V,$$

where S^kV is the k-th symmetric power of V. We also define $\overline{S}V$ to omit the degree-zero component:

$$\overline{S}V = \bigoplus_{k>0} S^k V.$$

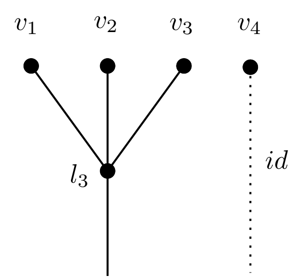
Definition 2.1. *An L_∞-**algebra** structure on V is a sequence of linear maps $\{l_k\}_{k\geqslant 1}$, called the higher brackets, where each map $l_k : S^kV \to V$ has degree 1. These maps collectively induce a degree-1 linear operator $\hat{l} : \overline{S}V \to \overline{S}V$ defined on an element $v_1 \otimes \cdots \otimes v_n \in S^nV$ by*

$$\hat{l}(v_1, \ldots, v_n) = \sum_{k=1}^{n} \sum_{\sigma\in Sh(k,n-k)} \epsilon(\sigma) l_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \otimes v_{\sigma(k+1)} \otimes \cdots \otimes v_{\sigma(n)},$$

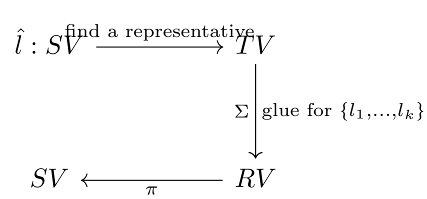
where $Sh(k, n-k)$ is the set of $(k, n-k)$-shuffles (permutations σ of $\{1, \ldots, n\}$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$), and $\epsilon(\sigma)$ is the Koszul sign. The defining condition for an L_∞-algebra is that this induced map squares to zero:

$$\hat{l}^2 = 0.$$

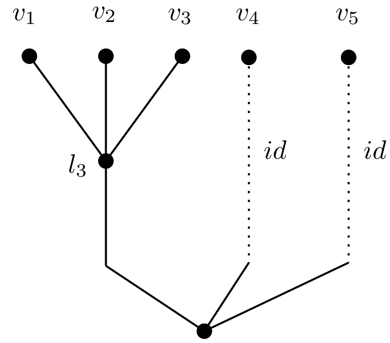
There also exists a pictorial description of $\hat{l}$ that is often easier to work with:



The action of $\hat{l}$ can be conceptualized as a procedure: an element is lifted from SV to a space of representatives TV, the operations $\{l_k\}$ are applied via a gluing map Σ to a space RV, and the result is projected back to SV via π. This is summarized by the diagram:



Using this, we can reformulate this equation diagramatically:

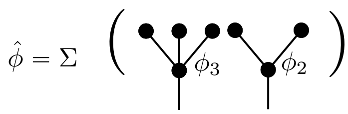


equals 0.

Now, we can define a L_∞-morphism.

Definition 2.2. *An L_∞-**morphism** from $(V, \{l^1_k\})$ to $(W, \{l^2_k\})$ is a collection of maps $\{\phi_k : S^kV \to W\}_{k\geqslant 1}$ that induces an operator $\hat{\phi}$ satisfying the intertwining relation $\hat{\phi} \circ \hat{l}^1 = \hat{l}^2 \circ \hat{\phi}$.*

Diagramaticaly, this condition can be drawn as:



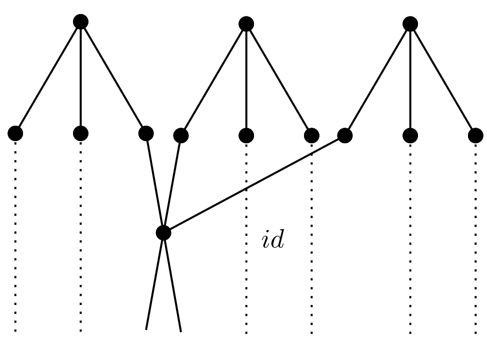
Exercise 2.3. *Define the composition of L_∞-morphisms and show that it is also an L_∞-morphism.*

2.1.2 BL^∞ and IBL^∞-algebras

For the intended applications, we require a more general structure that allows for operations with multiple outputs.

Definition 2.4. *A BL_∞-**algebra** on V is a collection of maps $\{p^{k,l} : S^kV \to S^lV\}_{k\geqslant 1, l\geqslant 0}$. Let $EV = \overline{S}(SV)$. These maps assemble into an operator $\hat{p} : EV \to EV$ that is required to satisfy $\hat{p}^2 = 0$.*

Diagramatically:



The operations $\{p^{k,l}\}$ of the BL_∞-algebra can be constructed through a map Σ. This map acts on a space of representatives, denoted $\overline{T}TV$, for elements in EV. The map Σ "glues" the basic operations $\{p^{k,l}\}$ acyclically, and its output is then mapped back to EV.

Proposition 2.5. *If $p^{k,0} = 0$ for $k = 1, 2$, then the following hold:*

1. *$p^{1,1}$ is a differential on V.*
2. *$p^{2,1}$ induces a Lie bracket on the homology $H_*(V, p^{1,1})$.*
3. *$p^{1,2}$ induces a co-Lie algebra structure on $H_*(V)$.*

4. The composition $p^{1,2} \circ p^{2,1}$ vanishes on homology.

Proposition 2.6. *If $p^{k,l} = 0$ for all $k > l$, the BL_∞-structure reduces to an L_∞-structure.*

Exercise 2.7. *Let $\hat{p}^1 = \hat{p}|_{SV} : SV \to SV$. Show that $(\hat{p}^1)^2 = 0$, implying $(SV, \hat{p}^1)$ is a differential graded algebra. Show that this induces an L_∞-structure on SV with brackets $l^1 = \hat{p}^1$.*

We have a trivial BL_∞ structure on $\{0\} = V$, which is an initial object:

$$0 \xrightarrow{\varphi} V, \quad \phi^{k,l} = 0.$$

What about the converse?

Definition 2.8. *A BL_∞ **augmentation** is a BL_∞ morphism*

$$(V, T) \xrightarrow{\mathcal{E}} \{0\},$$

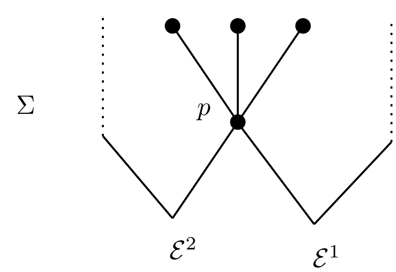
i.e., a collection of maps

$$\mathcal{E}^k : S^k V \to S^0\{0\} = \mathbb{Q}$$

such that

$$\hat{\mathcal{E}} \circ \hat{p} = 0.$$

Diagramatically, this is equivalent to:



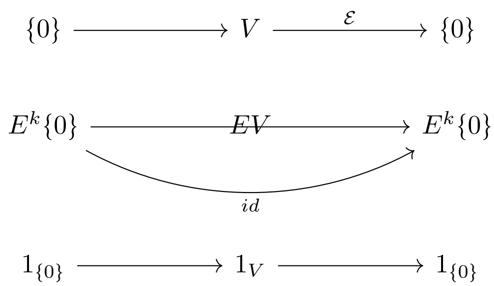
An augmentation may not exist for a given BL_∞-algebra. To study the obstruction to its existence, we introduce an invariant derived from the homology of a sequence of truncated complexes.

Definition 2.9. *For a BL_∞-algebra $(V, \hat{p})$, we define the following:*

- *For each $k \geqslant 0$, the **truncated space** is $E^k V = \bigoplus_{i=0}^k S^i(SV)$. The operator $\hat{p}$ restricts to a differential $\hat{p}|_{E^k V}$ on this space, making $(E^k V, \hat{p}|_{E^k V})$ a chain complex.*
- *For the trivial algebra $\{0\}$, the corresponding complex is $E^k\{0\} = \bigoplus_{i=0}^k \mathbb{Q}$, whose homology is $H_*(E^k\{0\}) \cong \bigoplus_{i=0}^k \mathbb{Q}$.*
- *The unique algebra morphism $i : \{0\} \to V$ induces a chain map $i_* : E^k\{0\} \to E^k V$. The **unit element** $1_V \in H_*(E^k V, \hat{p})$ is the image of the generator $1 \in H_0(E^k\{0\})$ under this map, i.e., $1_V = i_*(1)$.*

Proposition 2.10. *If a BL_∞-algebra V has an augmentation, then its unit 1_V is non-zero in $H_*(E^k V, \hat{p})$ for all $k \geqslant 0$.*

Proof. An augmentation $\mathcal{E} : V \to \{0\}$ induces a chain map on the truncated complexes. The composition of the morphisms induced by $i : \{0\} \to V$ and $\mathcal{E} : V \to \{0\}$ is the identity on $E^k\{0\}$. Consequently, the composition of the induced maps on homology, $\mathcal{E}_* \circ i_*$, is the identity on $H_*(E^k\{0\})$. This is summarized by the diagram:



Since $1 \in H_*(E^k\{0\})$ is non-zero, its image under i_*, the element 1_V, must also be non-zero for the composition to be the identity. $\square$

This proposition motivates the following definition, which quantifies the failure of 1_V to persist.

Definition 2.11. *The **torsion** of a BL_∞-algebra $(V, \hat{p})$ is*

$$T(V) = \inf\{k \in \mathbb{N} \cup \{0\} \mid 1_V = 0 \text{ in } H_*(E^{k+1}V, \hat{p})\} \in \mathbb{N} \cup \{\infty\}.$$

(Here, the infimum of the empty set is taken to be ∞.)

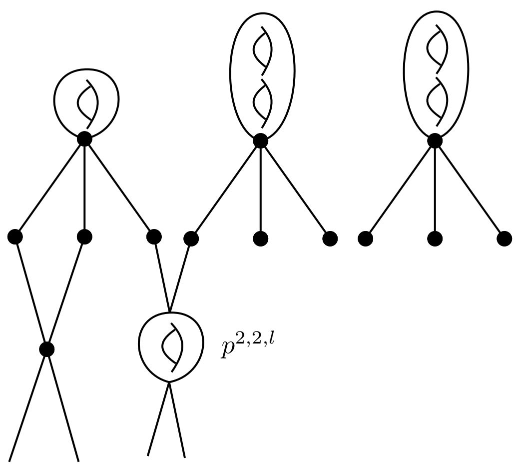
Exercise 2.12. *Show that if there exists a BL_∞-morphism $\phi : V \to W$, then $T(V) \geqslant T(W)$.*

Finally, we introduce a version of this algebra that includes a genus count.

Definition 2.13. *An IBL_∞**-algebra** is given by a collection of maps $p^{k,l,g} : S^kV \to S^lV$ for $k \geqslant 1, l \geqslant 0, g \geqslant 0$.*

These maps determine an operator on $\overline{S}(SV)[\hbar]$.

Pictorially, it's almost the same as previously, but now we count genus:



2.1.3 Curves in SFT

Now, we aim to answer the question: what is symplectic field theory?

Definition 2.14. *A $(2n-1)$-dimensional manifold Y equipped with a hyperplane distribution $\xi \subset TY$ is a **contact manifold** if there exists a 1-form $\alpha \in \Omega^1(Y)$, called the contact form, such that $\xi = \ker \alpha$ and the volume form condition $\alpha \wedge (d\alpha)^{n-1} \neq 0$ is satisfied everywhere. The choice of α is called a co-orientation.*

The contact form determines a canonical vector field on the manifold.

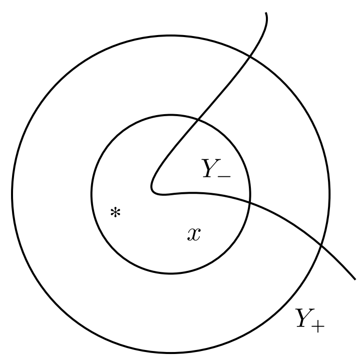
Definition 2.15. *The **Reeb vector field** R on a contact manifold (Y, α) is the unique vector field defined by the conditions $\alpha(R) = 1$ and $\iota_R d\alpha = 0$, where ι_R denotes the interior product with respect to R.*

The morphisms between contact manifolds are given by Liouville cobordisms.

Definition 2.16. *A **Liouville cobordism** from a contact manifold (Y_-, α_-) to (Y_+, α_+) is a compact manifold with boundary W equipped with a 1-form λ such that:*

- *The boundary is oriented as $\partial W = Y_+ \sqcup (-Y_-)$.*
- *The 2-form $\omega = d\lambda$ is a symplectic form on W.*
- *The **Liouville vector field** X, defined by $\iota_X \omega = \lambda$, points outwards along Y_+ and inwards along Y_-.*
- *The restriction of λ to the boundaries recovers the contact distributions: $ker\lambda|_{Y_\pm} = \xi_\pm$.*

Example 2.17. *Let M be a Stein manifold with boundary $Y = \partial M$. The manifold Y inherits a natural contact structure where the contact distribution is given by $\xi_p = T_pY \cap J(T_pY)$ for $p \in Y$. The Stein manifold M itself serves as a Liouville cobordism from Y to the empty set, known as a Stein filling.*



Remark 2.18. *Liouville cobordisms can be composed up to a natural equivalence known as Liouville homotopy. This observation establishes the **symplectic cobordism category**, whose objects are contact manifolds and whose morphisms are Liouville homotopy classes of cobordisms.*

Definition 2.19. ***Symplectic Field Theory (SFT)** is a contravariant functor from the symplectic cobordism category to a suitable algebraic category $\mathcal{C}$ (e.g., the category of IBL_∞-algebras).*

This functor is constructed, following the work of Eliashberg, Givental, and Hofer, by defining invariants from counts of pseudo-holomorphic curves.

Later, for technical reasons, we will later replace the symplectic cobordism category with the **strong symplectic cobordism category**, where morphisms are required to possess a Liouville structure near the boundary.

2.1.4 Analysis of Curves

To study pseudo-holomorphic curves, we must equip the symplectization $\hat{Y} = (\mathbb{R}_s \times Y, d(e^s\alpha))$ with a suitable almost complex structure J that is compatible with both the symplectic form and the underlying contact geometry. We choose a class of almost complex structures that satisfy the following three conditions:

1. Relation to the Reeb field: J maps the translation vector field ∂_s to the Reeb vector field R.
$$J(\partial_s) = R$$
2. Compatibility with the contact structure: J preserves the contact distribution ξ and is compatible with the 2-form $d\alpha$ on it. This means:
   - $J(\xi_p) = \xi_p$ for all points $p \in Y$.
   - The bilinear form $d\alpha(\cdot, J|_\xi \cdot)$ defines a Riemannian metric on the vector bundle ξ.
3. Invariance: J is invariant under translations in the $\mathbb{R}_s$ coordinate, i.e. it is s-invariant.

A J-holomorphic curve $u : \Sigma \to \hat{Y}$ from a closed Riemann surface Σ must be constant, as $\int_\Sigma u^*(d(e^s\alpha)) = 0$. To obtain a non-trivial theory, we must therefore consider curves from Riemann surfaces with punctures.

Example 2.20. *A periodic Reeb orbit γ of period T gives rise to a J-holomorphic cylinder $u : \mathbb{R}_s \times S^1_t \to \hat{Y}$ via the map $(s,t) \mapsto (Ts, \gamma(Tt))$. The energy of this cylinder is infinite:*

$$E(u) = \int_{\mathbb{R}\times S^1} u^*(d(e^s\alpha)) = \infty.$$

We must restrict our attention to curves with finite energy.

Definition 2.21. *The **Hofer energy** of a curve $u = (a,v) : \Sigma \to \mathbb{R} \times Y$ is defined as*

$$E(u) = \int_\Sigma v^*(d\alpha) + \sup_\phi \int_\Sigma u^*(d(\phi(s)\alpha)),$$

where the supremum is taken over all smooth, non-decreasing functions $\phi : \mathbb{R} \to [0,1]$.

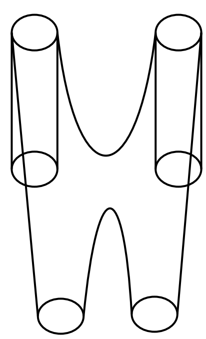
Theorem 2.22 (Hofer-Wysocki-Zehnder)**.** *Let α be a non-degenerate contact form on Y. If a J-holomorphic curve $u : \Sigma \to \hat{Y}$ has finite Hofer energy, then at each puncture of the Riemann surface Σ, the map u converges exponentially to a trivial cylinder over a periodic Reeb orbit.*

More precisely, in cylindrical coordinates (s,t) near a puncture, there exists a periodic Reeb orbit γ of period T, a constant $c \in \mathbb{R}$, and positive constants C and δ such that the following estimate holds:

$$\|u(s,t) - (Ts + c, \gamma(Tt))\| \leqslant Ce^{-\delta|s|} \quad \text{as } s \to \pm\infty.$$

The norm is measured with respect to a product metric on the symplectization $\mathbb{R} \times Y$.

The Hofer-Wysocki-Zehnder theorem allows us to draw pictures such as:



In particular, it guarantees that the cylindrical ends corresponding to punctures are asymptotically modeled by trivial cylinders over periodic Reeb orbits.

2.2 Lecture 2

We begin by explaining many analogies between SFT and the more familiar framework of Morse theory.

In Morse theory, the central object is a smooth function $f : M \to \mathbb{R}$ on a finite-dimensional manifold M. The dynamics are governed by its critical points and the gradient flow lines connecting them.

In SFT, the setting is infinite-dimensional. Let (Y, α) be a contact manifold. The analogue of the manifold is the free loop space $C^\infty(S^1, Y)$, and the Morse function is replaced by the **symplectic action functional** A:

$$A : C^\infty(S^1, Y) \to \mathbb{R}$$
$$\gamma \mapsto \int_{S^1} \gamma^*\alpha.$$

The critical points of this functional correspond to closed orbits of the Reeb vector field R (where R is defined by $i_R d\alpha = 0$ and $i_R\alpha = 1$). Specifically, a loop $\gamma(t)$ is a critical point of A if and only if its velocity vector $\gamma'(t)$ is everywhere proportional to the Reeb vector field $R(\gamma(t))$. These are precisely the constant loops, and the positive and negative parametrizations of closed Reeb orbits.

The analogy extends further. In Morse theory, one chooses a Riemannian metric g to define the gradient vector field $-\nabla_g f$. In SFT, one chooses a compatible almost complex structure J on the contact structure $\xi = \ker\alpha$. This choice induces an L^2-metric on the loop space, and the corresponding L^2-gradient of the action functional A at a loop γ is given by

$$\nabla A(\gamma) = -J(\pi_\xi \gamma'),$$

where π_ξ is the projection onto the contact hyperplanes along the Reeb direction. The "flow lines" of SFT, which are solutions to a perturbed Cauchy-Riemann equation, can be viewed as the gradient flow lines for this structure.

Let us now examine the local picture around a critical point, which we take to be a closed Reeb orbit $\gamma(t)$ with period T, so that $\gamma'(t) = TR(\gamma(t))$. The Hessian of A at γ, denoted $\mathrm{Hess}A(\gamma)$, is a symmetric operator acting on the tangent space $T_\gamma(C^\infty(S^1, Y)) = \Gamma(\gamma^*\xi)$. It is given by

$$\mathrm{Hess}A(\gamma)\eta = -J\pi_\xi\nabla_t\eta - J\pi_\xi\nabla_\eta(TR),$$

where ∇ is the Levi-Civita connection associated with a compatible metric. This operator is often referred to as the asymptotic operator A_γ.

Exercise 2.23. *Show that the asymptotic operator A_γ is a self-adjoint operator with respect to the L^2-metric. A Reeb orbit γ is called non-degenerate if* $\ker A_\gamma = \{0\}$.

Upon choosing a symplectic trivialization of the contact bundle $\gamma^*\xi \cong S^1 \times \mathbb{C}^{n-1}$, the operator A_γ takes the more familiar form

$$A_\gamma\eta = -J_0\frac{d\eta}{dt} - S(t)\eta,$$

where J_0 is the standard complex structure on $\mathbb{C}^{n-1}$ and $S(t)$ is a path of symmetric matrices in $\mathrm{sp}(2n-2, \mathbb{R})$. The spectrum of this operator determines the local behavior of the SFT "gradient flow." The linearized flow of the Reeb vector field along γ is the path of symplectic matrices $\Phi(t) \in \mathrm{Sp}(2n-2)$ solving $\Phi'(t) = J_0 S(t)\Phi(t)$.

The gradient flow lines of Morse theory, which solve the equation $\gamma'(s) + \nabla f(\gamma(s)) = 0$, have an analogue in Symplectic Field Theory (SFT). These are pseudo-holomorphic curves in the **symplectization** $\hat{Y} = Y \times \mathbb{R}$. For a map from a Riemann surface (Σ, j) to the symplectization, written as $\tilde{u} = (u, a) : \Sigma \to Y \times \mathbb{R}$, the condition to be pseudo-holomorphic is equivalent to the system:

$$\begin{cases} (\pi_\xi du)^{0,1} = 0 \\ da = u^*\alpha \circ j \end{cases}$$

where $a : \Sigma \to \mathbb{R}$ is the coordinate in the $\mathbb{R}$ factor and π_ξ is the projection onto the contact structure $\xi = \ker\alpha$. A crucial technical condition is that these curves have finite energy, $\int_\Sigma u^* d\alpha < \infty$, which controls the symplectic area of the curve's projection into ξ. For example, Hofer rules out the map $e^z : \mathbb{C}^\times \to \mathbb{C}^\times$.

The asymptotic behavior of these curves near punctures mirrors the exponential convergence of gradient flow lines in Morse theory. In Morse theory, a flow line $\gamma(s)$ approaches a critical point p as $|\gamma(s) - p| < Ce^{-\lambda|s|}$. Near the critical point, the flow line can be approximated by $\gamma(s) \approx p + e^{\lambda s}\nu$, where λ is an eigenvalue of $-\mathrm{Hess}f|_p$ and ν is a corresponding eigenvector. In SFT, a finite energy curve $u(s,t)$ on a cylinder $(s,t) \in (-\infty, 0] \times S^1$ approaches the cylinder over a Reeb orbit $\gamma(t)$. The convergence is of the form $u(s,t) \approx \gamma(t) + e^{\lambda s}\nu(t)$, where $\lambda > 0$ is a positive eigenvalue of the asymptotic operator A_γ and $\nu(t)$ is the corresponding eigenfunction.

We now construct the central objects of study in Symplectic Field Theory: the moduli spaces of pseudo-holomorphic curves. The construction requires several pieces of data:

- Domain: A closed, connected Riemann surface (Σ, j) of a fixed genus g.
- Punctures: Disjoint, finite, and ordered sets of points $\Sigma_+ \subset \Sigma$ (positive punctures) and $\Sigma_- \subset \Sigma$ (negative punctures).
- Asymptotic Data: For each set of punctures, we fix a corresponding ordered multiset of closed Reeb orbits, denoted $[\Gamma_+]$ and $[\Gamma_-]$, such that $|\Gamma_\pm| = |\Sigma_\pm|$. We also fix a base point on each of these Reeb orbits.
- Asymptotic Markers: At each puncture $p \in \Sigma_+ \cup \Sigma_-$, we fix an **asymptotic marker**, which is a non-zero vector $v_p \in T_p\Sigma$. These markers are used to resolve rotational symmetries in the convergence of a curve to a Reeb orbit.

With this data, we can define the moduli space.

Definition 2.24. *The **moduli space of pseudo-holomorphic curves**, denoted $M_Y(g, \Gamma_+, \Gamma_-)$, is the set of equivalence classes of pairs (j, u), where j is a complex structure on a genus-g surface Σ, and $u : \Sigma \backslash (\Sigma_+ \cup \Sigma_-) \to \hat{Y}$ is a map to the symplectization $\hat{Y} = Y \times \mathbb{R}$, satisfying the following conditions:*

1. *u is pseudo-holomorphic (i.e., $\bar{\partial}_J(u) = 0$) and has finite energy.*
2. *At each positive puncture $p_i \in \Sigma_+$, the map u is asymptotic to the corresponding Reeb orbit $\gamma_i \in \Gamma_+$.*
3. *At each negative puncture $p_j \in \Sigma_-$, the map u is asymptotic to the corresponding Reeb orbit $\gamma_j \in \Gamma_-$.*
4. *The convergence at each puncture respects the chosen asymptotic marker and base point.*

The equivalence relation is given by biholomorphic reparametrizations of the domain Σ that preserve the ordering of the punctures, modulo the natural translation action of $\mathbb{R}$ on the target $\hat{Y}$.

The moduli space $M_Y(g, \Gamma_+, \Gamma_-)$ can be described analytically as the zero set of a Fredholm section.

Proposition 2.25. *The space of maps (before quotienting by reparametrizations and the $\mathbb{R}$-action) can be framed as the zero set of a Fredholm section s of a Banach bundle $E \to B$.*

$$\begin{array}{c} E \\ \downarrow \uparrow s \\ B \end{array}$$

The virtual dimension of the moduli space can be computed:

Proposition 2.26. *The virtual dimension of $M_Y(g, \Gamma_+, \Gamma_-)$ is given by:*

$$\begin{aligned} \textit{virdim } M_Y(g, \Gamma_+, \Gamma_-) &= \textit{Ind}(s) - \dim(\textit{Aut}(\Sigma, \Sigma_\pm)) - 1 \\ &= (n-3)(2 - 2g - |\Gamma_+| - |\Gamma_-|) + \sum_{\gamma \in \Gamma_+} CZ_\tau(\gamma) - \sum_{\gamma \in \Gamma_-} CZ_\tau(\gamma) \\ &\quad + 2\langle c_1(\xi, \tau), A \rangle - 1, \end{aligned}$$

where $n = (\dim Y + 1)/2$, $CZ_\tau(\gamma)$ is the Conley-Zehnder index of an orbit γ, and the term $\langle c_1(\xi, \tau), A \rangle$ represents the evaluation of the first Chern class of the contact bundle ξ (relative to the trivialization τ) on the homology class represented by the curve.

In a SFT, we need compactness. Unlike in simpler Floer theories, a sequence of pseudo-holomorphic curves can degenerate by "breaking" into a multi-level object called a holomorphic building. The following example illustrates this phenomenon.

Example 2.27. *Consider the one-parameter family of maps $u_\delta : \mathbb{C}^\times \to \mathbb{C}^2 \backslash \{0\}$, given by*

$$u_\delta(z) = (z^3 + z^2 + \delta z, z^3 + 2z^2 + 3\delta z).$$

Here, the domain $\mathbb{C}^\times \cong \mathbb{R} \times S^1$ represents a cylindrical domain, and the target $\mathbb{C}^2 \backslash \{0\}$ is the symplectization of the standard contact sphere (S^3, α_{std}). We analyze the limiting behavior of this family as $\delta \to 0$ in the C^∞_{loc} topology.

If we take the limit directly as $\delta \to 0$, the sequence u_δ converges to

$$u_0(z) = \lim_{\delta \to 0} u_\delta(z) = (z^3 + z^2, z^3 + 2z^2).$$

This limiting curve connects a Reeb orbit of asymptotic period 3 (at the positive end, $z \to \infty$) to a Reeb orbit of period 2 (at the negative end, $z \to 0$).

To reveal more of the structure, we can perform a rescaling to "zoom in" on the behavior near the puncture at $z = 0$. We can reparametrize the domain of u_δ. For instance, a particular choice leads to the expression:

$$(\delta^3 z^3 + \delta^2 z^2 + \delta^2 z, \delta^3 z^3 + 2\delta^2 z^2 + 3\delta^2 z).$$

Following the reparametrization, we apply a translation in the target space to recenter the map. This yields the new family of maps $\tilde{u}_\delta$:

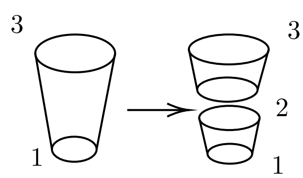
$$\tilde{u}_\delta(z) = (\delta z^3 + z^2 + z, \delta z^3 + 2z^2 + 3z).$$

Now, taking the limit of this rescaled sequence as $\delta \to 0$ in the C^∞_{loc} topology, we obtain a completely different limiting curve:

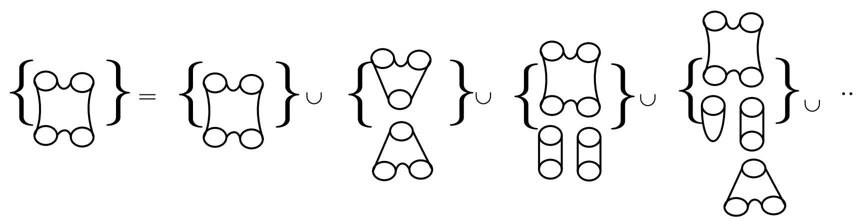
$$v_0(z) = \lim_{\delta \to 0} \tilde{u}_\delta(z) = (z^2 + z, 2z^2 + 3z).$$

This second curve, extracted from the same original sequence u_δ, connects a Reeb orbit of period 2 to one of period 1.

The existence of these two distinct limits from a single sequence is problematic and shows us why we need compactness.



Theorem 2.28 (Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder)**.** *The moduli space $M_Y(g, \Gamma_+, \Gamma_-)$ admits a natural compactification $\overline{M}_Y(g, \Gamma_+, \Gamma_-)$ by adding stable buildings:*



A building is stable if each level has non-zero energy $\int u^* d\alpha$ or has a stable component (i.e., a branched cover over a trivial cylinder).

2.2.1 Contact Homology and RSFT

Let α be a non-degenerate contact form. Let V be the vector space generated by "good" Reeb orbits, with a generator q_γ for each such orbit γ. The grading is given by $|q_\gamma| = CZ(\gamma) + n - 3 \pmod 2$.

Definition 2.29. *The **contact homology differential** is a linear map $\partial : S(V) \to S(V)$ of degree -1, which is defined on the generators $q_\gamma \in V$ by the formula:*

$$\partial(q_\gamma) = \sum_{[\Gamma]} \frac{\#\overline{M}_Y(\gamma, \Gamma)}{m_\Gamma k_\Gamma} q^\Gamma.$$

This map is extended to all of the symmetric algebra $S(V)$ by requiring that it satisfies the graded Leibniz rule, $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$. The terms in the formula are defined as follows:

- *The sum is taken over all multisets $[\Gamma]$ of "good" Reeb orbits.*
- *$\#\overline{M}_Y(\gamma, \Gamma)$ is the algebraic count of rigid (i.e., virtual dimension 0) pseudo-holomorphic buildings in the symplectization $\hat{Y}$. These buildings have one positive puncture asymptotic to γ and a set of negative punctures asymptotic to the orbits in the multiset Γ.*
- *q^Γ denotes the product in $S(V)$ of the generators corresponding to the orbits in Γ.*
- *The denominator is a symmetry factor. If the multiset Γ consists of n_i copies of an orbit γ_i for $i = 1, \dots, k$, then $m_\Gamma = n_1!n_2! \cdots n_k!$. The term k_Γ is the product of the multiplicities of the orbits themselves.*

The crucial property that makes this construction a homology theory is as follows:

Proposition 2.30. *If the moduli spaces $\overline{M}_Y$ are cut out transversally, then $\partial^2 = 0$.*

Remark 2.31. *For a generic choice of the almost complex structure J, the moduli spaces $\overline{M}_Y$ are regular, meaning they are orbifolds of the expected dimension. This is the "lucky" case where transversality holds.*

*In practice, to construct a robust invariant that is independent of the choice of J, one must employ more advanced "virtual" techniques. The algebraic count $\#\overline{M}_Y$ is rigorously defined using either the **virtual fundamental cycle (VFC)** machinery, developed in this context by Pardon, or the theory of **semi-global Kuranishi structures**, developed by Bourgeois and Hofer. Both of these approaches yield a well-defined count and ensure that $\partial^2 = 0$ holds in general.*

The resulting homology is a powerful invariant of the contact structure, and its construction is functorial.

Theorem 2.32 (Bourgeois-Hofer; Pardon)**.** *The homology of the chain complex $(S(V), \partial)$, denoted $CH(Y)$, is an invariant of the contact structure (Y, ξ) and is independent of the auxiliary choices (e.g., the contact form α and almost complex structure J) used in its definition.*

Furthermore, the assignment $(Y, \xi) \mapsto CH(Y)$ defines a functor from the symplectic cobordism category to the category of $\mathbb{Z}/2$-graded algebras.

2.2.2 RSFT as BL_∞-Algebras

Let V be the graded vector space generated by "good" Reeb orbits. We can define a family of multi-linear operations $p^{k,l} : V^{\otimes k} \to V^{\otimes l}$ for $k \geqslant 0, l \geqslant 0$ with $k + l \geqslant 1$. These operations are defined by counting rigid (i.e., virtual dimension 0) pseudo-holomorphic curves of genus zero.

Let q_{Γ_+} be an element in $V^{\otimes k}$ corresponding to an ordered multiset of k Reeb orbits. The operation $p^{k,l}$ is defined by:

$$p^{k,l}(q_{\Gamma_+}) = \sum_{[\Gamma_-]} \frac{\#M_Y(0, \Gamma_+, \Gamma_-)}{m_{\Gamma_-} k_{\Gamma_-}} q^{\Gamma_-}$$

Here, the sum is over all possible ordered multisets $[\Gamma_-]$ of l Reeb orbits, and the coefficient is the algebraic count of rigid genus-zero curves with k positive punctures asymptotic to Γ_+ and l negative punctures asymptotic to Γ_-.

The collection of operations $\{p^{k,l}\}$ satisfies a set of quadratic relations, endowing the graded vector space V with the structure of a BL_∞-algebra.

Most importantly, the assignment $(Y, \xi) \mapsto (V, \{p^{k,l}\})$ is not a functor to the category of BL_∞-algebras. A functor to this category would require a symplectic cobordism between two contact manifolds to induce a strict homomorphism between their associated algebras. Curve counting in a cobordism does not, in general, satisfy this strong condition.

Instead, curve counting in a cobordism gives rise to a weaker, homotopy-theoretic map: a BL_∞-morphism. This means SFT provides a functor to the infinity-category of BL_∞-algebras, whose morphisms are the BL_∞-morphisms themselves.

2.2.3 Simpler Invariants

The full BL_∞-algebra is a complex object. For many applications, it is useful to extract simpler invariants from this structure.

Definition 2.33. *The* ***algebraic planar torsion*** *of a contact manifold* (Y, ξ)*, denoted* $APT(Y)$*, is an invariant derived from the algebraic structure* $(V, \{p^{k,l}\})$ *generated by genus-zero curves. Formally,*

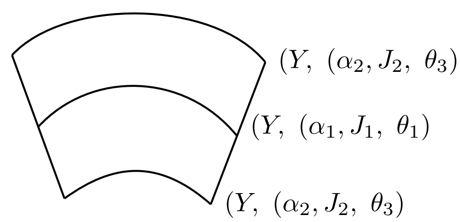
$$APT(Y) := T(V, \{p^{k,l}\}),$$

where T *is a specific algebraic construction that measures the "torsion" of the planar SFT algebra.*

This simpler invariant is well-behaved with respect to cobordisms.

Proposition 2.34. *The assignment* $(Y, \xi) \mapsto APT(Y)$ *defines a functor from the symplectic cobordism category to the partially ordered set* $(\mathbb{N} \cup \{\infty\}, \leqslant)$*.*

Proof. We have a proof by picture:



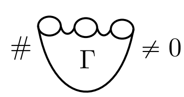
where the middle is the auxiliary data, where we stretch $c >> 0$. □

Proposition 2.35. *If* $APT(Y) = \infty$*, then* (Y, ξ) *does not admit a strong Liouville filling.*

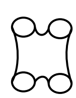
Exercise 2.36. *Show that* $APT(Y) = \infty$ *if and only if the unit element vanishes in the contact homology algebra, i.e.,* $1 = 0$ *in* $CH(Y)$*. This implies that the homology itself is trivial,* $CH(Y) = 0$*.*

Theorem 2.37 (Mei-Lin Yau, Bourgeois, etc.)**.** *The contact homology of any overtwisted contact manifold is trivial. That is, if* (Y, ξ) *is overtwisted, then* $CH(Y) = 0$*.*

We have



where we have $k+1$ punctures. In particular, we have no subset of Γ



This implies:

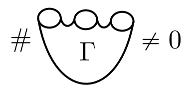
$$APT(Y) \leqslant k.$$

2.3 Lecture 3

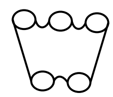
2.3.1 Examples

Recall the characterization of the invariant $APT(Y)$ for a contact manifold (Y, α): $APT(Y) \leqslant k$ if and only if there exists a set of Reeb orbits $\Gamma = \{r_1, \ldots, r_{k+1}\}$ satisfying a specific algebraic condition in RSFT, while no subset of Γ satisfies a related, simpler condition. These conditions are typically represented by diagrams corresponding to the moduli spaces of certain holomorphic curves.

Specifically, $APT(Y) \leqslant k$ is established by the existence of a set $\Gamma = \{r_1, \ldots, r_{k+1}\}$ for which the following configuration exists:



In particular,



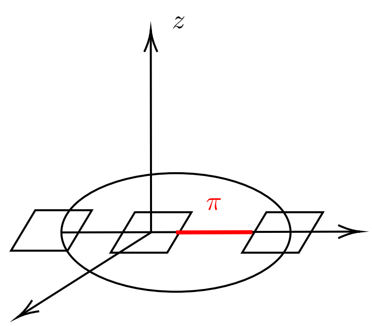
does not appear as a subset of Γ.

Theorem 2.38 (Mei-Lin Yau, Buk)**.** *For an overtwisted contact structure Y_{OT}, the cylindrical contact homology vanishes, $CH(Y_{OT}) = 0$. This vanishing corresponds to $APT(Y_{OT}) = 0$.*

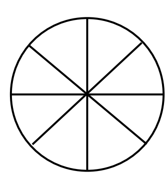
Theorem 2.39 (Latschev-Wendl)**.** *For any integer $k \geqslant 0$, there exists a contact 3-manifold Y_k such that $APT(Y_k) = k$.*

Example 2.40. *The canonical model for an overtwisted contact structure is given by the manifold $(\mathbb{R}^3, \alpha_{OT})$, where the contact form in cylindrical coordinates (r, θ, z) is $\alpha_{OT} = \cos r\, dz + r \sin r\, d\theta$*

The first diagram below provides a global picture of the contact planes $\xi = \ker(\alpha_{OT})$ in the ambient space. The planes rotate as one moves away from the z-axis, completing a full twist at $r = \pi$. The red line segment labeled π indicates the singular set of the projection of the contact planes onto the xy-plane.



The defining feature of this structure lies in the characteristic foliation on an embedded disk. Consider the disk D in the plane $\{z = 0\}$. The characteristic foliation $\mathcal{F}$ is the singular line field on D given by the intersection of the tangent space of the disk with the contact planes, $\mathcal{F} = TD \cap \xi$. Diagramatically, this possesses a unique singular point at the origin, around which the foliation spirals.



The boundary of the disk is a Legendrian curve. This embedded disk, endowed with this specific characteristic foliation, is the standard overtwisted disk.

The existence of such a disk is the defining property of an important class of contact manifolds.

Definition 2.41. *A contact 3-manifold (Y^3, ξ) is* ***overtwisted*** *if there exists an embedded disk $D \subset Y$ such that its boundary ∂D is Legendrian and the characteristic foliation $T\partial D \cap \xi$ has a unique singular point.*

Proposition 2.42 (Giroux)**.** *The characteristic foliation on a surface determines the contact structure in a neighborhood of the surface (the contact germ).*

Theorem 2.43 (Eliashberg)**.** *A contact 3-manifold (Y^3, ξ) contains an overtwisted disk if and only if it contains a Legendrian unknot with Thurston-Bennequin number $tb = 0$.*

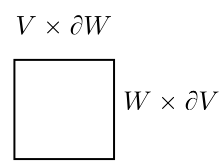
Now, we move on to discuss convex hypersurfaces, a special type of embedded surface.

Definition 2.44. *A hypersurface $\Sigma \subset Y^3$ is* ***convex*** *if there exists a contact vector field X that is transverse to Σ.*

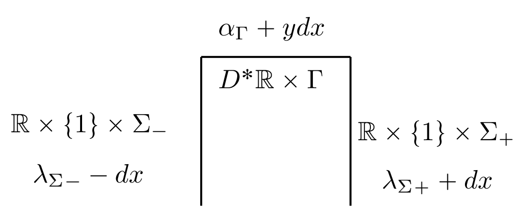
For a convex surface Σ, the contact form α can be chosen such that its Lie derivative $\mathcal{L}_X\alpha = 0$. This implies that Σ decomposes into regions $\Sigma_+ = \{p \in \Sigma \mid (d\alpha)|_p > 0\}$ and $\Sigma_- = \{p \in \Sigma \mid (d\alpha)|_p < 0\}$, separated by the dividing set $\Gamma = \{p \in \Sigma \mid (d\alpha)|_p = 0\}$. The regions Σ_+ and Σ_- can be viewed as Liouville fillings of the contact manifold Γ.

The local model near a convex hypersurface is determined by the Liouville flows on Σ_+ and Σ_-. A crucial result by Giroux connects the geometry of this decomposition to the property of being overtwisted.

A local model for the neighborhood of a convex hypersurface can be constructed from Liouville domains. Let (V, λ_V) and (W, λ_W) be two Liouville domains. Their symplectic product $(V \times W, \lambda_V \oplus \lambda_W)$ is also a Liouville domain. A neighborhood of the dividing set can be modeled on this product structure, as depicted below:

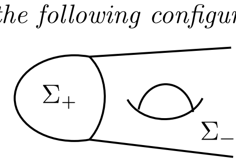


Consider the specific case where the dividing set Γ is diffeomorphic to S^1. A neighborhood of Γ inside the symplectization $\mathbb{R} \times Y$ can be described by the cotangent bundle of $\mathbb{R}$, $D^*\mathbb{R} \cong \mathbb{R}_x \times \mathbb{R}_y$, crossed with Γ. The contact form in this neighborhood is locally $\alpha_\Gamma + y\,dx$. The hypersurface itself is the slice at $x = 0$. The regions on either side are modeled by $\mathbb{R} \times \{1\} \times \Sigma_+$ and $\mathbb{R} \times \{-1\} \times \Sigma_-$, with corresponding 1-forms related to the Liouville forms on Σ_+ and Σ_-. Here is an diagram of a local model for the neighborhood of a convex hypersurface where the slice $x = 0$ corresponds to the hypersurface itself:

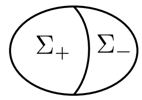


The geometry of the dividing set Γ and the characteristic foliation on Σ determine the contact structure in a neighborhood of the surface. A theorem by Giroux connects this local picture to the global property of being overtwisted.

Theorem 2.45 (Giroux)**.** *The contact germ of a convex hypersurface is overtwisted if and only if the characteristic foliation on the surface has the following configuration:*



rather than the following configuration:



This geometric criterion can be understood in terms of the dynamics of the Liouville flows on the regions Σ_+ and Σ_-. The overtwisted case corresponds to a situation where the dynamics on the two sides of the dividing set are opposites of one another.

Our goal is to demonstrate that if a contact manifold (Y, ξ) contains an overtwisted convex hypersurface, then its cylindrical contact homology vanishes, i.e., $CH(Y) = 0$. The strategy involves creating a specific Reeb orbit whose SFT differential is the unit element.

Goal: $CH(Y) = 0$. If Y contains an overtwisted convex hypersurface.

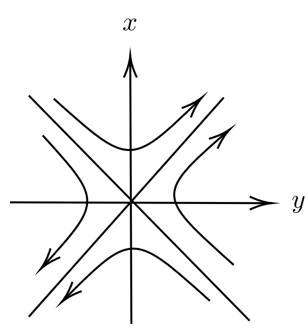
Step 1: Perturb the contact form to be non-degenerate. We begin by perturbing the contact form in a local neighborhood modeled on $S^1 \times \mathbb{R}^2$ with coordinates (t, x, y). The initial form is $\alpha = dt + y\,dx$. We introduce a new form $\alpha' = f(x, y)\,dt + y\,dx$, where $(f - 1)$ is C^2-small. Specifically, near the origin $(x, y) = (0, 0)$, we choose

$$f(x, y) = 1 + \epsilon(x^2 - y^2)$$

for some small $\epsilon > 0$. The Reeb vector field R_f for the form α' is parallel to $\partial_t - X_f$, where X_f is the Hamiltonian vector field of f with respect to the symplectic form $\omega = dx \wedge dy$. The Hamiltonian vector field X_f is defined by $i_{X_f}\omega = -df$. A direct calculation yields

$$-df = -2\epsilon x\,dx + 2\epsilon y\,dy \implies X_f = 2\epsilon y\,\partial_x + 2\epsilon x\,\partial_y.$$

The flow of this vector field describes a saddle point at the origin, as shown in the figure below:



This local perturbation creates a hyperbolic closed Reeb orbit $r = (0, 0) \times S^1$. The goal is to show that this orbit is the boundary of a pseudo-holomorphic plane.

Lemma 2.46. *For the generator q_r corresponding to the orbit r, we have $\partial(q_r) = 1$.*

Proof. We analyze the asymptotic operator at r to find a pseudo-holomorphic curve with one positive puncture at r and no negative punctures. The linearization of the Reeb flow in the contact planes normal to the orbit r is given by the Hessian of f. Identifying the tangent space with $\mathbb{C} = \mathbb{R}_x + i\mathbb{R}_y$, the asymptotic operator acts on sections of the trivial bundle over S^1 and has the form

$$A_r = -J_0 \left(\frac{d}{dt} - S(t) \right)$$

where $S(t)$ is the matrix of the linearized flow. In our case, this is simply the Hessian of f at the origin:

$$S = \nabla^2 f(0, 0) = \begin{pmatrix} 2\epsilon & 0 \\ 0 & -2\epsilon \end{pmatrix}.$$

The relevant operator governing the Fredholm theory is associated with the linearization of the Reeb flow, whose matrix has eigenvalues $\pm 2\epsilon$. For an appropriate choice of almost complex structure J, we can find a family of pseudo-holomorphic disks with a single positive puncture at r. The existence of such a disk is guaranteed by the local analysis of the linearized Cauchy-Riemann equations. The leading term in the asymptotics of such a curve $u(s,t) : (-\infty, 0] \times S^1 \to \mathbb{R} \times Y$ as $s \to -\infty$ will be of the form

$$u(s,t) \approx \gamma_r(t) + e^{\lambda s}\nu(t) + \text{higher order terms},$$

where $\lambda > 0$ is a positive eigenvalue of the asymptotic operator A_r and $\nu(t)$ is a corresponding eigenfunction. The existence of a rigid holomorphic plane establishes that $\partial(q_r) = 1$, which in turn forces the contact homology to be trivial, $CH(Y) = 0$. □

2.3.2 Detour to Intersection Theory

To justify the preceding claim, we briefly review the intersection theory of punctured pseudoholomorphic curves. Let U and V be two such curves in the symplectization $\mathbb{R} \times Y$. Let A_γ be the asymptotic operator at a Reeb orbit γ. Assume its eigenvalues $\cdots < a_{-2} < a_{-1} < 0 < a_1 < a_2 < \ldots$ are simple, with corresponding eigenfunctions η_i. The intersection number of U and V can be computed as:

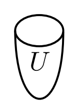
$$U \cdot V = \#(U \cap V) + I_{asy}(U, V)$$

where $\#(U \cap V)$ is the algebraic count of intersection points in the interior, and I_{asy} is an asymptotic contribution. At a positive puncture, this contribution is given by the difference of winding numbers of the asymptotic limits of U and V relative to the eigendirections of A_γ. For example,

$$I_{asy}(U, V) = \begin{cases} \text{wind}(\eta_{-1}) - \text{wind}(u - v) & \text{at positive punctures} \\ \text{wind}(u - v) - \text{wind}(\eta_1) & \text{at negative punctures} \end{cases}$$

where u, v are asymptotic markers for U, V. This framework allows one to show that certain moduli spaces are non-empty by demonstrating that intersection numbers must be negative, which is impossible for distinct pseudoholomorphic curves. By constructing appropriate foliations by curves (leaves), one can force intersections and establish the existence of connecting trajectories, thereby proving claims such as $\partial(\gamma) = 1$.

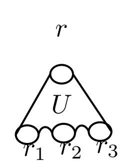
1. **Excluding Degenerations:** First, one must rule out certain degenerations or unexpected curves. Consider a pseudoholomorphic curve U with a single positive puncture, as depicted below.

   By constructing a local foliation of the ambient manifold by trivial cylinders, we can always find a "leaf" curve V whose asymptotic behavior relative to U is controlled. The algebraic intersection number $U \cdot V$ between such distinct curves is zero. This number decomposes into a sum of local intersection numbers and an asymptotic contribution: $U \cdot V = \#(U \cap V) + I_{asy}(U, V)$.

   With a careful choice of V, one can ensure that the asymptotic contribution is bounded, for instance $I_{asy}(U, V) > -1$. Since the local intersection count $\#(U \cap V)$ must be non-negative, the relation $0 = \#(U \cap V) + I_{asy}(U, V)$ leads to a contradiction.

2. Next, one must show that there are no urves that have a positive puncture at γ but also have one or more negative punctures. Consider a curve U with one positive puncture at γ and negative punctures at a set of Reeb orbits $\{r_i\}$.

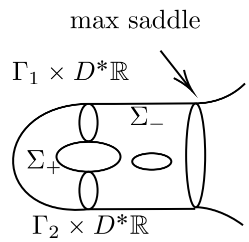
   

   Under suitable topological assumptions (e.g., that the linking numbers between γ and the r_i are zero), the intersection theory argument implies that U cannot be an solution. The condition $U \cdot V = 0$ forces U to be a leaf itself, meaning it does not contribute to the SFT differential.

Exercise 2.47. *Prove Giroux criterion, assuming that Γ^n represents the collection of Reeb orbits.*

2.3.3 Higher ADT

Generalizing the notion of overtwistedness to higher dimensions is difficult. A naive construction mimicking the 3-dimensional case often fails due to dimensional constraints. For instance, a configuration intended to produce a special holomorphic curve might have a virtual dimension of -1, meaning the corresponding moduli space is empty. As shown in the diagram below, a configuration involving a saddle point might be expected to yield a curve of dimension 0 only after including a sufficient number of additional punctures. For a surface with only one positive puncture, the virtual dimension may be -1, implying that the moduli space is empty for a generic choice of almost complex structure.



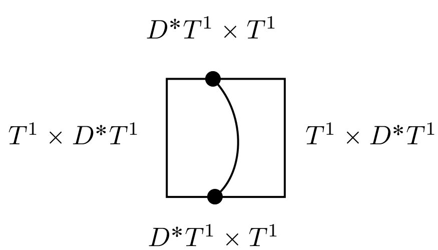
A higher-dimensional analogue of overtwistedness is provided by the concept of Giroux torsion.

Definition 2.48. *A contact manifold (Y^{2n+1}, ξ) is said to have **Giroux torsion** if it contains an embedded domain diffeomorphic to $[0,1]_z \times S^1_\theta \times S^{2n-2}$ with a contact form that is locally modeled by $\alpha = (\cos 2\pi z)\, d\theta + (\sin 2\pi z)\, \beta$. Here, β is a connection 1-form on a principal bundle over M.*

The existence of Giroux torsion imposes strong constraints on SFT-based invariants and has several implications for the existence of symplectic cobordisms. The following theorems establish some of these:

Theorem 2.49 (Latschev-Wendl)**.** *If a contact manifold Y contains Giroux torsion, then $APT(Y) \leqslant 1$.*

This can be visualized as follows, where D^* is the cotangent bundle:



Theorem 2.50 (Moreno-Zhou)**.** *If there exists a strong cobordism from (Y_-, ξ_-) to (Y_+, ξ_+), and if $APT(Y_+) < \infty$, then $APT(Y_-) < \infty$.*

Theorem 2.51 (Wendl)**.** *If a contact manifold Y has planar torsion, then Y admits a strong cobordism to an overtwisted contact manifold.*

3 Aliakbar Daemi: Atiyah-Floer conjecture

There were three lectures:

- Lecture 1: I will begin this talk by reviewing some foundational material in gauge theory, including connections on principal bundles, curvature of a connection and the action of the gauge group on connections. I will then explain how gauge theory on low dimensional manifolds provides a rich source of symplectic manifolds and Lagrangians in them. More specifically, flat connections on Riemann surfaces give rise to symplectic manifolds, and flat connections on 3-manifolds can be used to produce Lagrangians. Next, I will discuss how Lagrangian Floer homology of these Lagrangians can be used to define a 3-manifold invariant, called symplectic instanton homology.
- Lecture 2: Instanton Floer homology is a topological invariant of 3-manifolds, which is obtained by applying methods of Morse homology to the Chern—Simons functional. This invariant, along with its variations for knots and links, has recently found many interesting applications in low dimensional topology. In this talk, I will review the definition of instanton homology and various algebraic structures on this invariant, which will be useful for the third talk.
- Lecture 3: Atiyah-Floer conjecture predicts a connection between gauge theory and symplectic topology. Morse specifically, it proposes that instanton Floer homology and symplectic instanton homology are isomorphic invariants of 3-manifolds. In this talk, I will review the proof of the Atiyah-Floer conjecture for admissible bundles. In particular, this shows that framed instanton homology (introduced by Floer, Kronheimer and Mrowka) and its symplectic variant (defined by Wehrheim and Woodward) are isomorphic to each other. The key geometric ingredient in the proof is the mixed equation, relating ASD equation for connections and holomorphic curve equation.

3.1 Lecture 1

3.1.1 Connections, Curvature, and Gauge Groups

Throughout, let X be a smooth manifold and G be a Lie group. Our primary example of interest will be the special orthogonal group $G = SO(3)$. The fundamental geometric object is a principal G-bundle.

Definition 3.1 (Principal Bundle)**.** *A **principal** G**-bundle** over X is a smooth manifold P equipped with a smooth right action of G, such that X is the quotient space P/G and the action is free. We represent this structure as:*

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow \pi \\ & & X \end{array}$$

Definition 3.2 (Gauge Group)**.** *The **gauge group of** P, denoted $G(P)$, is the group of fiber-preserving diffeomorphisms $F : P \to P$ that are G-equivariant. That is, $\pi \circ F = \pi$ and $F(p \cdot g) = F(p) \cdot g$ for all $p \in P, g \in G$. This is equivalent to the set of maps $u : P \to G$ satisfying $u(p \cdot g) = g^{-1}u(p)g$, where $F(p) = p \cdot u(p)$.*

To understand the structure of the gauge group and related objects, we use the construction of an associated bundle.

Definition 3.3 (Associated Bundle)**.** *Let $P \to X$ be a principal G-bundle and let φ be a left action of G on a manifold F. The **associated bundle**, denoted $P \times_\varphi F$, is the quotient of $P \times F$ by the equivalence relation $(p \cdot g, f) \sim (p, \varphi(g)f)$. The projection $\pi_\varphi([p, f]) = \pi(p)$ makes this a fiber bundle over X with fiber F.*

Example 3.4 (Adjoint Action)**.** *Consider the adjoint action of G on itself, $Ad_g(h) = ghg^{-1}$.*

Exercise 3.5. *Following the construction, check that the gauge group $G(P)$ is isomorphic to the space of sections of the associated bundle $P \times_{Ad} G$.*

Example 3.6 (Determinant One Gauge Group)**.** *Consider $SO(3) = SU(2)/\{\pm I\}$. The group $SU(2)$ acts on itself via the Adjoint action. For a principal $SO(3)$-bundle P, the group $\tilde{G}(P) := \Gamma(X, P \times_{Ad} SU(2))$ is the* ***determinant one gauge group****.*

If the fiber is a vector space V on which G has a representation, then $P \times_\varphi V$ is a vector bundle.

Example 3.7. *The adjoint action of G on its Lie algebra $\mathfrak{g}$ gives the* ***adjoint bundle****, $ad(P) := P \times_{ad} \mathfrak{g}$, which is a vector bundle over X. For $G = SO(3)$, this corresponds to the standard representation of $SO(3)$ on $\mathfrak{so}(3) \cong \mathbb{R}^3$.*

Exercise 3.8. *Show that $G(P)$ (or $\tilde{G}(P)$) is an infinite-dimensional Lie group with Lie algebra $\Omega^0(X, ad(P))$, the space of sections of the adjoint bundle.*

A connection provides a notion of differentiation on the bundle. It can be defined as a splitting of the tangent bundle sequence:

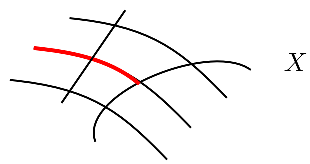
$$0 \to T_{\text{ver}}P \hookrightarrow TP \xrightarrow{\pi_*} TX \to 0$$

where $T_{\text{ver}}P = \ker(\pi_*)$ is the vertical tangent bundle.

Definition 3.9 (Connection)**.** *A* ***connection*** *is a G-equivariant splitting of this exact sequence. Equivalently, it is specified by a projection onto the vertical component. Let $\omega_A : TP \to T_{ver}P \cong P \times \mathfrak{g}$ be the projection map onto the vertical tangent space for a connection A. This map must satisfy:*

1. *For the inclusion $i : T_{ver}P \to TP$, we have $\omega_A \circ i = id$.*
2. *The kernel of ω_A, called the horizontal space, maps isomorphically to TX via π_*.*

Diagramatically:



The space of all connections on P is denoted $\mathcal{A}(P)$.

Exercise 3.10. *Show that $\mathcal{A}(P)$ is an affine space modeled on $\Omega^1(X, ad(P))$. That is, for a fixed connection $A_0 \in \mathcal{A}(P)$ and any $a \in \Omega^1(X, ad(P))$, $A_0 + a$ is another connection. The connection forms are related by $\omega_{A_0+a}(v) = \omega_{A_0}(v) + a(\pi_*(v))$ for $v \in TP$.*

- *For $u \in G(P)$ and $A \in \mathcal{A}(P)$, the action is given by the pullback u^*A.*
- *The* ***curvature*** *of A, denoted $F_A \in \Omega^2(X, ad(P))$, is the obstruction to integrability of the horizontal distribution. It is defined by $F_A(\eta, \eta') := \omega_A([\tilde{\eta}, \tilde{\eta}'])$, where $\tilde{\eta}, \tilde{\eta}'$ are the horizontal lifts of vector fields η, η' from X.*

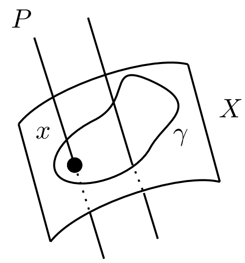
In a local trivialization of the bundle, a connection A can be written as the exterior derivative plus a $\mathfrak{g}$-valued 1-form, $A = d + a$, where $a \in \Omega^1(U) \otimes \mathfrak{g}$.

Example 3.11. *In a local trivialization, the curvature and the action of a gauge transformation u are given by:*

$$F_A = da + \frac{1}{2}[a, a] \quad \textit{and} \quad u^*A = d + u^{-1}au + u^{-1}du$$

Definition 3.12. *A connection A is* ***flat*** *if its curvature vanishes, $F_A = 0$.*

A flat connection allows for path-independent parallel transport. Given a path $\gamma : [0, 1] \to X$, the connection A defines a unique horizontal lift $\tilde{\gamma}$ starting at any point p_0 in the fiber over $\gamma(0)$. The endpoint $\tilde{\gamma}(1)$ will be in the fiber over $\gamma(1)$. If γ is a closed loop, the map from p_0 to $\tilde{\gamma}(1)$ defines an element of G, known as the holonomy of the connection along γ.



The flatness condition $F_A = 0$ implies that the holonomy depends only on the homotopy class of the loop. This gives rise to a homomorphism from the fundamental group of X into G.

Theorem 3.13. *The moduli space of flat connections on P is in one-to-one correspondence with the character variety of X:*

$$\{A \in \mathcal{A}(P) \mid F_A = 0\}/G(P) \cong Hom(\pi_1(X), G)^{\rho}/\sim$$

where $\sim$ is the adjoint action of G. A representation $\rho \in Hom(\pi_1(X), G)$ gives rise to a flat bundle $P_\rho = (\tilde{X} \times G)/\pi_1(X)$, where the action is $(\tilde{x}, g) \cdot \gamma = (\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} g)$ for $\gamma \in \pi_1(X)$.

3.1.2 Moduli Space of Flat Connections on a Riemann Surface

Let Σ be a Riemann surface of genus g, and let $G = SO(3)$. Principal $SO(3)$-bundles over Σ are classified by the second Stiefel-Whitney class $w_2(P) \in H^2(\Sigma; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Let P_i be a bundle with $w_2(P_i) = i$.

Definition 3.14 (Even Character Variety)**.** *The moduli space for the trivial bundle P_0 (with $w_2(P_0) = 0$) is the* ***even character variety****:*

$$M_{even}(\Sigma) = \{A \in \mathcal{A}(P_0) \mid F_A = 0\}/G(P_0) = \{\varphi : \pi_1(\Sigma) \to SU(2)\}/SU(2).$$

Definition 3.15 (Odd Character Variety)**.** *The moduli space for the non-trivial bundle P_1 (with $w_2(P_1) = 1$) is the* ***odd character variety****. It is described using representations of the fundamental group of the punctured surface, $\Sigma' = \Sigma \backslash \{pt\}$:*

$$\begin{aligned} M_{odd}(\Sigma) &= \{A \in \mathcal{A}(P_1) \mid F_A = 0\}/\tilde{G}(P_1) \\ &= \{\varphi : \pi_1(\Sigma') \to SU(2) \mid \varphi(\mu) = -I\}/SU(2) \\ &= \{(A_1, \ldots, B_g) \in SU(2)^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = -I\}/SU(2). \end{aligned}$$

where μ is a loop around the puncture.

Exercise 3.16. *Show that $M_{odd}(\Sigma)$ is a smooth manifold of dimension $6g - 6$. In contrast, $M_{even}(\Sigma)$ is singular.*

3.2 Lecture 2

3.2.1 More on Moduli Space of Flat Connections

Let $P \to \Sigma$ be a principal $SO(3)$-bundle. The space of all connections on P is denoted by $\mathcal{A}(P)$. A connection $A \in \mathcal{A}(P)$ is said to be flat if its curvature F_A vanishes. The group of gauge transformations is denoted by $\mathcal{G}(P)$, which acts on $\mathcal{A}(P)$. We are interested in the space of gauge equivalence classes of flat connections.

Definition 3.17. *The moduli space of flat connections on a principal $SO(3)$-bundle P over Σ is the quotient space*

$$M(\Sigma) = \{A \in \mathcal{A}(P) \mid F_A = 0\}/\tilde{\mathcal{G}}(P)$$

where $\tilde{\mathcal{G}}(P)$ is the group of based gauge transformations.

The bundle P is determined, up to isomorphism, by its second Stiefel-Whitney class $w_2(P) \in H^2(\Sigma; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Thus, for a given genus g surface, there are two distinct $SO(3)$-bundles to consider.

$$\begin{array}{ccc} SO(3) & \longrightarrow & P \\ & & \downarrow \\ & & \Sigma \end{array}$$

This diagram illustrates the structure of the principal bundle P over the surface Σ. The fibers of the bundle are copies of the structure group $SO(3)$.

The moduli space $M(\Sigma)$ can also be described in terms of group homomorphisms. Specifically, for a suitable choice of $SO(3)$-bundle, $M(\Sigma)$ is related to the space of homomorphisms from the fundamental group of a punctured surface to $SU(2)$, modulo conjugation. This leads to the following algebraic description:

$$M(\Sigma) = \{(A_1, \ldots, A_g, B_1, \ldots, B_g) \in SU(2)^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = -1\}/SU(2)$$

This space is a smooth manifold of dimension $6g - 6$.

Example 3.18. *When $g = 0$, Σ is a 2-sphere. The moduli space of flat connections on a sphere is empty, as there are no non-trivial flat bundles. Thus, $M(\Sigma) = \varnothing$.*

Exercise 3.19. *When $g = 1$, Σ is a torus. Show that $M(\Sigma)$ consists of a single point. Hint: Cnsider $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and $B_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.*

Example 3.20. *When $g = 2$, the moduli space $M(\Sigma)$ is the complete intersection of two quadrics.*

Remark 3.21. *The moduli space of flat connections is also a central object of study in algebraic geometry. In that context, it is referred to as the moduli space of stable holomorphic vector bundles of rank 2 and degree 1 with a fixed determinant. We won't study anything via this viewpoint, but it is still good to know.*

The tangent space to the moduli space at a point represented by a flat connection A can be characterized using cohomology. Let $ad(P_1)$ denote the adjoint bundle of the principal bundle. The tangent space is given by

$$T_A M(\Sigma) = \{\zeta \in \Omega^1(\Sigma, ad(P_1)) \mid d_A\zeta = 0\}/\{d_A\beta \mid \beta \in \Omega^0(\Sigma, ad(P_1))\}$$

Using Hodge theory, this is isomorphic to the space of harmonic 1-forms with values in the adjoint bundle:

$$T_A M(\Sigma) \cong \{\zeta \in \Omega^1(\Sigma, ad(P_1)) \mid d_A\zeta = 0, d_A^*\zeta = 0\}$$

The moduli space $M(\Sigma)$ is endowed with a natural symplectic structure. The symplectic form ω is defined on the tangent space by the following expression.

Definition 3.22. *Let $\zeta, \zeta' \in \Omega^1(\Sigma, ad(P_1))$. The symplectic form ω is given by*

$$\omega(\zeta, \zeta') = \int_\Sigma \langle \zeta \wedge \zeta' \rangle$$

where $\langle \cdot, \cdot \rangle$ is induced by the inner product on the Lie algebra $\mathfrak{su}(3) \cong \mathbb{R}^3$.

Exercise 3.23. *Show that the action of the group of based gauge transformations $\tilde{\mathcal{G}}(P_1)$ on the space of connections $\mathcal{A}(P_1)$ is Hamiltonian, with the curvature map $A \mapsto F_A$ serving as the moment map. Deduce that the moduli space is isomorphic to the space of connections quotiented by the full gauge group, i.e., $M(\Sigma) \cong \mathcal{A}(P_1)/\mathcal{G}(P_1)$.*

3.2.2 Moduli Space of Flat Connnections on 3-Manifolds

The study of moduli spaces on 3-manifolds is a powerful method for defining topological invariants. The central idea is that the moduli space of flat connections on a 3-manifold with boundary determines a Lagrangian submanifold within the symplectic moduli space of its boundary.

Let Y be a compact 3-manifold with boundary $\partial Y = \Sigma$. We need to fix an $SO(3)$-bundle on Y, denoted by $Q \to Y$.

$$\begin{array}{ccc} SO(3) & \longrightarrow & Q \\ & & \downarrow \\ & & Y \end{array}$$

Q is determined up to isomorphism by $w_2(Q) = PD([\gamma])$, where γ is a properly embedded 1-manifold in Y. The bundle Q is determined, up to isomorphism, by its second Stiefel-Whitney class $w_2(Q)$, which is the Poincaré dual of a properly embedded 1-manifold γ in Y. The diagram shows the structure of this principal bundle Q over the 3-manifold Y.

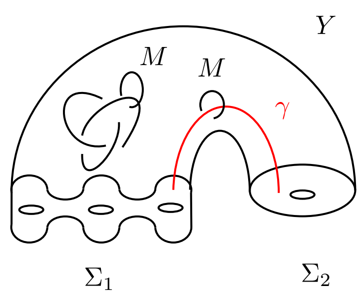
For a 3-manifold Y with a boundary, the moduli space of flat connections on Y is a subset of the moduli space on its boundary. Consider the case where the boundary consists of two components, $\partial Y = \Sigma_1 \sqcup \Sigma_2$. The moduli space of flat connections on Y is given by

$$L_{(Y,\gamma)} = \{A \in \mathcal{A}(Q) \mid F_A = 0\}/\tilde{\mathcal{G}}(Q)$$

This can also be described as the set of holonomies of flat connections on Y.

$$L_{(Y,\gamma)} = \{p : \pi_1(Y \backslash \gamma) \to SU(2) \mid P(\mu) = -I \text{ for any } M\}/SU(2)$$

The following diagram shows a 3-manifold Y with two boundaries Σ_1 and Σ_2, and a properly embedded 1-manifold γ shown in red. The moduli space $L_{(Y,\gamma)}$ associated with this setup is a Lagrangian submanifold of $\mathcal{M}(\Sigma_1) \times \mathcal{M}(\Sigma_2)$.



Theorem 3.24 (Herald)**.** *After a small perturbation, the moduli space $L_{(Y,\gamma)}$ is an immersed Lagrangian submanifold in $\mathcal{M}(\Sigma)$. This immersed Lagrangian is well-defined up to Lagrangian cobordisms.*

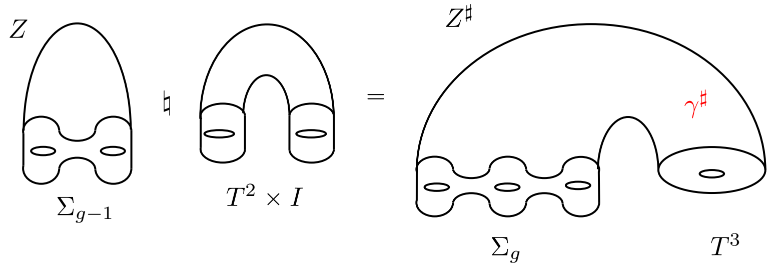
Let $L : L^{(Y,\gamma)} \hookrightarrow M(\Sigma)$ be the restriction to the boundary after a small perturbation. This embedding has two key properties:

- $L^*\omega = 0$, where ω is the symplectic form on $M(\Sigma)$.
- $\dim(L_{(Y,\gamma)}) = \dim(M(\Sigma))/2 = 3g - 3$.

Exercise 3.25. *Prove the first part of Herald's theorem. Hint: Use Stokes's Theorem.*

Let Z be a 3-manifold with boundary $\partial Z = \Sigma_{g-1}$. Consider a new 3-manifold $Z^{\#} = Z \natural T^2_X \times I$ formed by taking a connected sum with a thickened torus, where the connected sum is taken along a point. The associated 1-manifold is $\gamma^{\#} = \{pt\} \times I$. This construction yields a new Lagrangian submanifold $L^{\#}_Z := L_{(Z^{\#},\gamma^{\#})}$ embedded in $\mathcal{M}(\Sigma_g) \times \mathcal{M}(T^2) = \mathcal{M}(\Sigma_g)$. For instance, if $Z = H_{g-1}$ is a handlebody of genus $g-1$, then the associated Lagrangian is a product of spheres, $L^{\#}_{H_{g-1}} \cong (S^3)^{g-1}$, which is a Lagrangian submanifold of $\mathcal{M}(\Sigma_g)$.

The diagram below illustrates this handlebody connected sum. The leftmost figure represents the initial handlebody Z with boundary Σ_{g-1}. The middle figure shows the connected sum operation with a thickened torus $T^2 \times I$. The result, on the right, is a new handlebody $Z^{\#}$ whose boundary is a surface of genus g, denoted Σ_g, and contains a properly embedded 1-manifold $\gamma^{\#}$ (in red).



Exercise 3.26. *Prove that* $L_Z^{\#} \cong S^3 \times ... \times S^3$.

3.2.3 Symplectic Instanton Homology

Given two 3-manifolds (Y, γ) and (Y', γ') with the same boundary Σ, their associated moduli spaces $L_{(Y,\gamma)}$ and $L_{(Y',\gamma')}$ are embedded Lagrangians in $\mathcal{M}(\Sigma)$. We can define a Floer homology, $HF(L_{(Y,\gamma)}, L_{(Y',\gamma')})$, by studying the intersection points of these two Lagrangians.

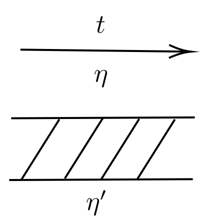
The homology is defined as the homology of a chain complex (C_*, d).

Definition 3.27. *The chain group* C_* *is a free abelian group generated by the intersection points of the two Lagrangians,* $L_{(Y,\gamma)} \cap L_{(Y',\gamma')}$.

The differential d is a map from C_* to C_* defined by counting specific solutions to a partial differential equation. For any two intersection points α_- and α_+, the coefficient of α_+ in $d\alpha_-$ is given by

$$\langle d\alpha_-, \alpha_+ \rangle = \#\{u : \mathbb{R} \times [0,1] \to \mathcal{M}(\Sigma) \mid u_\eta \subseteq L_{Y,\gamma},\ u_{\eta'} \subseteq L_{Y',\gamma'}, \lim_{t\to\infty} u(\cdot, s) = \alpha_\pm, \overline{\partial}_J u = 0\}/\mathbb{R}$$

The domain for the map u is the cylinder $\mathbb{R} \times [0,1]$. The term u_η denotes the map restricted to the boundary of the cylinder at a fixed time, while $\overline{\partial}_J u = 0$ is the Cauchy-Riemann equation for a J-holomorphic curve, where J is a chosen compatible almost complex structure on $\mathcal{M}(\Sigma)$. The quotient by $\mathbb{R}$ accounts for translational symmetry in the domain. The geometric setup for this is shown in the following figure:

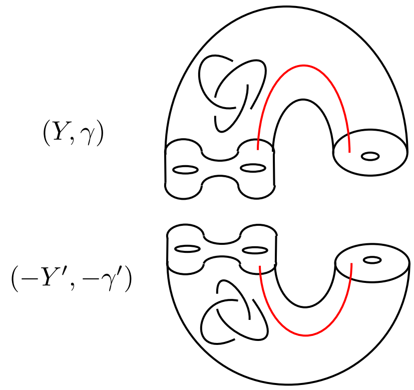


$\mathbb{R} \times [0,1]$

This diagram shows the domain of the maps u used to define the Floer differential. It is a cylinder with ends extending infinitely in either direction, representing a flow between two intersection points α_- and α_+. The flow lines are constrained to lie within the two Lagrangian submanifolds $L_{Y,\gamma}$ and $L_{Y',\gamma'}$ at the boundaries.

Theorem 3.28 (Daemi-Fukaya-Lipyankij)**.** *The map* d *satisfies* $d^2 = 0$*, so* (C_*, d) *is a chain complex. Its homology* $HF(L_{(Y,\gamma)}, L_{(Y',\gamma')})$ *is a topological invariant of the pair* (N, ω)*, where* N *is a closed 3-manifold and* ω *is a properly embedded 1-manifold.*

This result follows from gluing the two 3-manifolds along their boundaries to obtain a closed 3-manifold. The geometric gluing process is depicted below:



This figure illustrates the topological gluing of two 3-manifolds, (Y,γ) and $(-Y',-\gamma')$, where the minus sign denotes a reversal of orientation. They are glued along their common boundary Σ. The resulting object is a closed 3-manifold N which contains a closed 1-manifold ω.

The homology can then be identified as an invariant of the closed 3-manifold.

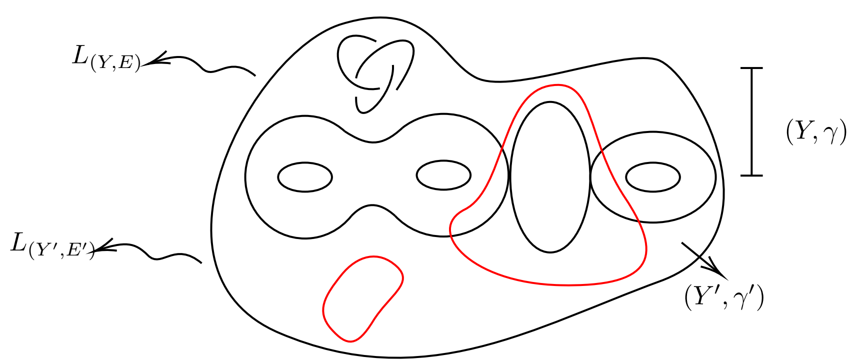
Theorem 3.29 (Wehrheim-Woodward)**.** *The symplectic framed instanton homology, denoted $SI^{\#}(N)$, is an invariant of the closed 3-manifold N with a properly embedded 1-manifold ω. It is defined by the Floer homology of the Lagrangians associated with a Heegaard splitting of N:*

$$SI^{\#}(N) = HF(L_H^{\#}, L_{H'}^{\#})$$

where $N = H \cup_{\Sigma_{g-1}} H'$ is a Heegaard splitting of genus $g-1$.

3.3 Lecture 3

The moduli space of flat connections on a closed 3-manifold N can be understood as the intersection of two Lagrangian submanifolds derived from a Heegaard splitting of N. Given a pair of 3-manifolds (Y,γ) and (Y',γ') with a common boundary Σ, the intersection points $L_{(Y,\gamma)} \cap L_{(Y',\gamma')}$ correspond to pairs of flat connections on the respective manifolds that agree on the boundary. Gluing the two 3-manifolds along their boundary, we obtain a closed 3-manifold $N = Y \cup_\Sigma Y'$. A flat connection on this closed manifold corresponds to an equivalence class of such pairs.



The diagram above illustrates how the intersection of two Lagrangians $L_{(Y,\gamma)}$ and $L_{(Y',\gamma')}$ corresponds to the gluing of the two 3-manifolds (Y,γ) and (Y',γ'), yielding a closed 3-manifold N. The intersection points of the Lagrangians represent the flat connections on this closed manifold.

The moduli space of flat connections on the closed 3-manifold N with a fixed $SO(3)$-bundle $Q \to N$ (determined by $w_2(Q) = PD(\omega)$) is denoted by $R(N,w)$. The expected dimension of this moduli space is zero. We will now provide another justification for this fact by considering the Chern-Simons functional.

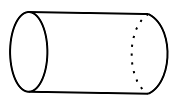
3.3.1 Chern-Simons Functional

The Chern-Simons functional provides a real-valued invariant of connections on a 3-manifold.

Definition 3.30. *Let $Q \to N$ be a principal $SO(3)$-bundle over a 3-manifold N. The **Chern-Simons functional** $CS : \mathcal{A}(Q) \to \mathbb{R}$ is defined by fixing a reference connection $A_0 \in \mathcal{A}(Q)$. For any other connection $A \in \mathcal{A}(Q)$, we choose an auxiliary connection $\mathbb{A}$ on the cylinder $[-1, 1] \times N$ such that $\mathbb{A}|_{\{-1\}\times N} = A$ and $\mathbb{A}|_{\{1\}\times N} = A_0$. Then, the functional is given by*

$$CS(A) = \frac{1}{8\pi^2} \int_{[-1,1]\times N} \langle F_{\mathbb{A}} \wedge F_{\mathbb{A}} \rangle$$

The diagram below visualizes the cylinder over the 3-manifold N which is used as the domain for the auxiliary connection $\mathbb{A}$.



The Chern-Simons functional has several key properties:

- The value of $CS(A)$ depends only on the connection A (and the reference connection A_0), not on the choice of the auxiliary connection $\mathbb{A}$.
- It is not gauge invariant in general. However, for a gauge transformation $u \in \tilde{\mathcal{G}}(P)$, the value changes by an integer, i.e., $CS(u^*A) \equiv CS(A) \pmod{Z}$.
- As a consequence, CS descends to a map $CS : \mathcal{A}(Q)/\hat{\mathcal{G}} \to \mathbb{R}/\mathbb{Z}$, where $\hat{\mathcal{G}}$ is the group of gauge transformations that are based, but not necessarily trivial at infinity.

Another important result is that the critical points of the Chern-Simons functional are precisely the flat connections on N. Thus, the set of critical points is isomorphic to the moduli space of flat connections, $Crit(CS) \cong R(N, w)$. This provides a functional-theoretic justification for the expected zero-dimensionality of the moduli space.

Definition 3.31. *A pair (N, w) is called **admissible** if there exists a properly embedded surface $S \hookrightarrow N$ such that the pullback of the second Stiefel-Whitney class, s^*w, is non-zero in $H_2(S, \partial S; \mathbb{Z}_2)$. This condition ensures that the moduli space is non-empty and well-behaved.*

Example 3.32. *The pair $(N, w) = (Y, \gamma) \cup (Y', \gamma')$, formed by gluing two 3-manifolds as described in the previously, is an admissible pair.*

Exercise 3.33. *Prove that the stabilizer of any flat connection $A \in R(N, w)$ under the action of the group of based gauge transformations $\tilde{\mathcal{G}}(Q)$ is trivial, consisting only of $\{\pm 1\}$.*

The instanton homology $I(N, w)$ is defined as the Morse homology of the Chern-Simons functional, $I(N, w) = H(C_*, d)$. The chain complex C_* is a free abelian group generated by the critical points of CS, which are the flat connections in $R(N, w)$. The differential d counts the number of unparameterized downward gradient flow lines of the functional, connecting critical points.

The gradient flow lines of CS are closely related to the anti-self-duality (ASD) equation. The flow line equation $\frac{dA(t)}{dt} = -\nabla CS$ is equivalent to the ASD equation, $F_A + *F_A = 0$, for a connection $\mathbb{A}$ on the cylinder $\mathbb{R} \times N$ with the product metric. This establishes a connection between the Morse theory of the Chern-Simons functional and solutions to a gauge-theoretic PDE.

3.3.2 Atiyah-Floer Conjecture for Admissible Bundles

The Atiyah-Floer conjecture is the central result that links the two homologies we have discussed. It states that the instanton homology and the symplectic instanton homology are isomorphic.

Theorem 3.34 (Daemi, Fukaya, Lipyanskiy)**.** *For any admissible pair (N, w), there is a natural isomorphism between the instanton homology and the symplectic instanton homology:*

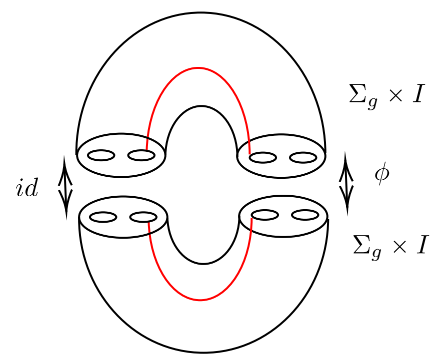
$$I_*(N, w) \cong SI_*(N, w)$$

This theorem provides a powerful statement that a homology theory defined by counting solutions to the anti-self-duality equations on a closed 3-manifold is isomorphic to a Floer homology theory defined by counting J-holomorphic curves in a symplectic moduli space on a surface.

Corollary 3.35 (Dostoglou, Salamon)**.** *Let $\phi : \Sigma_g \to \Sigma_g$ be a diffeomorphism, and let $M_\phi = \Sigma_g \times I/(x, 1) \sim (\phi(x), 0)$ be the mapping torus. The diffeomorphism induces a symplectomorphism $\phi_* : \mathcal{M}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$. The graph of this map, $\Gamma_{\phi_*} \subseteq -\mathcal{M}(\Sigma_g) \times \mathcal{M}(\Sigma_g)$, is a Lagrangian submanifold. The instanton homology of the mapping torus is isomorphic to the Floer homology of the diagonal and the graph of the induced map:*

$$I_*(M_\phi, w_\phi) \cong HF(\Delta, \Gamma_{\phi_*})$$

The diagram below shows the mapping torus construction. The two boundary components of the cylinder $\Sigma_g \times I$ are identified by the diffeomorphism ϕ. This gluing process creates a closed 3-manifold M_ϕ and is the context for the above corollary.



Corollary 3.36. *There is a natural isomorphism between framed instanton homology and symplectic framed instanton homology:*

$$SI^\#(N) \cong I^\#(N) := I(N\#T^3, w = \{pt\} \times S^1)$$

where the first term is the symplectic framed instanton homology, and the second is the framed instanton homology as defined by Kronheimer, Mrowka, and Floer.

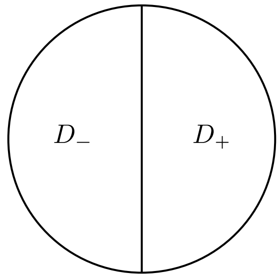
Exercise 3.37. *Conclude the previous corollary from the main Theorem by considering the specific case of the connected sum with a 3-torus.*

The proof of the main theorem is quite complicated and we do not have enough time to explain it in depth. However, there are two main approaches:

1. The adiabatic approach, which involves analyzing the limiting behavior of the ASD equation as a parameter is sent to infinity.
2. The functorial approach, which constructs an explicit chain map $\Phi : (C_*, d) \to (C'_*, d')$ that is a quasi-isomorphism, proving that the two homology theories are isomorphic. The map Φ is defined using solutions to a "mixed equation" that combines features of both the ASD and Cauchy-Riemann equations.

The differential d is defined by counting solutions to the ASD equation, while the differential d' is defined by counting solutions to the Cauchy-Riemann equation. The map Φ is constructed by counting solutions to a hybrid equation defined on a domain that interpolates between the two.

Consider the following:



The diagram shows a disk split into two semicircles, D_+ and D_-. This provides the domain for the mixed equation, allowing for a connection on one half and a map to the moduli space on the other.

The mixed equation for a pair (A, u) consists of a connection A on $D_- \times \Sigma$ and a map $u : D_+ \to \mathcal{M}(\Sigma)$, satisfying the following conditions:

- A is a solution to the ASD equation, $F_A + *F_A = 0$, on $D_- \times \Sigma$.
- u is a J-holomorphic curve, satisfying the Cauchy-Riemann equation $\overline{\partial} u = 0$, on D_+.
- On the shared boundary between D_+ and D_-, the two solutions are required to match: the holonomy of the connection A must equal the value of the map u in the moduli space, i.e., $A_q = A|_{\{q\}\times\Sigma}$ is flat, and its gauge equivalence class $[A_q]$ must be equal to $u(q)$ for all q on the boundary.

RESEARCH TALKS

There were 9 research talks, each one hour long.

1. An Invitation to Seidel/Shift Operators by Eduardo Gonzalez (UMass Boston)
2. A Symplectic Look at Contractible Affine Surfaces of Log Kodaira Dimension One by Yin Li (Uppsala)
3. The Mapping Class Group Action on the Odd Character Variety is Faithful by Aliakbar Daemi (Washington University in St. Louis)
4. Immersed Exact Lagrangian Fillings and Augmentations to Arbitrary Fields by Zijun Li (Duke)
5. Sectorial Decompositions of Symmetric Products of Surfaces and Homological Mirror Symmetry by Xinle Dai (Harvard)
6. Towards the HZ- and Multiplicity Conjectures for Dynamically Convex Reeb Flows by Basak Gurel (University of Central Florida)
7. On Wrapped Floer Homology Barcode Entropy and Hyperbolic Sets Restricted to the Hyperbolic Set by Rafeal Fernandez (UC Santa Cruz)
8. Symplectic vs. Algebraic Log Maps by Mohammad Farajzadeh Tehrani (University of Iowa)
9. Manin Configurations of Lagrangians in del Pezzos by Chris Woodward (Rutgers)

I have scribed notes for all four of the chalkboard talks. The remaining talks were given via beamer slides, which I do not have any notes for.

4 Eduardo Gonzalez: An Invitation to Seidel/Shift Operators

Abstract: We will review several applications of Seidel/shift operators in quantum cohomology, including recent joint work with D. Pomerleano and C. Y. Mak on its relation to Coulomb branches.

4.1 Introduction to Quantum Cohomology

Let (M, ω) be a symplectic manifold. We consider a specific case where the symplectic form ω is a multiple of the first Chern class, i.e., $\omega = \lambda c_1$ for some constant $\lambda > 0$. In this context, the quantum cohomology ring, denoted $QH(M)$, can be described as a tensor product $H(M) \otimes \Lambda$, where $H(M)$ is the ordinary cohomology ring of M and $\Lambda = k[q, q^{-1}]$ is a Novikov ring with formal variable q.

The quantum cohomology ring is a deformation of the ordinary cup product. This deformed product, often denoted by $\times$, is defined via Gromov-Witten invariants. For cohomology classes $a, b, c \in H(M)$, the structure coefficients of the deformed product are given by

$$(a \times b, c) = \sum_{d \in H_2(M)} \langle a, b, c \rangle_{0,3} q^d,$$

where $\langle a, b, c \rangle_{0,3}$ is the three-point Gromov-Witten invariant counting the number of genus-0 curves in a given homology class d that pass through representatives of the cohomology classes a, b, and c.

Example 4.1. *Consider the complex projective space $\mathbb{P}^n$. Its ordinary cohomology ring is $H(\mathbb{P}^n) = \frac{k[p]}{p^{n+1}}$, where p is a generator of $H^2(\mathbb{P}^n)$. The quantum multiplication of powers of p is given by*

$$(p^l \times p^m, p^k) = \begin{cases} q^0 & \text{if } l + m + k = n \\ q^n & \text{if } l + n + k = 2n + 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, the integer n corresponds to the first Chern class. The term $\langle a, b, c \rangle_{0,3,d}$ is the number of curves in a class d passing through a, b, c.

4.2 Toric Varieties

Quantum cohomology can be applied to toric varieties, which can be expressed as a symplectic quotient $M = \mathbb{C}^n // T^Y$. In 1995, Seidel introduced an operator S_γ associated with a loop $\gamma : S^1 \to \mathrm{Aut}(M)$ in the group of symplectomorphisms of M. This operator acts on the quantum cohomology ring $QH(M)$ and is defined via a count of holomorphic disks. The operator is defined using a pairing $(S_\gamma(a), b) = \sum_{d \in H_2(M)} \langle a, b \rangle_{0,2,\sigma+d} q^d$ and is invertible on $QH(M)$. The relationship between loops in the automorphism group of M and quantum cohomology is captured by a homomorphism from $\pi_1(\mathrm{Aut}(M))$ to the group of units in the quantum cohomology ring, $QH(M)^*$.

Theorem 4.2 (McDuff-Tolman)**.** *For certain symplectic manifolds, the Seidel operator has a leading term given by the homology class of the maximal fixed point component.*

$$S_\gamma(a) = [F_{max}] + \textit{lower order terms},$$

where $F_{max} \subset M^{v_i}$ is the maximal fixed point set.

A key challenge is that the lower order terms are not well-understood.

Theorem 4.3 (Batyrev)**.** *For a toric variety X, the quantum cohomology ring $QH(X)$ can be expressed as an algebraic structure.*

$$QH(X) = Bat\left(\frac{\Lambda[w_1, \ldots, w_N]}{\textit{additive relations}} \right),$$

This algebra is subject to multiplicative relations of the form $\prod_{\langle D, d \rangle > 0} q^d \prod_{\langle D_i, d \rangle \leqslant 0} w_i^{-\langle D_i, d_i \rangle}$.

Theorem 4.4. *The operators S_i satisfy the multiplicative relations but do not satisfy the additive relations.*

Letting $W_i = S_i$ only works for Fano varieties where the anticanonical class $-K_M$ is positive. For cases where $-K_M \geqslant 0$, we have an additional curve correction term. This is similar to the approach taken by Giventhal using mirror maps. This raises the question of whether S_i is equal to the mirror map itself.

Example 4.5. *Consider the projective line $\mathbb{P}^1$. Its Jacobian ring is given by $Jac(\mathbb{P}^1) = \frac{\langle q,z\rangle}{z^2=q}$. This relation is derived from the mirror map $w = z + q/z$ by setting $z\partial_z = 0$, which implies $z - q/z = 0$, or $z^2 = q$.*

4.3 Equivariant Quantum Cohomology and Coulomb Branches

The theory can be extended to an equivariant setting by considering a Hamiltonian action of a torus T on a symplectic manifold M. This leads to the definition of equivariant quantum cohomology $QH_T(M)$. The Seidel operator S_γ can be generalized to this setting, acting on the equivariant quantum cohomology ring $S_\gamma : QH_{\hat{T}}(M) \to QH_{\hat{T}}(M)$, where $\hat{T} = U_1 \times T$.

A result by Iritani and Liebenschutz-Jones proves that, under very good conditions, the Jacobian ring of the equivariant mirror potential, $Jac(W^{eq})$, is isomorphic to the equivariant quantum cohomology ring, $QH_T(M)$.

Example 4.6. *Consider a $U(1)$ action on $\mathbb{P}^1$. If $U(1)$ acts on $\mathbb{C}^2$ with weight $(0,1)$, then the equivariant cohomology of $\mathbb{P}^1$ is $H_{U(1)}(\mathbb{P}^1) = \frac{k[p,u]}{p(p+u)}$. This corresponds to the relation $p(p+u) = 0$. The equivariant potential is related to the non-equivariant one by $W^{eq} = W^{Hv} + u \log z$. Substituting the non-equivariant potential for $\mathbb{P}^1$, we get $W^{eq} = z + q/z + u \log z$. Setting the derivative with respect to z to zero, we obtain $z^2 - q + uz = 0$, which simplifies to $z(z+u) = q$.*

At his ICM lectures, Teleman proposed a connection between the quantum cohomology of a manifold and a Coulomb branch of a gauge theory: If a group G acts on a manifold M, then there exists a Lagrangian with certain categorical aspects that lives inside a Coulomb branch $M(G^\vee, 0)$.

Coulomb branches for specific groups are known.

1. $M(T, 0) = T^*T_{\mathbb{C}}$.
2. $(T^*G_{\mathbb{C}}^{\text{reg}} = G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}^{\text{reg}})//\mathbb{G}_{\mathbb{C}}$.
3. $M(G^\vee, 0) = \text{Spec}(H_*^G(\Omega(g)))$.

For $U(1)$, its Coulomb branch is $\mathbb{C} \times \mathbb{C}^\times$. A result by Mak-Pomerleano demonstrates a connection between the equivariant quantum cohomology and the homology of the based loop group. Specifically, there is an action of $H_*^{\hat{T}}(LG/T)$ on $QH_{\hat{T}}(M)$, where LG/T is the flat affine variety. This action allows the spectrum of the equivariant quantum cohomology to be viewed as a sheaf over the spectrum of the equivariant homology of the based loop group. The action is given by a convolution product, where the homology of the loop group is decomposed as $H_*^{\hat{T}}(LG/T) = \bigoplus R[S_{\sigma w}^*]$ based on a decomposition of $LG/T = \bigcup_{\sigma W} S_{\sigma W}$. This gives a module structure via the action $S_{\sigma_1,w_1}^F \circ S_{\sigma_2,w_2}^F = S_{(\sigma_1,w_1)\circ(\sigma_2,w_2)}^F$ on fixed points.

This theory provides a way to construct the Coulomb branch $M(U(1), V \oplus V^\vee)$ via two charts whose gluing is determined by the Seidel operators. This construction yields the space $\mathbb{C}^2 \backslash \{(0,0)\}$. This framework can also be developed in a K-theory setting. The concepts have natural applications in other fields, such as symplectic reduction, where $T^*(G/H)$ can be obtained by studying $T^*//H$. In this context, $QH(X//U(1))$ can be recovered from $QH_{U(1)}(X)$ by setting the Seidel operator to the identity.

5 Aliakbar Daemi: The Mapping Class Group Action on the Odd Character Variety is Faithful

Abstract: The odd character variety of a Riemann surface is a moduli space of $SO(3)$ representations of the fundamental group which can be interpreted as the moduli space of stable holomorphic rank 2 bundles of odd degree and fixed determinant. This is a symplectic manifold, and there is a homomorphism from (a finite extension of) the mapping class group of the surface to the symplectic mapping class group of this moduli space. In this talk, I will discuss a result establishing that this homomorphism is injective when the genus is at least 2. This answers a question posed by Dostoglou and Salamon and generalizes a theorem of Smith from the genus 2 case to arbitrary genus. Our appraoch also yields a result on the faithfulness of the action on the Fukaya category of the odd character variety. The proofs use instanton Floer homology, a version of the Atiyah-Floer Conjecture, and aspects of a strategy used by Clarkson in the Heegaard Floer setting.

5.1 The Odd Character Variety

Let Σ be a Riemann surface of genus g. We recall the definition of the odd character variety, denoted M_g or $\mathcal{M}(\Sigma)$, as the space of certain $SU(2)$-representations of the fundamental group of the punctured surface $\pi_1(\Sigma_g \backslash \{\text{pt}\})$. Specifically, we have

$$M_g = \mathcal{M}(\Sigma) = \{p : \pi_1(\Sigma_g \backslash \{\text{pt}\}) \to SU(2) \mid p(\mu) = -I\}/SU(2)$$

where the relation is conjugation, and μ is any meridian around the puncture.

This variety has several equivalent descriptions. It can also be seen as the set of connections A on a principal $SO(3)$-bundle P over Σ_g with a non-trivial second Stiefel-Whitney class, $w_2(P) \neq 0$, all modded out by gauge transformations.

$$\begin{array}{ccc} SO(3) & \longrightarrow & P \\ & & \downarrow \\ & & \Sigma_g \end{array}$$

Furthermore, M_g is equivalent to the moduli space of holomorphic stable bundles of rank 2 and degree 1 with a fixed determinant. The space M_g is known to be a smooth symplectic manifold of dimension $6g - 6$ with trivial fundamental group, i.e., $\pi_1(M_g) = 0$.

Example 5.1. *For $g = 1$, the odd character variety is a single point, $M_g = \{pt\}$. For $g = 2$, M_g is the complete intersection of two quadrics in $\mathbb{P}^1$.*

An important object of study is the mapping class group of the surface Σ_g. A diffeomorphism $\Phi : (\Sigma_g, p) \to (\Sigma_g, p)$ naturally induces a diffeomorphism $\Phi^* : M_g \to M_g$ via pullback. A basic property is that Φ^* depends only on the isotopy class of Φ and, importantly, Φ^* preserves the symplectic structure of M_g.

Definition 5.2. *Let $MCG(\Sigma_g, p)$ be the group of diffeomorphisms of (Σ_g, p) up to isotopy. The group $\pi_0 Symp(M_g)$ consists of the symplectomorphisms of M_g considered up to symplectic isotopy.*

The induced action described above gives a map $\rho : MCG(\Sigma_g, p) \to \pi_0\text{Symp}(M_g)$. This map fits into a Birman exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\Sigma_g) & \xrightarrow{\text{push map}} & MCG(\Sigma_g, p) & \longrightarrow & MCG(\Sigma_g) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow j & & & & \\ 0 & \longrightarrow & H^1(\Sigma_g; \mathbb{Z}/2) & \longrightarrow & \hat{\Gamma}_g & \longrightarrow & MCG(\Sigma_g) & \longrightarrow & 0 \end{array}$$

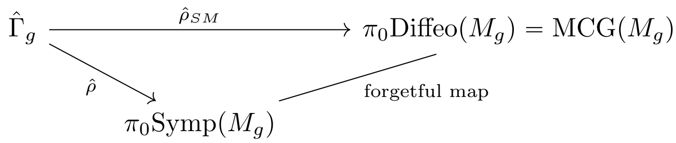
The maps in this diagram are related as follows: The map from $MCG(\Sigma_g, p)$ to $MCG(\Sigma_g)$ is the forgetful map that forgets the puncture. The map from $\pi_1(\Sigma_g)$ to $MCG(\Sigma_g, p)$ is the push map. The group $\hat{\Gamma}_g$ is an extension of $MCG(\Sigma_g)$ by $H^1(\Sigma_g; \mathbb{Z}/2)$. The action on the coordinates of the character variety is given by $p(\gamma)((A_i, B_i)_{i=1}^g) = (-1)^{\gamma A_i} A_i, ..., (-1)^{\sigma B_i} B_i$. This leads to a map $\hat{\rho} : \hat{\Gamma}_g \to \pi_0\text{Symp}(M_g)$ where $\rho = \hat{\rho} \circ j$.

Problem 5.3 (Dostoghan-Salamon, 1993)**.** *For $g \geqslant 2$, is the map $\hat{\rho}$ injective?*

Theorem 5.4 (Smith, 2012)**.** *The answer is yes for $g = 2$.*

Theorem 5.5 (Daemi-Scaduto)**.** *The answer is yes for any g.*

The action of $\hat{\Gamma}_g$ on M_g can be viewed in two ways: via its symplectic action and its smooth action. This leads to the following diagram:



There is also a related result about the kernel of the smooth action:

Theorem 5.6 (Daemi-Scaduto)**.** *For $g = 2$, the kernel of the smooth action,* $\ker(\hat{\rho}_{smooth})$*, is not finitely generated.*

The proof of this theorem relies on the work of Kreck-Su which provides tools to study $MCG(M_g)$ for manifolds with $\pi_1 = 0$ and a specific dimension and cohomology ring.

5.2 Lagrangian Floer Theory and the Proof Strategy

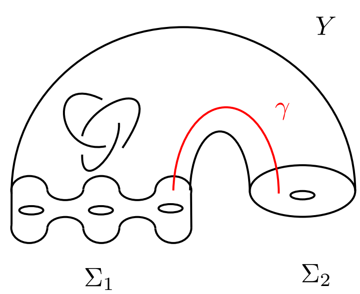
To prove the main theorem concerning the injectivity of $\hat{\rho}$, the strategy is to use Lagrangian Floer theory. The core idea is to show that any non-trivial element $\phi \in \hat{\Gamma}_g \backslash \{0\}$ acts non-trivially on the space of Lagrangians in M_g.

The strategy can be summarized as follows: it is sufficient to show that for any $\phi \in \hat{\Gamma}_g \backslash \{0\}$, there exist Lagrangians $L, L' \in M_g$ such that their Lagrangian Floer cohomology groups are not isomorphic after applying the action of ϕ.

$$\text{HF}(L, L') \ncong \text{HF}(L, \rho(\phi)(L'))$$

This result implies that $\rho(\phi)$ is not Hamiltonian isotopic to the identity, which in turn proves injectivity.

To do this, we need to find a source of Lagrangians in M_g. These Lagrangians are constructed from three-manifolds with boundary. Consider a three-manifold Y with a boundary that is a surface Σ, and a knot γ inside it.



In this diagram, we see a 3-manifold Y with a boundary composed of two surfaces, Σ_1 and Σ_2. A knot γ is shown passing through the manifold, intersecting the boundary at a point on each surface.

The Lagrangian $L_{(Y,\gamma)}$ is defined as the moduli space of $SU(2)$-representations of the fundamental group of the three-manifold complement, $Y \backslash \gamma$, with a specific condition on the meridian μ of the knot γ:

$$L_{(Y,\gamma)} = \{\rho : \pi_1(Y \backslash \gamma) \to SU(2) \mid \rho(\mu) = -I \text{ for any meridian } \mu \text{ of } \gamma\}/\text{conj}.$$

By restricting these representations to the boundary surfaces, we obtain a map $L : L_{(Y,\gamma)} \to M_g \times M_{g'}$.

Theorem 5.7 (Herald)**.** *The space $L_{(Y,\gamma)}$ is an immersed Lagrangian submanifold in the product manifold $M_g \times M_{g'}$, possibly after a perturbation.*

Theorem 5.8 (Daemi, Fukaya, Lipyanskiy)**.** *The Lagrangian Floer cohomology* $\mathrm{HF}(L, L')$ *of any two embedded Lagrangians* L *and* L' *is a well-defined invariant.*

Let us define a subgroup $G \subseteq \hat{\Gamma}_g$.

$$G = \{\varphi \in \hat{\Gamma}_g \mid \text{ for any pair of embedded 3-manifold Lagrangians } L, L', \mathrm{HF}(L, L') \cong \mathrm{HF}(L, \hat{\rho}(\varphi)(L'))\}.$$

The main result is a classification of this subgroup G.

Theorem 5.9. *For* $g \geqslant 3$*, the subgroup* G *is trivial, i.e.,* $G = 1$*. For* $g = 2$*,* G *is a subgroup of* $\{1, \tau\}$*, where* τ *is the hyperelliptic involution on* Σ_g*.*

Corollary 5.10. *The action of* $\hat{\Gamma}_g$ *on the Fukaya category Fuk*(M_g) *is non-trivial for* $g \geqslant 3$*.*

5.3 Proof of Main Theorem

The proof proceeds by establishing several key properties of the subgroup G:

1. G is a normal subgroup of $\hat{\Gamma}_g$.
2. The intersection of G with the normal subgroup $H^1(\Sigma, \mathbb{Z}/2) \subseteq \hat{\Gamma}_g$ is trivial, i.e., $G \cap H^1(\Sigma, \mathbb{Z}/2) = 0$.
3. We relate the group G to the Torelli subgroup of the mapping class group. The Torelli subgroup $I(\Sigma_g) \subseteq MCG(\Sigma_g)$ is defined as the subgroup of mapping classes that act trivially on the first cohomology group $H^1(\Sigma_g; \mathbb{Z})$. If an element $\varphi \in \hat{\Gamma}_g$ has a projection $\pi(\varphi)$ that is a pseudo-Anosov element of the Torelli group, then $\varphi \notin G$.
4. A result on subgroups of $\hat{\Gamma}_g$ that satisfy properties (1)-(3) implies that G is trivial for $g \geqslant 3$ and is a subgroup of $\langle 1, \tau \rangle$ for $g = 2$. This deduction requires showing that the projection of G to $MCG(\Sigma_g)$ is trivial, which follows from the works of Ivanov and Long.

The proof of property (3) utilizes the Atiyah-Floer conjecture, applied to admissible bundles. The conjecture relates the Lagrangian Floer cohomology of our Lagrangians to the instanton Floer homology of a specific 3-manifold. Specifically, for Lagrangians $L_{(Y,E)}$ and $L_{(Y',E')}$ constructed from three-manifolds Y and Y', we have

$$\mathrm{HF}(L_{(Y,E)}, L_{(Y',E')}) = I_*(-Y \#_\Sigma Y', \gamma \# \gamma')$$

Here, $-Y \#_\Sigma Y'$ denotes the connected sum of the three-manifolds along their boundary, and $\gamma \# \gamma'$ is the connected sum of the knots. We denote $N = -Y \#_\Sigma Y'$ and $\omega = \gamma \# \gamma'$.

Theorem 5.11 (Kronheimer-Mrawka)**.** *The instanton Floer homology* $I_*(N, w)$ *is non-zero if the 3-manifold* N *is irreducible.*

The action of $\hat{\rho}(\varphi)$ on the Lagrangian $L_{(Y,E)}$ corresponds to an action on the underlying 3-manifold, which transforms the instanton Floer homology.

$$\mathrm{HF}(L^{\hat{\rho}(\varphi)}_{(Y,E)}, L_{(Y',E')}) = I_*(-Y \#^{\hat{\rho}(\varphi)}_\Sigma Y', \gamma \# \gamma')$$

We denote the transformed manifold and knot as N_φ and η_φ.

To complete the proof of property (3), we must find a suitable pair of 3-manifolds with knots, (Y, γ) and (Y', γ'), that satisfy two conditions:

1. The Lagrangians are disjoint, leading to zero Floer cohomology:

$$L_{(Y,\gamma)} \cap L_{(Y',\gamma')} = \varnothing \implies \mathrm{HF}(L_{(Y,\gamma)}, L_{(Y',\gamma')}) = 0.$$

2. After the action of $\hat{\rho}(\varphi)$, the Floer cohomology becomes non-zero: The manifold N_φ is irreducible, which by the Kronheimer-Mrawka theorem implies $I_*(N_\varphi, \eta_\varphi) \neq 0$, and thus $\mathrm{HF}(L^{\hat{\rho}(\varphi)}_{(Y,E)}, L_{(Y',E')}) \neq 0$.

The existence of such pairs (Y, γ) and (Y', γ') is established using the works of Clarkson, which themselves rely on the foundational work of Hemples. Hemples's work on defining a distance and Heegaard-splittings of 3-manifolds was used by Clarkson to guarantee the existence of the pseudo-Anosov and Torelli elements needed for the proof.

6 Xinle Dai: Sectorial Decomposition of Symmetric Products and Homological Mirror Symmetry

Abstract: Symmetric products of Riemann surfaces play a important role in symplectic geometry and low-dimensional topology. For example, they are essential ingredients in the definition of Heegaard Floer homology and serve as important examples of Liouville manifolds when the surfaces are open. In this talk, I will discuss work in progress on the symplectic topology of these spaces using Liouville sectorial methods.

6.1 Symmetric Products of Surfaces

We begin with the concept of a symmetric product of a topological surface, a fundamental construction in topology and algebraic geometry.

Definition 6.1. *Let Σ be a topological surface. The **n-th symmetric product** of Σ, denoted $Sym^n(\Sigma)$, is the quotient space of the Cartesian product $\Sigma \times \cdots \times \Sigma$ (with n factors) by the action of the symmetric group S_n permuting the factors.*

$$Sym^n(\Sigma) = \Sigma \times \cdots \times \Sigma / S_n$$

This construction provides a natural way to consider unordered n-tuples of points on a surface. The geometric properties of $Sym^n(\Sigma)$ depend on the structure of Σ:

1. *If Σ is a Riemann surface, then $Sym^n(\Sigma)$ inherits a natural complex structure, making it a complex manifold.*
2. *If Σ is a compact Riemann surface, $Sym^n(\Sigma)$ is a projective variety.*
3. *If Σ is an open quasi-projective Riemann surface, $Sym^n(\Sigma)$ is a quasi-projective variety.*

Example 6.2. *A classic example is the symmetric product of the complex plane $\mathbb{C}$.*

1. *$Sym^n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$. The isomorphism is given by mapping an unordered n-tuple of complex numbers $\{z_1, \ldots, z_n\}$ to the elementary symmetric polynomials $(\sigma_1, \ldots, \sigma_n)$ in these variables.*
2. *For any Riemann surface Σ, $Sym^n(\Sigma)$ is a complex manifold.*

The space $Sym^n(\mathbb{P}^1 \backslash \{(n+2) \text{ pts}\})$ is isomorphic to $\mathbb{P}^n \backslash \{(n-2) \text{ hyperplanes}\}$. These spaces are a type of n-dimensional pair of pants.

6.2 Liouville Manifolds and Sectors

Liouville manifolds form a special class of symplectic manifolds characterized by a particular vector field or form. These structures provide a natural way to define an "exact at infinity" condition, which is important for applications in Floer theory.

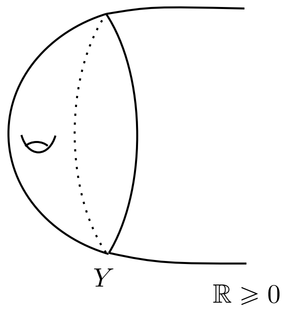
Definition 6.3. *A **Liouville vector field** on a symplectic manifold (X, ω) is a vector field Z that satisfies the condition $L_Z \omega = \omega$.*

The condition $L_Z \omega = \omega$ implies that the symplectic form ω is not only preserved but scaled by the flow of Z. This concept has a dual formulation in terms of differential forms.

Definition 6.4. *A **Liouville form** λ on a symplectic manifold (X, ω) is a primitive for ω, meaning that $\omega = d\lambda$. This form is related to the Liouville vector field Z by the identity $\lambda = \iota_Z \omega$, where ι_Z denotes the interior product.*

Definition 6.5. *A **Liouville manifold** is an exact symplectic manifold $(X, \omega = d\lambda)$ that, near infinity, is modeled on the product $(\mathbb{R}_{2n} \times Y, d(e^s \alpha))$, where Y is a compact manifold and α is a contact form on Y.*

The following image shows a Liouville manifold, where the end is modelled on a product of a ray and a compact manifold Y.

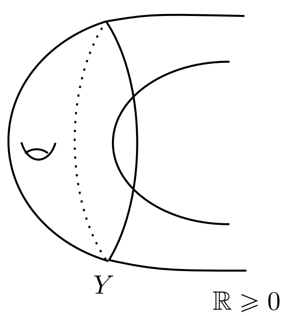


Example 6.6. *Every cotangent bundle or Stein manifold is a Liouville manifold.*

The notion of a Liouville manifold can be extended to include boundaries. This leads to the concept of a Liouville sector, which is a foundational object in the study of wrapped Fukaya categories.

Definition 6.7. *A **Liouville sector** X is a Liouville manifold with boundaries, ∂X. The Liouville vector field Z must be tangent to ∂X near infinity. Additionally, there exists a function $I : \partial X \to \mathbb{R}$ such that $ZI = I$ near infinity and the Hamiltonian vector field X_I is transverse to ∂X and points outward along it. This condition is equivalent to the positivity of the differential of I restricted to the characteristic foliation, i.e., $dI_{char\ fol} > 0$.*

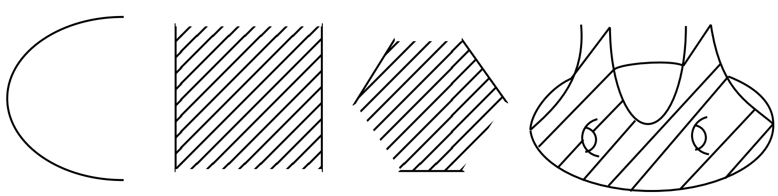
The image shows a Liouville sector, which is a Liouville manifold with a boundary. The shaded region denotes the interior of the sector.



Example 6.8.

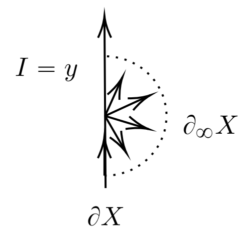
1. *The cotangent bundle of any manifold with boundary is a Liouville sector.*
2. *Any punctured bounded Riemann surface with no boundary components is a Liouville sector, which is homeomorphic to S^1.*

The following image displays various examples of Liouville sectors. These are: a half-plane, a square, a hexagon, and a more complex genus two surface with a boundary. The shaded regions denote the interior of each sector.



Example 6.9. *Consider the manifold $X = \mathbb{C}_{Re\geqslant 0}$, the right half-plane in the complex numbers. A Liouville form is $\lambda = \frac{x\,dy - y\,dx}{2}$, which gives the standard symplectic form $\omega = d\lambda = dx \wedge dy$. The Liouville vector field is $Z = \frac{x\partial x + y\partial y}{2}$.*

This image illustrates the Liouville sector on the right half-plane. The central curved dashed line represents the boundary at infinity, $\partial_\infty X$, while the vertical line is the finite boundary, ∂X. Arrows indicate the direction of the Liouville vector field.



Every Liouville sector X possesses a **wrapped Fukaya category**, denoted $\mathcal{W}(X)$. For an inclusion of Liouville sectors $X \hookrightarrow X'$, this structure induces a map $\mathcal{W}(X) \to \mathcal{W}(X')$.

6.3 Sectorial Decomposition and Homological Invariants

This section delves into the main topic of the talk: using a specific structure on a Riemann surface to decompose its symmetric product into Liouville sectors. This decomposition provides a powerful tool to study the geometry of the symmetric product and its relationship to other algebraic objects through the lens of Homological Mirror Symmetry (HMS).

Definition 6.10. *Let Σ be a Riemann surface and φ a proper plurisubharmonic function on Σ. Let $\{s_i\}_{i\in I}$ be the saddles (critical points of Morse index 1) of φ. For each saddle s_i, let γ_i be the stable manifold of s_i. If the function φ is quadratic in a neighborhood of each saddle, i.e., $\varphi|_{N(\gamma_i)}$ is of the form $ax^2 + by^2$ with $ab < 0$, we say that (Σ, φ) is a Riemann surface with a* ***quadratic Stein structure****.*

An important result establishes the existence of such structures on a wide class of surfaces.

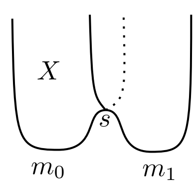
Proposition 6.11. *For any orientable topological surface Σ with a set of disjoint proper embedded arcs $\{\gamma_i\}_{i\in I}$, one can construct a quadratic Stein structure φ on Σ. This structure is built such that φ has a saddle s_i on each arc γ_i and one minimum m_j on each component of the complement $\Sigma\setminus\bigcup_{i\in I}\gamma_i$.*

This existence result allows us to apply the theory of Liouville sectors to the symmetric product. The main theorem establishes a decomposition of $Sym^2(\Sigma)$ based on this structure.

Theorem 6.12 (Sectorial Decomposition)**.** *In the setting of a Riemann surface with a quadratic Stein structure as described above, the structure determines a decomposition of the second symmetric product into Liouville sectors:*

$$Sym^2(\Sigma) = \bigcup_{H_{s_i,m_j}} U_{m_j,m_k}$$

where the sets U_{m_j,m_k} are Liouville sectors (with corners) for $j \leqslant k$. These sectors are separated by smooth hypersurfaces H_{s_i,m_j} which intersect at corners C_{s_i,s_j}.



The image above shows a decomposition of a space X with two minima, m_0 and m_1, and a saddle s. This structure leads to a decomposition of $Sym^2(X)$.

Let's consider the specific case of a 2-dimensional Liouville sector X. A related corollary simplifies the structure of its symmetric product.

Corollary 6.13. *For a 2-dimensional Liouville sector X, the space $Sym^2(X)$ is deformation equivalent to a Liouville sector Y.*

$$Sym^2(X) \simeq Y$$

where $\simeq$ denotes deformation equivalence.

The sectorial decomposition allows for a local-to-global approach, where a larger space is understood by piecing together simpler, more manageable sectors.

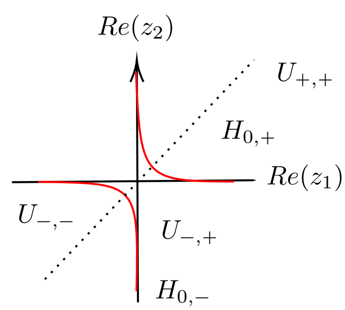
To further understand this decomposition, we can study a local model.

1. Let φ be a plurisubharmonic Morse function on $\mathbb{C}$. The associated symplectic form and Liouville form are $\omega = dd^c\varphi$ and $\lambda = d^c\varphi$ respectively. Let g_φ be the Riemannian metric and $Z = \nabla_{g_p}\varphi$ the gradient vector field.
2. Assume φ has only one index 1 critical point at 0, with a local form $\varphi(x + yi) = -\frac{1}{4}x^2 + \frac{3}{4}y^3$.
3. The gradient flow of this function has specific asymptotic properties. Under the gradient flow of a related function $\hat{I}$, as $|z_1 - z_0| \to +\infty$, the real part of the Liouville form $R(\omega) \to +\infty$ and the imaginary part of the symplectic form $Im(w) \to 0$ as $t \to \infty$.
4. The gradient flow moves faster near the origin.
5. For large time t, the gradient flow will align with one of the diagonals.

The local model provides a guide for the decomposition. The different asymptotic regions of the space correspond to the Liouville sectors in the decomposition. Specifically, for points (z_1, z_2):

- $(-\infty, -\infty)$ corresponds to the sector $U_{-,-}$.
- $(-\infty, 0)$ corresponds to the hypersurface $H_{0,-}$.
- $(-\infty, +\infty)$ corresponds to the sector $U_{-,+}$.
- $(0, +\infty)$ corresponds to the hypersurface $H_{0,+}$.
- $(+\infty, +\infty)$ corresponds to the sector $U_{+,+}$.

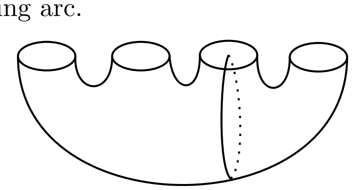
This correspondence is visualized in the following diagram.



6.4 Homological Mirror Symmetry of a Pair of Pants

The main application of this decomposition is in HMS. We explore a concrete example of this conjecture for a 2-dimensional pair of pants.

The geometric setting is $Sym^2(\mathbb{P}^1\{p_0, p_1, p_2, p_3\})$, a symmetric product of a sphere with four punctures. This surface can be cut by a single separating arc.



The separating arc divides the surface into two components: Σ_-, which is a pair of pants, and Σ_+, which is a cylinder.

By applying the sectorial decomposition to this specific surface, we find that the Liouville sectors have familiar geometric forms:

- $U_{-,-}$ is deformation equivalent to Sym^2(pants), which is isomorphic to the Liouville manifold $(\mathbb{C}^\times)^2$.
- $H_{0,-}$ is deformation equivalent to pants $\times\ \mathbb{R}$.

A central claim links the wrapped Fukaya category of these sectors to derived categories of coherent sheaves on algebraic varieties.

Proposition 6.14. *The wrapped Fukaya category of the sector $U_{-,-}$, which is isomorphic to $((\mathbb{C}^\times)^2, W = z_1 + z_2)$, corresponds under Homological Mirror Symmetry to the derived category of coherent sheaves on the algebraic variety $\mathbb{C}^2$ defined by the relation $\{xy = 0\}$.*

We can learn more about the mirror correspondence by studying the inclusion maps between sectors.

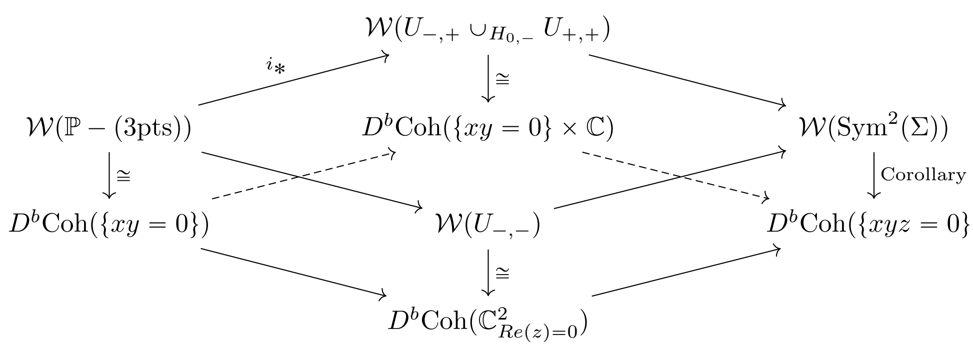
$$\begin{array}{ccc} \mathcal{W}(F_{0,-}) & \xrightarrow{\ i_+\ } & \mathcal{W}(U_{-,-}) \\ {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ D^b(\{xy=0\}) & \xrightarrow{\ i_-\ } & D^b(\mathbb{C}^2) \end{array}$$

The wrapped Fukaya category of the hypersurface $F_{0,-}$ maps to the derived category of the line $\{xy = 0\}$, while the wrapped Fukaya category of the sector $U_{-,-}$ maps to the derived category of $\mathbb{C}^2$.

Extending this analysis to other sectors in the decomposition yields a more general mirror symmetry:

- $U_{-,+}$ is deformation equivalent to pants $\times$ cylinder. This is conjectured to be mirror to $\{xy = 0\} \times \mathbb{C}$.
- $U_{+,+}$ is deformation equivalent to cylinder $\times$ torus.

The complete picture of the sectorial decomposition and its mirror correspondence is summarized in the following diagram.



This diagram illustrates the correspondence between the symplectic side (top, wrapped Fukaya categories) and the algebraic side (bottom, derived categories). The key maps, such as the inclusions of sectors, are mirrored by maps of derived categories. The top part of the diagram represents the geometric objects and their inclusions, ending with $Sym^2(\Sigma)$, while the bottom part represents their mirror duals. The horizontal arrows show how smaller pieces are glued together, and the vertical arrows represent the mirror map s.

7 Chris Woodward: Disk Counting for Tropical Lagrangians

Abstract: Manin assigned to any del Pezzo surface (compact complex surface with positive first Chern class) a root system, given by the set of second homology classes with square -2, perpendicular to the symplectic form. For example, the del Pezzo surface of degree one, which is obtained by blowing up the projective plane at eight points, corresponds to the root system E8. I will explain how to realize these homology classes as Lagrangian spheres, and outline the proof that they split-generate all of the Fukaya eigencategories with integer eigenvalues with non-maximal modulus.

7.1 Introduction

The central object of study in this section is the disk count for tropical Lagrangians, which provides a way to relate the geometric properties of a symplectic manifold to combinatorial data. We begin by setting up the necessary definitions and a key theorem.

Let $\Phi : X \to B$ be a compact almost toric manifold. Let $L \subset X$ be a tropical Lagrangian that is both compact and oriented spin. We assume that both X and L are monotone. Given these conditions, we can define a number $w_L \in \mathbb{Z}$ as a disk count of holomorphic, Maslov index two disks. This integer count is related to a polyhedral decomposition of the base manifold B. Let $R = \{P \subset B\}$ be a good polyhedral decomposition. We then choose a dual complex $B^\vee = \bigcup_{P \in \mathcal{P}} P^\vee$.

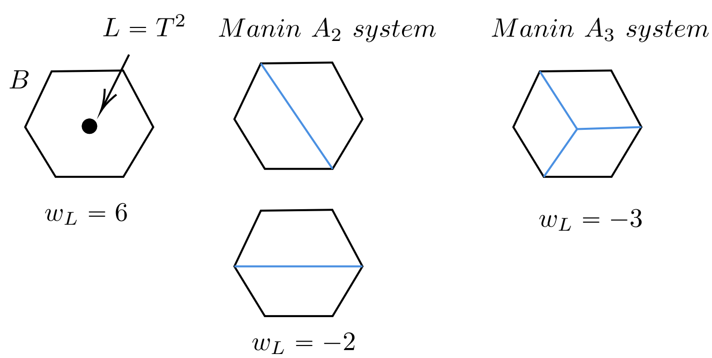
Theorem 7.1. *The disk count w_L can be expressed as a sum over tropical graphs in the dual complex $B^\vee$:*

$$w_L = \sum_{tropical\ graphs\ in\ B^\vee} \frac{1}{|Aut(\Gamma)|} m(\Gamma)$$

In "good cases," the multiplicity term $m(\Gamma)$ simplifies to a product over the vertices of the graph: $m(\Gamma) = \prod_{V \in Ver(\Gamma)} m(v)$. The vertex multiplicities $m(v)$ are given by explicit formulas.

An important application of this theory is found in the study of Fukaya categories. Specifically, the Fukaya category of a del Pezzo surface with a monotone symplectic form is split generated by monotone tori and Manin configurations of spheres.

Example 7.2. *Let $X = Bl^3(\mathbb{P}^2)$ be the blow-up of $\mathbb{P}^2$ at three points. Manin observed that a certain configuration of spheres in this space can be used to visualize the ADE root system. The following diagram illustrates this for the A_2 and A_3 systems, showing a tropical graph representation within a hexagon-shaped base manifold. Each graph is associated with a specific disk count w_L.*



The spectrum of the quantum cohomology ring $QH(X)$ of the manifold $X = Bl^3(\mathbb{P}^2)$ is given by the set of disk counts $\{6, -2, -3\}$. This set split generates the Fukaya category, $Fuk(X)$.

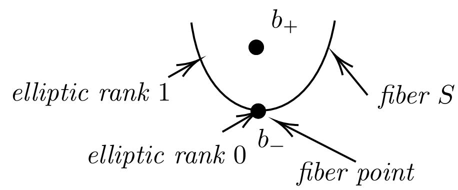
7.2 Almost Toric Manifold

To understand tropical Lagrangians, we first need to define an almost toric manifold.

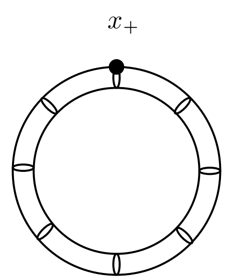
Definition 7.3. *A symplectic manifold X is an* ***almost toric manifold*** *if there exists a map $\Phi : X \to \mathbb{R}^n$ with a certain local structure. The components of this map, Φ_i, are required to commute with respect to the Poisson bracket, i.e., $\{\Phi_i, \Phi_j\} = 0$ for all i, j. The local structure of this map is a product of three types of components:*

- *A* ***regular*** *component: $(q, p) \mapsto p_0$.*
- *An* ***elliptic*** *component: $(q, p) \mapsto (q^2 + p^2)/2$.*
- *A* ***focus-focus*** *component: $(q_1, p_1, q_2, p_2) \mapsto (q_1p_2 - q_2p_1, q_1p_1 + q_2p_2)$.*

Example 7.4. *Let $X = T^*S^2 \to \mathbb{R}^2$ represent a spherical pendulum. The map Φ can be chosen to be the energy and angular momentum. The behavior of the system can be visualized in the base space $B = \mathbb{R}^2$ via the image of the map Φ. The following diagram shows the image of the momentum map for a spherical pendulum. The upper point b_+ corresponds to an elliptic rank 1 singularity, while the lower point b_- corresponds to an elliptic rank 0 singularity.*



The diagram shows two critical points: an unstable equilibrium point x_+ corresponding to a focus-focus singularity in the fiber, and a stable equilibrium point x_- corresponding to an elliptic rank 0 singularity. The fiber over $\Phi(x_+)$ is given by the manifold X_+ shown in the next diagram.

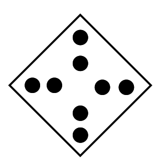


The diagram shows a double torus-like structure. The central point corresponds to the unstable equilibrium x_+.

The existence of almost toric structures is known for a broad class of manifolds.

Example 7.5 (Vienna)**.** *All monotone del Pezzo surfaces are almost toric. This follows from work by McDuff, which showed that the monotone symplectic form on such a surface is unique up to isomorphism.*

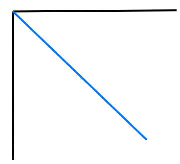
Example 7.6. *The blow-up of $\mathbb{P}^2$ at four points, $Bl^4\mathbb{P}^2$, is also an almost toric manifold. The base diagram for such a manifold is shown below. This diagram represents a square with eight interior points, a central point, and four points on the edges.*



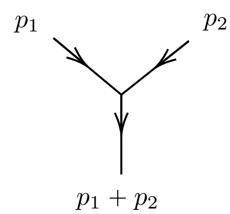
7.3 Tropical Lagrangians

Tropical Lagrangians are a class of Lagrangian submanifolds whose projections onto the base of a toric fibration have a specific combinatorial structure.

Theorem 7.7 (Mikhalkin)**.** *Let $\Phi : X \to B$ be a toric variety. Let $\Pi \subset B$ be a tropical graph. We assume for simplicity that* $\dim(X) = 4$*. Then the univalent vertices of the tropical graph are bisectrices, as shown in the diagram.*

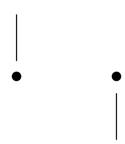


The vertices can also be trivalent. The following diagram shows a trivalent vertex with vectors p_1, p_2 and their sum $p_1 + p_2$.



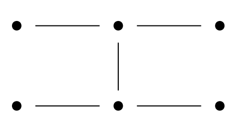
The vectors p_1 and p_2 must satisfy the condition $|\det p_1 p_2| = 1$.

An addendum to this theorem states that for almost toric manifolds, vanishing thimbles are also allowed, as represented by a simple graph.



Roughly speaking, this implies the existence of a family of Lagrangians L whose image under the map Φ approaches the tropical graph Π.

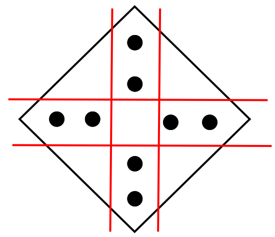
Example 7.8. *Returning to the blow-up of $\mathbb{P}^2$ at five points, $Bl^5\mathbb{P}^2$, we obtain the affine Dynkin diagram of type $\hat{D}_5$.*



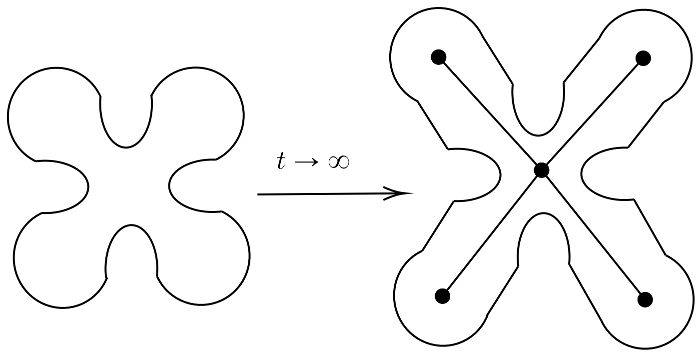
7.4 Holomorphic Disk Counts

The computation of disk counts relies on techniques developed by Venugopalan and Woodward, building on earlier works by Ianel, B. Parker, and Tehrani. This method involves constructing a dual complex from a polyhedral decomposition of the base manifold.

Let $\mathcal{P} = \{P\}$ be a good polyhedral decomposition of the base manifold. We choose a dual complex $P^\vee$ for each P. For instance, the following diagram shows a polyhedral decomposition of a diamond shape with its dual complex lines in red.



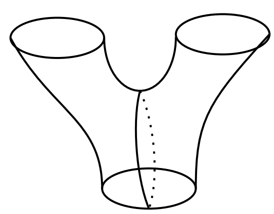
The process of holomorphic disk counting can be understood by considering a degeneration process where we obtain long necks, as shown in the diagram. This corresponds to the limit as $t \to \infty$.



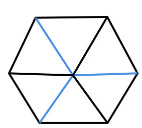
This framework also applies to Lagrangians that are locally invariant on neck regions, mapping to U_V. In this case, the moduli space splits as a product of components associated to the vertices, and the vertex multiplicity $m(v)$ is the number of maps from a compact Riemann surface with boundary c_V to the manifold $X_{P(v)}$ associated with the vertex v.

$$m(v) = \#u_V : c_V \to X_{P(v)}$$

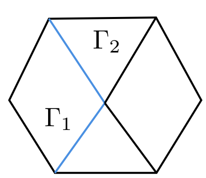
The multiplicities are known for certain special cases. For example, for a type $(d, 0)$ singularity, the Bryan-Pandharipande formula states that $m(v) = (-1)^{d-1}/d^2$. The multiplicity for a half pair of pants with type $(d, 0)$ is also known to be $(-1)^d$, as illustrated by the diagram showing a half-pants-like shape.



Example 7.9. *For $X = Bl^3(\mathbb{P}^2)$, the spectrum of the first Chern class c_1 in the quantum cohomology is $c_1 \circlearrowright QH(Bl^3\mathbb{P}^2)) = \{6, -2, -3\}$. This is consistent with the disk counts. The graph for $w_L = 6$ is the following.*

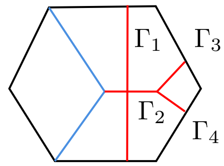


The value $w_L = -2$ is explained by the following tropical graph $\Gamma \subset B^\vee$.



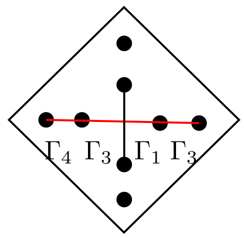
The total disk count is the sum of the multiplicities of the components: $w_L = -1 + -1 = -2$.

Example 7.10. *The value* $w_L = -3$ *from the quantum cohomology spectrum is explained by the following tropical graph.*



In this case, the total disk count is given by the sum of multiplicities of four components: $-1 + -1 + -1/2 + -1/2 = -3$.

Example 7.11. *For the monotone torus, the disk count is* $w_L = 12$. *The corresponding tropical graph is shown below, represented by lines on a diamond-shaped base manifold.*



The diagram shows a cross-shaped tropical graph on the dual complex. We can count in a similar way to obtain -12.