Topics in Symplectic Geometry: Foundational Aspects

Rutgers Symplectic Summer School

August 2024

Abstract

From August 19 to August 23, Rutgers University ran a summer school on symplectic geometry that aimed to provide graduate students and advanced undergraduate students tutorials in various advanced topics in symplectic geometry and introductions to recent developments. This year was focused on, but was not restricted to, the foundational aspects, including the theory of global Kuranishi charts, integer-valued curve-counting invariants, Hamiltonian dynamics, and contact topology.

This is an unofficial set of notes scribed by Gary Hu, who is responsible for all mistakes. None of these notes have been endorsed by the original lecturers. All of the course content is owned by their respective institutions and their researchers, while mistakes should be attributed solely to me. If you do find any errors, please report them to: gh7@williams.edu

Contents

MINICOURSES

There were three minicourses, each three hours long:

1. Global Kuranishi Charts by Mohan Swaminathan (Stanford)
2. Introduction to Contact Homology by Erkao Bao (University of Minnesota)
3. Quantitative Symplectic Geometry by Mike Usher (University of Georgia)

1 Mohan Swaminathan: Global Kuranishi Charts

There were three lectures:

1. Day 1: Local Structure of Holomorphic Curve Moduli Spaces

   We discuss how the implicit function theorem and gluing analysis give rise to 'local Kuranishi charts' for holomorphic curve moduli spaces. We also explain what it means for two or more such local Kuranishi charts to be 'compatible' on their overlap and briefly discuss how an atlas of Kuranishi charts allows us to 'virtually count' points in a compact moduli space of expected dimension 0.

2. Day 2: Global Kuranishi Charts: Definitions and Preliminaries

   We introduce the notion of a 'global Kuranishi chart' and explain how having one of these substantially simplifies the previous discussion. We also explain what it means for two global Kuranishi charts to be 'equivalent', which is analogous to the notion of compatibility for local Kuranishi charts. For the remainder, we discuss some geometric preliminaries necessary to understand the construction of global Kuranishi charts for moduli spaces of closed holomorphic curves of genus 0.

3. Day 3: Global Kuranishi Charts: Construction

   Following Abouzaid–McLean–Smith 2021, we explain the construction of global Kuranishi charts for genus 0 Gromov–Witten moduli spaces and show that the outcome of the construction is unique up to equivalence. Time permitting, we will also briefly discuss how one can extend this construction to settings beyond genus 0 GW theory.

1.1 Local Structure of Holomorphic Curve Moduli Spaces

1.1.1 The Moduli Space of Stable Maps

Let (X^{2n}, ω) be a closed symplectic manifold and J be an almost complex structure on X tamed by ω, meaning $\omega(v, Jv) > 0$ for all non-zero $v \in T_xX$. Let $A \in H_2(X; \mathbb{Z})$ be a homology class. We are interested in studying the space of J-holomorphic maps from genus-zero Riemann surfaces to X.

To obtain a compact space, one must consider not just maps from the smooth Riemann sphere $\mathbb{CP}^1$, but also maps from **nodal genus-zero curves**. A nodal curve is formed by gluing several copies of $\mathbb{CP}^1$ together at pairs of points, forming a tree-like structure.

Definition 1.1. *Let $m \geqslant 0$ be an integer. The **moduli space of genus-zero, m-marked, J-holomorphic stable maps** in the class A, denoted $\overline{\mathcal{M}}_{0,m}(X, A, J)$, is the set of equivalence classes of tuples $(\Sigma, x_1, \ldots, x_m, u)$, where:*

1. *Σ is a nodal genus-zero curve (a tree of $\mathbb{CP}^1$s).*
2. *$x_1, \ldots, x_m \in \Sigma$ are distinct marked points located on the smooth part of Σ.*
3. *$u : \Sigma \to X$ is a J-holomorphic map such that $u_*[\Sigma] = A$.*
4. *The tuple is **stable**, meaning the group of automorphisms of $(\Sigma, x_1, \ldots, x_m)$ that are compatible with u is finite. This is equivalent to requiring that on any irreducible component of Σ where u is constant, there are at least three "special points" (marked points or nodes).*

Two such tuples $(\Sigma, \{x_i\}, u)$ and $(\Sigma', \{x_i'\}, u')$ are equivalent if there exists a biholomorphism $\phi : \Sigma \to \Sigma'$ such that $\phi(x_i) = x_i'$ for all i and $u = u' \circ \phi$.

We present two basic properties of this moduli space:

Theorem 1.2 (Gromov Compactness)**.** *The moduli space $\overline{\mathcal{M}}_{0,m}(X, A, J)$ is compact and Hausdorff.*

Theorem 1.3 (Virtual Dimension)**.** *The moduli space $\overline{\mathcal{M}}_{0,m}(X, A, J)$ has an **virtual dimension** given by*

$$d = 2(n + c_1(TX) \cdot A + m - 3).$$

Now, we move on to discuss a motivating problem:

Problem 1.4. *Can we define enumerative invariants by "counting" the number of points in $\overline{\mathcal{M}}_{0,m}(X, A, J)$ when its virtual dimension is zero?*

The primary obstruction is that, for a generic J, $\overline{\mathcal{M}}_{0,m}(X, A, J)$ is not a manifold of the expected dimension. It is typically a more complicated object, an orbifold with singularities.

1.1.2 The Transverse Case

We now investigate the local structure of $\overline{\mathcal{M}}_{0,m}(X, A, J)$ near a given point.

Problem 1.5. *Given an element* $(\Sigma, x_1, \ldots, x_m, u) \in \overline{\mathcal{M}}_{0,m}(X, A, J)$*, what is the local structure of the moduli space near this point?*

The analysis proceeds in two cases, depending on whether the domain Σ is smooth or nodal.

Σ is Smooth Here, "smooth" means $\Sigma \cong \mathbb{CP}^1$. The local analysis is best framed in the language of infinite-dimensional geometry. We define:

- The Banach manifold of maps $\mathcal{B} = C^\infty(\Sigma, X)_A = \{v : \Sigma \to X \mid v \text{ is } C^\infty, v_*[\Sigma] = A\}$.
- The infinite-rank Banach vector bundle $\mathcal{E} \to \mathcal{B}$, whose fiber over $v \in \mathcal{B}$ is $\mathcal{E}_v = \Omega^{0,1}(\Sigma, v^*TX) = \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, v^*TX)$.
- The section σ of $\mathcal{E}$ over $\mathcal{B}$ given by $v \mapsto \overline{\partial}_J v$, where $\overline{\partial}_J v = \frac{1}{2}(\underset{\cdot}{\mathrm{v}} + J(v) \circ \underset{\cdot}{\mathrm{v}} \circ j_\Sigma)$.

Note that the space of holomorphic maps is precisely the zero set of this section: $\sigma^{-1}(0) = \mathrm{Hol}(\Sigma, X, A, J)$. For a solution $u \in \sigma^{-1}(0)$, the linearization of σ at u is a well-defined operator

$$D_u\sigma : T_u\mathcal{B} \to \mathcal{E}_u.$$

More explicitly, identifying $T_u\mathcal{B}$ with $\Omega^0(\Sigma, u^*TX)$, this map is

$$D(\overline{\partial}_J)_u : \Omega^0(\Sigma, u^*TX) \to \Omega^{0,1}(\Sigma, u^*TX).$$

In local holomorphic coordinates $z = s + it$ on Σ, the operator $\overline{\partial}_J$ is

$$\overline{\partial}_J u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t}\right) \otimes (\underset{\cdot}{\mathrm{s}} - i\underset{\cdot}{\mathrm{t}}).$$

For a variation $\xi \in \Omega^0(\Sigma, u^*TX)$, its derivative is

$$D(\overline{\partial}_J)_u\xi = \frac{1}{2}\left(\frac{\partial \xi}{\partial s} + J(u)\frac{\partial \xi}{\partial t} + (\partial_\xi J)(u)\frac{\partial u}{\partial t}\right) \otimes (\underset{\cdot}{\mathrm{s}} - i\underset{\cdot}{\mathrm{t}}).$$

The term $\frac{\partial \xi}{\partial s} + J(u)\frac{\partial \xi}{\partial t}$ is a first-order differential operator, while $(\partial_\xi J)(u)\frac{\partial u}{\partial t}$ is a zeroth-order term. For notational convenience, let $D_u := D(\overline{\partial}_J)_u$. This operator is Fredholm, and by the Riemann–Roch theorem, its index is

$$\mathrm{ind}(D_u) = \dim(\ker D_u) - \dim(\mathrm{coker}\, D_u) = 2(n + c_1(TX) \cdot A).$$

The implicit function theorem for Banach manifolds states that if D_u is surjective, then near $(\Sigma, \{x_i\}, u)$, the moduli space $\mathcal{M}_{0,m}(X, A, J)$ is an orbifold of the expected dimension. In this case, we conclude that

$$\mathcal{M}_{0,m}(X, A, J) = \frac{\mathrm{Hol}(\mathbb{CP}^1, X, A, J) \times ((\mathbb{CP}^1)^m \backslash \Delta)}{\mathrm{PSL}_2(\mathbb{C})},$$

where $\Delta = \{(x_1, \ldots, x_m) \mid x_i = x_j \text{ for some } i \neq j\}$.

Σ is Nodal Here, Σ is a tree of $\mathbb{CP}^1$s. Let $\tilde{\Sigma}$ be the normalization of Σ, which is a disjoint union of spheres. The map $u : \Sigma \to X$ lifts to a map $\tilde{u} : \tilde{\Sigma} \to X$. The linearized operator D_u acts on the subspace of sections in $\Omega^0(\tilde{\Sigma}, \tilde{u}^*TX)$ that satisfy gluing conditions at the nodes, i.e., sections in $\Omega^0(\Sigma, u^*TX)$.

Exercise 1.6. *Check that ind$(D_u) = 2(n + c_1(TX) \cdot A)$.*

Theorem 1.7 (Gluing Theorem)**.** *If D_u is surjective, then $\overline{\mathcal{M}}_{0,m}(X, A, J)$ has a local chart near $(\Sigma, x_1, \ldots, x_m, u)$ of the form V/Γ, where V is a vector space of the expected dimension and Γ is a finite group acting linearly on V.*

1.1.3 Local Kuranishi Charts

Let $\mathcal{B}$ be a Banach manifold, $\mathcal{E} \to \mathcal{B}$ a Banach vector bundle, and $\overline{\partial}$ a smooth section whose linearizations are Fredholm operators. The object of study is the zero set of this section, $\overline{\mathcal{M}} = (\overline{\partial})^{-1}(0) \subset \mathcal{B}$.

Consider an element $u \in \overline{\mathcal{M}}$. In the case where the linearized operator $(D\overline{\partial})_u$ is surjective, the implicit function theorem for Banach spaces ensures that $\overline{\mathcal{M}}$ is a smooth manifold in a neighborhood of u. The tangent space at this point is given by $T_u\overline{\mathcal{M}} = \ker(D\overline{\partial})_u$. In this case, we are done.

So let's assume that the linearization $D_u : T_u\mathcal{B} \to \mathcal{E}_u$ is not surjective. The standard approach, known as the Kuranishi method, is to augment the problem. We choose a finite-dimensional vector space E and a linear map $\lambda : E \to \mathcal{E}_u$ such that the image of λ is a complement to the image of D_u, yielding a surjective operator $D_u \oplus \lambda : T_u\mathcal{B} \oplus E \to \mathcal{E}_u$. We then choose a neighborhood $\mathcal{U} \subset \mathcal{B}$ of u and extend λ to a smooth map $\lambda : \mathcal{U} \times E \to \mathcal{E}|_{\mathcal{U}}$.

This allows us to define a perturbed moduli space,

$$\mathcal{M}_{\mathcal{U},E,\lambda} = \{(v, e) \in \mathcal{U} \times E \mid \overline{\partial}v + \lambda(v, e) = 0\}.$$

The original moduli space, intersected with $\mathcal{U}$, embeds into this larger space: $\overline{\mathcal{M}} \cap \mathcal{U} \hookrightarrow \mathcal{M}_{\mathcal{U},E,\lambda}$. There is a natural projection map $s : \mathcal{M}_{\mathcal{U},E,\lambda} \to E$.

The linearization of the perturbed system at the point $(u, 0) \in \mathcal{M}_{\mathcal{U},E,\lambda}$ is given by the map from $T_u\mathcal{B} \oplus E$ to $\mathcal{E}_u$ defined by

$$(\xi, e) \mapsto D_u\xi + \lambda(u, e).$$

By construction, this operator is surjective. This ensures that $\mathcal{M}_{\mathcal{U},E,\lambda}$ is a finite-dimensional manifold near $(u, 0)$. This motivates the following definition:

Definition 1.8. *Suppose $\overline{\mathcal{M}}$ is a compact Hausdorff space. A* ***local Kuranishi chart of virtual dimension*** *d for $\overline{\mathcal{M}}$ is a quintuple $(\mathcal{M}_\alpha, E_\alpha, \Gamma_\alpha, s_\alpha, \psi_\alpha)$ where:*

- *$\mathcal{M}_\alpha$ is a finite-dimensional topological manifold.*
- *E_α is a finite-dimensional vector space such that* $\dim \mathcal{M}_\alpha = d + \dim E_\alpha$.
- *Γ_α is a finite group which acts on $\mathcal{M}_\alpha$ and E_α.*
- *$s_\alpha : \mathcal{M}_\alpha \to E_\alpha$ is a Γ_α-equivariant function.*
- *$\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \xrightarrow{\cong} U_\alpha \subset \overline{\mathcal{M}}$ is a homeomorphism onto an open subset.*

The upshot is that the entire moduli space $\overline{\mathcal{M}}_{0,m}(X, A, J)$ can be covered by a finite atlas of such local Kuranishi charts. A local Kuranishi chart induces a **local virtual fundamental class** on U_α through the composition of maps:

$$\begin{aligned}
\check{H}^d_c(U_\alpha; \mathbb{Q}) &\xrightarrow[\cong]{\frac{1}{|\Gamma_\alpha|}\psi_\alpha^*} \check{H}^d_c(s_\alpha^{-1}(0); \mathbb{Q})^{\Gamma_\alpha} \\
&\xrightarrow[\cong]{\text{Pardon}} H_{\dim E_\alpha}(\mathcal{M}_\alpha, \mathcal{M}_\alpha \backslash s_\alpha^{-1}(0); \mathbb{Q})^{\Gamma_\alpha} \\
&\xrightarrow{(s_\alpha)_*} H_{\dim E_\alpha}(E_\alpha, E_\alpha \backslash \{0\}; \mathbb{Q})^{\Gamma_\alpha} \\
&\xrightarrow[\cong]{\text{orientation}} \mathbb{Q}
\end{aligned}$$

where Pardon is the map found in [Pardon, 2016, Appendix A].

Definition 1.9. *The* ***local virtual fundamental class*** *is the map*

$$[v_\alpha]^{vir}_{local} : \check{H}^d_c(U_\alpha; \mathbb{Q}) \to \mathbb{Q}$$

defined by the composition above.

Example 1.10. *Consider a chart where* $\overline{\mathcal{M}}_\alpha = \mathbb{C}$, $E_\alpha = \mathbb{C}$, $\Gamma_\alpha = \{1\}$, *and the section is given by* $s_\alpha(z) = z^n$. *The local contribution to the virtual count of points is* n, *so*

$$[v_\alpha]^{vir}_{local} = n[pt].$$

1.2 Global Kuranishi Charts: Definitions and Preliminaries

1.2.1 Global Kuranishi Charts and Equivalence

Definition 1.11. *Let* $\overline{\mathcal{M}}$ *be a compact Hausdorff space. A* ***global Kuranishi chart of virtual dimension*** d *for* $\overline{\mathcal{M}}$ *consists of a tuple* $(G, \mathcal{T}, \mathcal{E}, s)$, *where:*

- $\mathcal{T}$ *is a finite-dimensional topological manifold, called the* ***thickening***.
- $\mathcal{E} \to \mathcal{T}$ *is a finite-rank vector bundle, called the* ***obstruction bundle***.
- $s : \mathcal{T} \to \mathcal{E}$ *is a section, called the* ***obstruction section***.
- G *is a compact Lie group, the* ***symmetry group***, *which acts on the bundle* $\mathcal{E} \to \mathcal{T}$ *such that the action on* $\mathcal{T}$ *has finite stabilizers.*
- *The dimensions satisfy the relation* $\dim \mathcal{T} = d + rank\,\mathcal{E} + \dim G$.
- *A homeomorphism* $s^{-1}(0)/G \xrightarrow{\sim} \overline{\mathcal{M}}$, *called the* ***footprint map***.

The allowance of a infinite Lie group G is very important. It turns out that allowing only finite groups is not flexible enough. For example, if we take $\mathbb{CP}^1$ and consider a disk $\mathbb{D}$ at the origin with a $\mathbb{Z}/2$ action, we need an infinite group G. On the other hand, the current condition is sufficient: every orbifold is a global quotient M/G for some manifold M and Lie group G.

A global Kuranishi chart, equipped with compatible orientations on the manifold $\mathcal{T}$, the bundle $\mathcal{E}$, and the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, induces a virtual fundamental class for $\overline{\mathcal{M}}$. This class is constructed via the following sequence of maps in equivariant (co)homology:

$$\begin{aligned}
\check{H}^*(\overline{\mathcal{M}}; \mathbb{Q}) &\xrightarrow{\text{Poincaré Duality}} H^G_{\operatorname{rank}\mathcal{E}}(\mathcal{T}, \mathcal{T} \backslash s^{-1}(0); \mathbb{Q}) \\
&\xrightarrow{s_*} H^G_{\operatorname{rank}\mathcal{E}}(\mathcal{E}, \mathcal{E} \backslash 0_{\mathcal{E}}; \mathbb{Q}) \\
&\xrightarrow{\tau^G_{\mathcal{E}}} H^G_0(\mathrm{pt}; \mathbb{Q}) \cong \mathbb{Q},
\end{aligned}$$

where the first map is an isomorphism and the final map is induced by the equivariant Thom class of $\mathcal{E}$.

Problem 1.12. *When are two global Kuranishi charts to be considered equivalent?*

The appropriate notion of equivalence is one that preserves the induced virtual fundamental class.

Proposition 1.13. *Two global Kuranishi charts are* ***equivalent*** *if one can be obtained from the other through a finite sequence of the following operations:*

1. ***Germ equivalence:*** *Replace* $(\mathcal{T}, \mathcal{E}, s)$ *with its restriction to a* G*-invariant open neighborhood* $\mathcal{U}$ *of* $s^{-1}(0)$ *in* $\mathcal{T}$, *yielding the chart* $(G, \mathcal{U}, \mathcal{E}|_{\mathcal{U}}, s|_{\mathcal{U}})$.
2. ***Group enlargement:*** *Given another compact Lie group* H *and a* G*-equivariant principal* H*-bundle* $p : P \to \mathcal{T}$, *form the new chart* $(G \times H, P, p^*\mathcal{E}, p^*s)$.
3. ***Stabilization:*** *Given a* G*-equivariant vector bundle* $\pi : \mathcal{W} \to \mathcal{T}$, *form the new chart* $(G, \mathcal{W}, \pi^*(\mathcal{E} \oplus \mathcal{W}), \pi^*s \oplus \Delta_{\mathcal{W}})$, *where* $\Delta_{\mathcal{W}}$ *is the diagonal section.*

These operations should be thought of as analogous to the Reidemeister moves in knot theory.

1.2.2 Complex Geometry Background

The construction of global charts relies on several basic results from complex geometry. On $\mathbb{CP}^n$, the tautological line bundle is denoted $\mathcal{O}(-1) \hookrightarrow \mathbb{CP}^n \times \mathbb{C}^{n+1}$, and its dual is the hyperplane bundle $\mathcal{O}(1)$. We define $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$.

Holomorphic Line Bundles on Curves (Riemann Surfaces)

Lemma 1.14. *Suppose Σ is a Riemann surface, $L \to \Sigma$ is a C^∞ complex line bundle, and ∇ is a $\mathbb{C}$-linear connection on it. Then the operator $\nabla^{0,1}$ defines a unique holomorphic structure on L.*

Proof. Given $p \in \Sigma$, choose a C^∞ section τ of L defined near p such that $\tau(p) \neq 0$. Then, $\nabla^{0,1}(\tau) = g \otimes \tau$ where $g \in \Omega^{0,1}(\Sigma)$. The $\bar{\partial}$-Poincaré lemma guarantees the existence of a local C^∞-function f such that $g = \bar{\partial} f$. We define a new section $\sigma = e^{-f}\tau$. A direct calculation shows that σ is annihilated by $\nabla^{0,1}$:

$$\begin{aligned}\nabla^{0,1}\sigma &= \nabla^{0,1}(e^{-f}\tau) = e^{-f}\nabla^{0,1}\tau + (\bar{\partial}e^{-f}) \otimes \tau \\ &= e^{-f}(g \otimes \tau) - fe^{-f}(\bar{\partial} f) \otimes \tau \\ &= e^{-f}(g \otimes \tau - g \otimes \tau) = 0.\end{aligned}$$

Thus, σ is a local holomorphic section, defining the holomorphic structure. $\square$

Lemma 1.15. *Suppose Σ is a nodal genus-zero curve. The isomorphism class of a holomorphic line bundle L on Σ is determined by the degree of L on each irreducible component of Σ.*

Corollary 1.16. *Consider $L \to \Sigma$ as above. If L has total degree d and has degree $\geqslant 0$ on each component, then $\dim_{\mathbb{C}} H^0(\Sigma; L) = d + 1$ and $\dim_{\mathbb{C}} H^1(\Sigma; L) = 0$.*

Genus-Zero Curves in $\mathbb{CP}^n$ A holomorphic map into projective space can be specified either geometrically or algebraically.

1. A holomorphic map $f : X \to \mathbb{CP}^n$.
2. A holomorphic line bundle $\mathcal{L} \to X$ with holomorphic sections $s_0, \ldots, s_n$ that have no common zero in X.

The correspondence between these two perspectives is given by:

$$(X \xrightarrow{f} \mathbb{CP}^n_{[x_0:\cdots:x_n]}) \mapsto (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$$

$$(\mathcal{L}, s_0, \ldots, s_n) \mapsto (X \xrightarrow{[s_0:\cdots:s_n]} \mathbb{CP}^n).$$

One important example where transversality holds without perturbation is the moduli space of maps into projective space.

Lemma 1.17. *For $n, d \geqslant 1$ and $m \geqslant 0$, the space $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n, d)$ is a complex orbifold of the expected dimension.*

Proof. Let $f : \Sigma \to \mathbb{CP}^n$ be a genus-zero nodal stable map. Transversality holds if the linearized operator D_f is surjective, which is equivalent to the vanishing of its cokernel, coker $D_f = H^1(\Sigma; f^*T\mathbb{CP}^n)$. The tangent bundle of $\mathbb{CP}^n$ fits into the Euler exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \to T\mathbb{CP}^n \to 0.$$

Pulling back this sequence by f and taking the long exact sequence in cohomology yields the segment

$$\cdots \to H^1(\Sigma; f^*\mathcal{O}(1))^{\oplus(n+1)} \to H^1(\Sigma; f^*T\mathbb{CP}^n) \to H^2(\Sigma; \mathcal{O}) \to \cdots$$

Since Σ is a genus-zero curve, $H^2(\Sigma; \mathcal{O}) = 0$. The line bundle $f^*\mathcal{O}(1)$ has degree $d \geqslant 1$ and non-negative degree on each component, so by the previous corollary, $H^1(\Sigma; f^*\mathcal{O}(1)) = 0$. It follows that $H^1(\Sigma; f^*T\mathbb{CP}^n) = 0$, establishing surjectivity. $\square$

1.3 Global Kuranishi Charts: Construction

1.3.1 The AMS Trick

Let (X^{2n}, ω) be a closed symplectic manifold, $A \in H_2(X;\mathbb{Z})$ a homology class, and J an ω-tame almost complex structure on X. These data define the moduli space $\overline{\mathcal{M}}_0(X, A, J)$, which consists of equivalence classes of J-holomorphic maps $u : \Sigma \to X$ from nodal genus-zero curves Σ such that $u_*[\Sigma] = A$. The main result is the following.

Theorem 1.18 (Abouzaid, McLean, Smith, 2021)**.** *The moduli space $\overline{\mathcal{M}}_0(X, A, J)$ admits a global Kuranishi chart. Although the construction involves choices, the resulting chart is unique up to equivalence.*

A key ingredient in the construction is the moduli space of non-degenerate maps into projective space.

Proposition 1.19. *The space*

$$\overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d) := \left\{ f : \Sigma \to \mathbb{CP}^d \mid f \text{ is a degree } d \text{ genus-0 stable map and } f \text{ is non-degenerate} \right\}$$

is a smooth quasi-projective variety of the expected dimension. A map f is non-degenerate if its image is not contained in any hyperplane of $\mathbb{CP}^d$.

Example 1.20. *The rational normal curve provides a canonical example of a non-degenerate map:*

$$\begin{aligned} \mathbb{CP}^1 &\to \mathbb{CP}^d \\ [u : v] &\mapsto [u^d : u^{d-1}v : \cdots : v^d]. \end{aligned}$$

Associated with this moduli space is a universal family, described by the diagram below, where $\mathcal{C}$ is the universal curve.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathbb{CP}^d \\ \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d) & \end{tikzcd}$$

The space $\mathcal{C}$ is also a smooth, quasi-projective variety.

Recall that a map to projective space $Z \to \mathbb{CP}^n$ is induced by a line bundle $\mathcal{L} \to Z$ and a choice of global sections $s_0, \ldots, s_n \in H^0(Z;\mathcal{L})$ with no common zeros.

Definition 1.21. *A line bundle $\mathcal{L}$ is* ***very ample*** *if there exist sections $s_0, \ldots, s_n \in H^0(Z;\mathcal{L})$ that define an embedding $Z \hookrightarrow \mathbb{CP}^n$.*

Definition 1.22. *A line bundle $\mathcal{L}$ is* ***ample*** *if there exists an integer $m \geqslant 1$ such that $\mathcal{L}^{\otimes m}$ is very ample.*

Proposition 1.23. *If Z is a nodal curve, a line bundle $\mathcal{L}$ on Z is ample if and only if $\deg(\mathcal{L}|_C) > 0$ for every irreducible component $C \subset Z$.*

We now state the main technical lemma.

Lemma 1.24 (AMS Trick)**.** *Suppose Z is a compact complex manifold, $\mathcal{L} \to Z$ is an ample line bundle, and $\mathcal{E} \to Z$ is a holomorphic vector bundle. Endow these bundles with Hermitian metrics. For $k \gg 1$, define the finite-dimensional space of sections*

$$W_k := \operatorname{Im}\left(H^0(Z; \mathcal{E} \otimes \mathcal{L}^{\otimes k}) \otimes_{\mathbb{C}} \overline{H^0(Z; \mathcal{L}^{\otimes k})} \xrightarrow{\langle \cdot, \cdot \rangle} \Omega^0(Z, \mathcal{E}) \right).$$

As $k \to \infty$, the spaces W_k provide an L^2-dense subspace of $\Omega^0(Z, \mathcal{E})$. That is, for any $\xi \in \Omega^0(Z, \mathcal{E})$, there exists k_0 such that for all $k \geqslant k_0$, there is an $\eta \in W_k$ with $\langle \xi, \eta \rangle_{L^2} \neq 0$.

1.3.2 Construction

Line Bundles on X First, approximate the symplectic form ω by another symplectic form Ω which also tames J and satisfies $[\Omega] \in H^2(X;\mathbb{Q})$. By clearing denominators, we may assume $[\Omega] \in H^2(X;\mathbb{Z})$/torsion. This condition implies the existence of a C^∞ complex line bundle $L_\Omega \to X$ with first Chern class $c_1(L_\Omega) = [\Omega]$.

From Chern-Weil theory, we have the following lemma:

Lemma 1.25. *There exists a Hermitian metric and a compatible Hermitian connection ∇ on L_Ω such that its curvature form is $-2\pi i\Omega$.*

Let us fix the notation $d := [\Omega] \cdot A$.

Framed Genus-Zero Curves Consider a J-holomorphic stable map $u : \Sigma \to X$. The pullback bundle $u^*L_\Omega \to \Sigma$ is endowed with a holomorphic structure induced by the operator $(u^*\nabla)^{0,1}$. Since Ω tames J, the integral $\int_C u^*\Omega \geqslant 0$ for every component $C \subset \Sigma$, and this integral is strictly positive for any unstable component. Consequently, the line bundle u^*L_Ω has non-negative degree on each component of Σ and positive degree on each unstable component. From the results of the previous section, it follows that

$$\dim_{\mathbb{C}} H^0(\Sigma; u^*L_\Omega) = d + 1 \quad \text{and} \quad H^1(\Sigma; u^*L_\Omega) = 0.$$

A **framing** is a choice of basis $F = (f_0, \ldots, f_d)$ for $H^0(\Sigma; u^*L_\Omega)$. Such a choice induces a degree-d, genus-zero stable map

$$\Phi_F = [f_0 : \cdots : f_d] : \Sigma \to \mathbb{CP}^d.$$

If this map is non-degenerate, then (Σ, Φ_F) is a point in $\overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d)$, and the curve Σ can be identified with a fiber of the universal curve via an embedding i_F. The following diagram illustrates this relationship:

$$\begin{tikzcd}
\Sigma \arrow[r, hook, "i_F"] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & \mathbb{CP}^d \\
(\Sigma, \Phi_F) \arrow[r, phantom, "\in"] & \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d) &
\end{tikzcd}$$

Associated to a framed curve is the matrix

$$H(\Sigma, u, F) := \left(\int_\Sigma \langle f_i, f_j \rangle_{u^*L_\Omega} u^*\Omega \right)_{0 \leqslant i,j \leqslant d},$$

which is a Hermitian positive-definite $(d+1) \times (d+1)$ matrix.

Definition 1.26. *A **framed genus-zero curve in** X is a tuple (Σ, u, F) where*

1. *Σ is a nodal genus-zero curve.*
2. *$u : \Sigma \to X$ is a C^∞ map in the class A such that $\int_C u^*\Omega \geqslant 0$ on each component $C \subset \Sigma$ and $\int_C u^*\Omega > 0$ on each unstable component.*
3. *$F = (f_0, \ldots, f_d)$ is a basis of $H^0(\Sigma; u^*L_\Omega)$ such that the matrix $H(\Sigma, u, F)$ is positive-definite.*

Two framed curves (Σ, u, F) and (Σ', u', F') are equivalent if there exists a biholomorphism $\varphi : \Sigma \to \Sigma'$ such that $u = u' \circ \varphi$. The diagram describing this equivalence is as follows:

$$\begin{tikzcd}
\Sigma \arrow[dr, "u"] \arrow[dd, "\varphi"', "\cong"] & \\
& X \\
\Sigma' \arrow[ur, "u'"'] &
\end{tikzcd}$$

Achieving Transversality To achieve transversality in the construction, we make the following choices:

1. A relatively ample line bundle $\mathcal{L}$ on $\mathcal{C} \to \overline{\mathcal{M}}_0^*(\mathbb{CP}^d, d)$ equipped with a $U(d+1)$-invariant Hermitian metric.
2. A $U(d+1)$-invariant $\mathbb{C}$-linear connection on $T^{*0,1}\mathcal{C}$.
3. A $\mathbb{C}$-linear connection on TX (viewed as a complex vector bundle via J).
4. A sufficiently large integer $k \gg 1$.

For more details, see [Horschi, Swaminathan, 2021, Section 2.1]. With this data, the components of the global Kuranishi chart are defined as follows.

Proposition 1.27. *The* ***thickening*** $\mathcal{T}$ *is the space of tuples* (Σ, u, F, η), *where*

1. (Σ, u, F) *is a framed genus-zero curve in* X.
2. η *is an element of the finite-dimensional space*

$$H^0(\Sigma; u^*TX \otimes i_F^*(T^{*0,1}\mathcal{C} \otimes \mathcal{L}^{\otimes k})) \otimes_{\mathbb{C}} \overline{H^0(\Sigma; i_F^*\mathcal{L}^{\otimes k})}$$

satisfying the perturbed equation

$$\overline{\partial}_J u + \langle \eta \rangle \circ i_F = 0.$$

Proposition 1.28. *The* ***obstruction bundle*** $\mathcal{E} \to \mathcal{T}$ *is a vector bundle whose fiber over* (Σ, u, F, η) *is*

$$E_{(\Sigma,u,F)} \oplus \mathcal{H}_{d+1},$$

where $\mathcal{H}_{d+1}$ *is the space of* $(d+1) \times (d+1)$ *Hermitian matrices and* $E_{(\Sigma,u,F)}$ *is another finite-dimensional space of sections constructed via the AMS trick.*

Proposition 1.29. *The* ***obstruction section*** $s : \mathcal{T} \to \mathcal{E}$ *is given by*

$$s(\Sigma, u, F, \eta) = (\eta, \log H(\Sigma, u, F)).$$

Proposition 1.30. *The* ***symmetry group*** *is* $G = U(d+1)$, *which acts on the space of framings.*

The key point of the construction is that for $k \gg 1$, the AMS trick ensures that the defining equation for the thickening $\mathcal{T}$ is transverse.

2 Erkao Bao: Introduction to Contact Homology

There were three lectures:

1. Day 1: Moduli Spaces of J-holomorphic Curves and Compactness

   In this lecture, we begin with an introduction to basic contact geometry. We then introduce J-holomorphic curves as the gradient of the action functional. The focus will be on the moduli space of J-holomorphic curves, with a discussion on compactness. We will provide heuristic definitions of cylindrical contact homology and full contact homology.

2. Day 2: Cylindrical Contact Homology in Dimension Three via Obstruction Bundle Gluing

   This lecture addresses the transversality issues associated with the moduli space of J-holomorphic curves. We specifically focus on cylindrical contact homology in the 3-dimensional case. The lecture will cover the resolution of transversality issues using obstruction bundle gluing techniques.

3. Day 3: Semi-Global Kuranishi Structure and Full Contact Homology

   In this lecture, we introduce the semi-global Kuranishi structure. We explore its application in relation to obstruction bundle gluing, including computations of simple examples. The discussion will culminate in the rigorous definition of full contact homology, facilitated by the semi-global Kuranishi structure.

2.1 Moduli Spaces of J-Holomorphic Curves and Compactness

2.1.1 Introduction

Definition 2.1. *Let M be a manifold of dimension $2n+1$, and let ξ be a hyperplane distribution on M, i.e., a subbundle of the tangent bundle TM of rank $2n$. We say ξ is a* ***contact structure*** *if there exists a 1-form α on M, called the* ***contact form****, such that*

- $\xi = \ker \alpha$.
- $\alpha \wedge (d\alpha)^n \neq 0$.

The second condition is equivalent to the statement that $d\alpha$ is a symplectic form on each fiber of the bundle ξ. Together, these two conditions imply that the distribution ξ is non-integrable, which is the defining property of a contact structure.

Example 2.2. *Consider the standard contact structure on $\mathbb{R}^{2n+1}$, defined by the contact form $\alpha_{std} = dz - \sum_{i=1}^{n} y_i\, dx_i$. The corresponding contact planes $\xi_{std} = \ker \alpha_{std}$ at a point $(x, y, z) \in \mathbb{R}^{2n+1}$ are given by the equation $dz = \sum y_i\, dx_i$. When $n = 1$, at the origin, the contact form is $dz - y\, dx$, and the contact plane is given by $dz = 0$, a simple 2-plane in $\mathbb{R}^3$.*

The non-integrability of contact structures distinguishes them from their symplectic counterparts. However, locally, they all share the same canonical form, as shown by the Darboux Theorem.

Theorem 2.3 (Darboux Theorem)**.** *All contact structures on a $(2n+1)$-dimensional manifold are locally isomorphic to the standard contact structure.*

This result implies that the local geometry of contact manifolds is trivial. Thus, the interesting invariants of contact structures must be of a global nature. This is the idea behind the Gray Stability Theorem:

Theorem 2.4 (Gray Stability Theorem)**.** *Consider a one-parameter family of contact structures $\{\xi_t\}_{0\leqslant t\leqslant 1}$ on a closed manifold M. There exists a one-parameter family of diffeomorphisms ϕ^t of M such that $(\phi^t)_*\xi_t = \xi_0$ for all t, with $\phi_0 = id$.*

2.1.2 Contact Homology

Contact homologies are a class of invariants used to distinguish non-isotopic contact structures. They are constructed using geometric data related to the dynamics on the contact manifold. Before we define them, we consider some applications of these invariants.

- [Ustilovsky, 1999]: For S^{4m+1}, there exist infinitely many non-isotopic contact structures within each homotopy class of almost contact structures $(\xi, J, \xi \to \xi, J^2 = -\mathrm{id})$ where ξ is a hyperplane.
- [Bourgeois, 2004]: Similarly, T^5 and $T^2 \times S^3$ admit infinitely many non-isotopic contact structures in some homotopy class of almost contact structures.
- [Giroux, 1994; Eliashberg, Hofer, Givental, 2000]: On the 3-torus T^3, the contact forms $\alpha_n = \cos(2\pi nz)\,dx + \sin(2\pi nz)\,dy$ define contact structures $\xi_n = \ker \alpha_n$ that are pairwise non-isomorphic for different integers n.

One of the main objects in contact geometry is the Reeb vector field.

Definition 2.5. *Given a contact form α, the unique vector field R_α satisfying*

- $\alpha(R_\alpha) = 1.$
- $d\alpha(R_\alpha, \cdot) = 0.$

*is called the **Reeb vector field**. The first condition implies that R_α is positively transverse to the contact planes ξ, while the second means that the flow of R_α preserves the contact planes.*

Definition 2.6. *A periodic orbit of the Reeb vector field R_α is called a **Reeb orbit**.*

The existence of Reeb orbits is a basic question in contact geometry, formalized by the Weinstein Conjecture.

Conjecture 2.7 (Weinstein Conjecture)**.** *If a manifold M of dimension $2n+1$ is closed, then for any contact form α, there exists at least one Reeb orbit.*

Theorem 2.8 (Taubes, 2007)**.** *The Weinstein conjecture holds for $n = 1$.*

To construct a chain complex from Reeb orbits, we need to define a grading. This is achieved through the Conley-Zehnder index.

Definition 2.9. *Let γ be a Reeb orbit with period T, and let φ^t be the time-t flow of R_α. We say that γ is **non-degenerate** if the linearized return map $d\varphi^T : \xi_{\gamma(0)} \to \xi_{\gamma(T)}$ does not have 1 as an eigenvalue.*

Definition 2.10. *By symplectically trivializing the contact planes along a Reeb orbit γ, we obtain a path of symplectic matrices given by $d\varphi^t$. The **Conley-Zehnder index** $\mu_{cz}(\gamma)$ is an integer invariant defined for such a path.*

The general definition is complicated so we will only present simple examples:

Example 2.11. *In the case of a 3-dimensional manifold ($n = 1$), we can classify the Conley-Zehnder index based on the eigenvalues of the linearized return map.*

- *A positive hyperbolic orbit has eigenvalues that are positive real numbers. If the linearized flow $d\varphi^t(v)$ for an eigenvector v winds around the origin k times, the Conley-Zehnder index is $\mu_{cz} = 2k$.*
- *A negative hyperbolic orbit has negative real eigenvalues. If $d\varphi^t(v)$ winds around the origin $k + \frac{1}{2}$ times, the index is $\mu_{cz}(\gamma) = 2k + 1$.*
- *An elliptic orbit has non-real eigenvalues. If $d\varphi^t(w)$ for a vector w in the contact plane winds between k and $k + 1$ times, the index is $\mu_{cz}(\gamma) = 2k + 1$.*

Consider the actional functional $\mathcal{A} : C^\infty(s^t, M) \to \mathbb{R}, \gamma \mapsto \int_{S^1} \gamma^*(\alpha)$ with Reeb orbits crit A, with a complex structure $J : \xi$ acting on itself with $T^2 = -\mathrm{id}$ and $\langle u, v \rangle = d\alpha(u, Tv)$ for any $u, v \in \xi$. Then $\langle \cdot, \cdot \rangle$ defines an inner product which implies

- $d\alpha(u, v) = d\alpha(Ju, Jv)$
- $d\alpha(u, Ju) > 0$ for any $u \neq 0 \in \xi$.

Take $\eta_1, \eta_2 \in T_\gamma C^\infty(S^1, M)$ with

$$\langle \eta_1, \eta_2 \rangle = \int_{S^1} \langle \eta_1, \eta_2 \rangle + \alpha(\eta_1)\alpha(\eta_2)\,dt.$$

Take $u : \mathbb{R} \to C^\infty(S^1, M)$ with $s \in \mathbb{R}$ and $t \in C^\infty(S^1, M)$ with $\dim u(s)$ are Reeb orbits as $s \to \pm\infty$. Then

$$\frac{du}{ds} = -\text{grad } \mathcal{A}$$

with $u : \mathbb{R} \times S^1 \to M$. This gives

$$d(u^*\alpha \cdot j) = 0\pi_\xi u_s + J\pi_\xi u_t = 0$$

where j is a complex structure on $\mathbb{R} \times S^1$.

Let's require $u^* d\alpha \cdot j = da$ where $a : \mathbb{R} \times S^1 \to \mathbb{R}$. Let $\tilde{u} = (a, u) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$. Now, we extend J: $T(\mathbb{R} \times M) = \mathbb{R}(\partial a) \oplus \mathbb{R}(R_\alpha) \oplus \xi$ where $a \in \mathbb{R}$, and extend $J : \mathbb{R}(\partial_a) \to \mathbb{R}(R_\alpha)$. $\tilde{u}$ is J-holomorphic, i.e.

$$\overline{\partial}\tilde{u} = \frac{1}{2}(d\tilde{u} + J(\tilde{u}) d\tilde{u} \cdot j) = 0$$

or

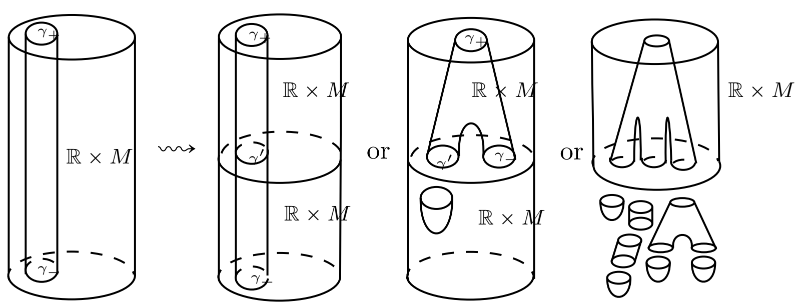
$$\tilde{u}_s + J(\tilde{u})\tilde{u}_t = 0$$

Next time, we will study the compactification of moduli spaces of J-holomorphic cylinders.

2.2 Cylindrical Contact Homology in Dimension Three via Obstruction Bundle Gluing

2.2.1 Compactification

Let $\tilde{\mathcal{M}}(\gamma_+, \gamma_-)$ denote the set of J-holomorphic curves from $\mathbb{R} \times S^1$ to $\mathbb{R} \times M$ that are asymptotic to the Reeb orbits γ_+ and γ_-. We are interested in the moduli space $\mathcal{M} = \tilde{\mathcal{M}}/\mathbb{R}$, where the action is by translation along the $\mathbb{R}$ factor. The compactness of these moduli spaces is crucial for counting them to define the boundary map in the chain complex. However, sequences of J-holomorphic cylinders can degenerate, or "break," into unions of simpler curves. This phenomenon is illustrated in the diagram below, showing a cylinder breaking into multiple components:



The leftmost figure represents a single J-holomorphic cylinder. The other figures represent a sequence of such cylinders converging to a "broken" curve, which consists of a concatenation of cylinders and spheres. The rightmost image depicts a single cylinder broken into multiple cylinders and spheres, with the bottom components mapping to $\mathbb{R}^n$.

The notion of Hofer energy provides a quantitative measure for the asymptotic behavior of these curves.

Definition 2.12. *Let (F, j) be a Riemann surface with a finite set of punctures $\dot{F} = F \backslash \{p_1, \ldots, p_k\}$, and let $u : (\dot{F}, j) \to (\mathbb{R} \times M, J)$ be a J-holomorphic curve. The **Hofer energy** of u is defined as*

$$E(u) = \sup_{\phi \in \mathcal{C}} \int_{\dot{F}} u^* d(\phi, \alpha),$$

where

$$\mathcal{C} = \left\{ \phi : \mathbb{R} \to [1,2] \;\middle|\; \begin{array}{l} \lim_{s \to -\infty} \phi(s) = 1, \\ \lim_{s \to +\infty} \phi(s) = 2, \\ \phi'(s) \geqslant 0 \quad \forall s \in \mathbb{R} \end{array} \right\}$$

Proposition 2.13. *The Hofer energy can be expressed in terms of the action of the asymptotic Reeb orbits:*

$$E(u) = 2\sum_{i=1}^{k_+}\mathcal{A}(\gamma_{+,i}) - \sum_{i=1}^{k_-}\mathcal{A}(\gamma_{-,i})$$
$$\geqslant 0.$$

The Hofer energy is finite if and only if the curve is well-behaved at its punctures.

Theorem 2.14. *If $u : (\dot{F}, j) \to \mathbb{R} \times M$ is a J-holomorphic curve, the following are equivalent:*

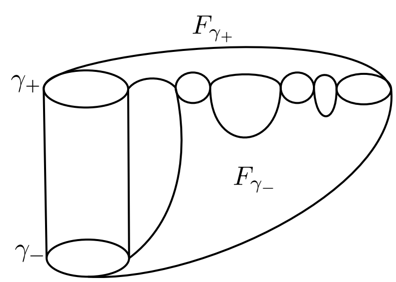
- *$E(u) < \infty$.*
- *For any puncture p of $\dot{F}$, either p is removable, or u converges to some Reeb orbit as the domain approaches p.*

2.2.2 Grading

To define a chain complex, we need a grading for the generators (Reeb orbits) and a well-defined boundary operator.

Assume $H_1(M;\mathbb{Z})$ is torsion-free. If $H_1(M) = 0$, then for each Reeb orbit γ, we can fix a capping surface $F_\gamma \subset M$ with $\partial F_\gamma = \gamma$. If $H_1(M;\mathbb{Z}) \neq 0$, we choose a basis $\{c_i\}$ for $H_1(M)$ and fix trivializations of ξ over these cycles. For any Reeb orbit γ, we choose a surface F_γ such that $[\partial F_\gamma] = [\gamma] - \sum n_i[c_i]$ in $H_1(M)$. A trivialization of $\xi|_\gamma$ that extends over F_γ and is compatible with the trivializations over the c_i allows for a well-defined Conley-Zehnder index $\mu_{CZ}(\gamma)$.

A J-holomorphic cylinder $u \in \mathcal{M}(\gamma_+, \gamma_-)$ together with capping disks F_{γ_+} and F_{γ_-} defines a closed surface whose image under u represents a homology class $A \in H_2(M)$.



Assuming there are no contractible Reeb orbits, we define a chain complex (C_*, ∂). The chain group C_* is the free $\mathbb{Q}[H_2(M)]$-module generated by "good" Reeb orbits. The grading of a generator γ is defined as $|\gamma| = \mu_{CZ}(\gamma) + n - 3$. The boundary operator is defined by

$$\partial\gamma_+ = \sum_{\gamma_-, A} \#\mathcal{M}^1_A(\gamma_+, \gamma_-)e^A \frac{1}{m(\gamma_-)}\gamma_-,$$

where $\mathcal{M}^1_A(\gamma_+, \gamma_-)$ is the moduli space of index 1 curves in the class A, $m(\gamma_-)$ is the multiplicity of γ_-, and e^A is the formal variable corresponding to $A \in H_2(M)$. The index is such that $|\gamma_+| - |\gamma_-| - |A| = 1$, where $|A| = -2c_1(\xi)[A]$.

Definition 2.15. *A Reeb orbit γ is* ***bad*** *if it is a multiple cover of an embedded orbit γ' and $\mu_{CZ}(\gamma) - \mu_{CZ}(\gamma')$ is odd. We exclude bad orbits from our set of generators.*

Example 2.16. *In dimension 3, an orbit γ is bad if and only if it is an even multiple cover of a negative hyperbolic orbit.*

If contractible Reeb orbits exist, the differential becomes more complex, counting configurations of curves with multiple outputs. This leads to the richer structure of (Rational) Symplectic Field Theory (SFT). The differential can take the form:

$$\partial\gamma_+ = \sum_{\gamma_-} \#\mathcal{M}(\gamma_+,\gamma_-)\gamma_- + \sum_{\gamma_{-1},\gamma_{-2}} \#\mathcal{M}(\gamma_+;\gamma_{-1},\gamma_{-2})\gamma_{-1}\gamma_{-2} + \dots$$

where the terms correspond to cylinders, pairs of pants, and other punctured Riemann surfaces.

Example 2.17. *In dimension three, if γ is a hyperbolic orbit, its Conley–Zehnder index is given by the formula*

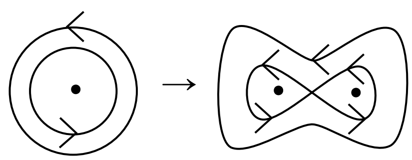
$$\mu_{cz}(\gamma,k) = k\mu_{cz}(\gamma),$$

where k is the covering number.

Let u be a J-holomorphic curve. Its Fredholm index is defined as $ind(u) = \mu(\gamma_+) - \mu(\gamma) - |A|$, where μ is the Conley–Zehnder index of the Reeb orbit and $|A|$ is the area of the curve. If u is a k-fold cover of a simple curve u' and there are no elliptic orbits, then the Fredholm index scales linearly with the covering number: $ind(u) = k \cdot ind(u')$.

If the index of the simple curve u' is greater than or equal to one and the curve intersects the index one manifold transversely, then $ind(u) \geqslant k$. From the definition of the boundary operator, an index of one for u implies that the covering number k must also be one.

We can always eliminate elliptic orbits up to any action.



The two illustrated contact manifolds have the same contact structure but are endowed with different contact forms.

2.3 Semi-Global Kuranishi Structure and Full Contact Homology

2.3.1 Introduction

When the moduli spaces of J-holomorphic curves are not regular (i.e., not smooth manifolds of the expected dimension), one cannot simply count their points to define homological invariants. The theory of Kuranishi structures provides a way to deal with such situations by constructing a virtual fundamental class for these moduli spaces.

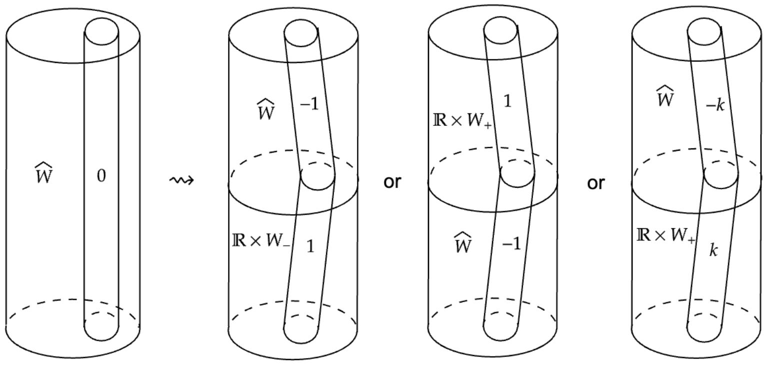
Consider an exact symplectic cobordism $(W, d\alpha)$ with boundary $\partial W = W_+ \cup W_-$, where $\alpha|_{W_\pm}$ is a contact form on $W_\pm$. Let J be a compatible almost complex structure on the completion $\widehat{W}$. Counting index-0 J-holomorphic cylinders in $\widehat{W}$ defines a chain map:

$$\Phi : C_*(W_+,\alpha_+,J_+) \to C_*(W_-,\alpha_-,J_-).$$

Given a 1-parameter family of such data $(W_t, d\alpha_t, J_t)$, one expects the induced maps Φ_0 and Φ_1 to be chain homotopic. Proving this requires analyzing the 1-dimensional moduli space of parameter-dependent solutions, which is given by

$$\mathcal{M}(\gamma_+,\gamma_-) = \bigsqcup_{t\in[0,1]} \{t\} \times \mathcal{M}^{\text{ind}=0}_{J_t}(\gamma_+,\gamma_-).$$

We aim to show the existence of a chain homotopy operator $K : C_*(W_+,\alpha_+,J_+) \to C_{*-1}(W_-,\alpha_-,J_-)$ such that $\Phi_0 - \Phi_1 = K\partial + \partial K$. This requires counting curves in a 1-dimensional moduli space whose boundaries correspond to broken configurations, as depicted in the figure below:



For the right-most cylinder shown in the figure, we can attach a configuration $\mathcal{U}$ to the top half and a configuration from a moduli space $\mathcal{M}$ to the bottom half. For each such $\mathcal{U}$, we have a map $\mathcal{S}^{-1} : \mathcal{O} \to [R, \infty) \times M$. A vector space is attached to the domain $\mathcal{O}$, defined by $\ker D_u^* / \mathbb{R}\langle Y \rangle$, where Y comes from the variation of the almost complex structure. The set $\mathcal{S}^{-1}(0)$ is then the set of curves in M that can be successfully glued with $\mathcal{U}$.

2.3.2 Semi-Global Kuranishi Structure

The concept is best introduced by analogy with the finite-dimensional case of Morse homology. Let (X, f, g) be a closed manifold with a Morse function and a Riemannian metric. The moduli space of gradient flow lines between critical points p, q is $\mathcal{M}(p, q) = (\mathcal{A}_q \cap \mathcal{D}_p)/\mathbb{R}$, where $\mathcal{A}_q$ is the ascending manifold of q and $\mathcal{D}_p$ is the descending manifold of p. When this intersection is not transverse, $\mathcal{M}(p, q)$ is not a manifold. An interior semi-global Kuranishi chart provides a local model for this space as the zero set of a section of a vector bundle.

Definition 2.18. *An **interior semi-global Kuranishi chart** for a space $\mathcal{M}$ is a quadruple $(K, \pi_V : E \to V, \mathcal{L}, \psi)$ where:*

1. *$K \subset \mathcal{M}$ is a compact subset.*
2. *$\pi_V : E \to V$ is a finite-rank vector bundle over a finite-dimensional manifold V.*
3. *$\mathcal{L} : V \to E$ is a smooth section.*
4. *$\psi : \mathcal{L}^{-1}(0) \to \mathcal{M}$ is a homeomorphism onto an open neighborhood of K.*
5. *The virtual dimension is given by $\dim V - \mathit{rank}\, E = \mathit{vir} \dim \mathcal{M}$.*

We order the moduli space $\mathcal{M}_1, \mathcal{M}_2, ...$ such that the energy increases. Let

$$\mathcal{M}_i = \mathcal{M}(p_i, q_i), \qquad E(\mathcal{M}_i) = f(p_i) - f(q_i).$$

and suppose we have an index tuple $I = (i_1, ..., i_n)$ such that $p_{i_m} = q_{i_{m+1}}$. Suppose $S \subset I$ is a subindex tuple. Then I/S is the index tuple obtained by replacing S by an integer which is the index of $\mathcal{M}(p_{s_1}, q_{s_k})$ if $S = (s_1, ..., s_k)$, and $\vec{S} \subset I$ is a disjoint union of subindex tuples. We say $I < J$ if there exists $\vec{S} \subset J$ such that $I = J/\vec{S}$.

Definition 2.19. *A **semi-global Kuranishi structure for** $\mathcal{M}_1, ..., \mathcal{M}_p$ for any $1 \leqslant i \leqslant p$ such that*

1. *For each index tuple I corresponding to a stratum of $\mathcal{M}_i$ (denoted $I/I = i$), there is an associated Kuranishi chart:*
$$C_I = (\pi_I : E_I \to V_I, \mathcal{L}_I : V_I \to E_I, \psi_I : \mathcal{L}_I^{-1}(0) \to \mathcal{M}_i).$$
2. *For each pair of index tuples $I' < I$, there exists a coordinate change map defined on an open subset $V_{I'I} \subset V_{I'}$. This map consists of a bundle map $\phi^{\#}_{I'I}$ and an embedding $\phi_{II'}$ that form the following*

commutative diagram:

$$\begin{tikzcd} & E_{I'}|_{V_{I'I}} \arrow[r, "\phi^{\#}_{I'I}"] \arrow[d] & E_I \arrow[d] \\ V_{I'} \supset & V_{I'I} \arrow[r, hook, "\phi_{II'}"] \arrow[u, bend left, "\mathcal{L}_{I'}"] & V_I \arrow[u, bend right, "\mathcal{L}_I"'] \end{tikzcd}$$

where the bundle map $\phi^{\#}_{I'I}$ is injective.

3. *The sections are compatible with the coordinate change:*

$$\mathcal{L}_I \circ \phi_{I'I} = \phi^{\#}_{I'I} \circ \mathcal{L}_{I'}|_{V_{I'I}}.$$

4. *The linearization of the section, $(d\mathcal{L}_I)_*$, descends to an isomorphism:*

$$(d\mathcal{L})_I : TV_I/TV_{I \cdot I} \xrightarrow{\cong} E_I/E_I.$$

5. *The composition of coordinate change maps, $(\phi_{I'I}, \phi^{\#}_{I'I})$, is associative.*
6. *The union of the zero sets of the charts covers the moduli space $\mathcal{M}_i$:*

$$\mathcal{M}_i = \bigcup_{I, I/I=i} \psi_I(\mathcal{L}_I^{-1}(0)).$$

2.3.3 Strata Compatibility

Definition 2.20. *For an index tuple $I = (i_1, \ldots, i_m)$, we form the product spaces corresponding to the unglued components:*

$$\begin{aligned} \mathbb{V}_I &= V_{i_1} \times \cdots \times V_{i_m} \times [R, \infty)^{m-1}, \\ \mathbb{E}_I &= E_{i_1} \oplus \cdots \oplus E_{i_m}. \end{aligned}$$

Suppose that G_I a diffeomorphism onto its image, $G_I^{\#}$ is a bundle isomorphism, and $\mathcal{L}_I$ are C'-close as $T_1, ..., T_{m-1} \to \infty$.

*For all i, a bundle map satisfies the **strata compatibility** conditions if the following map commutes*

$$\begin{tikzcd} \mathbb{E}_I \arrow[r, "G_I^{\#}"] \arrow[d] & E_I \arrow[d] \\ \mathbb{V}_I \arrow[r, "G_I"'] \arrow[u, bend left, "(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m)"] & V_I \arrow[u, bend right, "\mathcal{L}_I"'] \end{tikzcd}$$

Here, the perturbation section is

$$\sigma = \{G_I : V_I \to F_I \mid I/I = i\}$$

where σ_I transverses $\mathcal{L}_I$, σ_I is small, the section is compatible with $(\phi_{I'I}, \phi^{\#}_{I'I})$, and both σ_I and $(\sigma_{i_1, i_2, \ldots, i_n})$ are C^1-close as $T_1, ..., T_{n-1} \to \infty$. The resulting perturbed moduli space is

$$Z_i = \coprod_{I/I=i} \mathcal{L}_I^{-1}(\sigma_I)/\sim$$

2.3.4 Construction of Semi-Global Kuranishi Structures for Cylindrical Contact Homology

Suppose we have an interior chart $\mathcal{E} \to \mathcal{B}$ and $\bar{\partial}$ on the inverse.

Theorem 2.21. *Given any compact set $K \subset \mathcal{M}$, there exists a finite rank subbundle E of the relevant obstruction bundle over a neighborhood of K such that the section defined by the $\bar{\partial}$-operator is transverse to E.*

Sketch of Proof. The core idea is to model the non-linear equation $\bar{\partial}_J u = 0$ as the zero set of a section transverse to a finite-dimensional vector bundle.

First, we analyze the equation in the asymptotic region of a curve u. For a small constant $\epsilon > 0$, we can find $s_0 \in \mathbb{R}$ such that for $s > s_0$, the action $\mathcal{A}(u(s - s_0))$ is within ϵ of its asymptotic value $\mathcal{A}(\gamma_+)$. In this regime, for a suitable choice of J, the equation $\bar{\partial}_J u = 0$ can be approximated by its linearization:

$$\partial_s u + Au = 0,$$

where A is a linear, self-adjoint operator acting on sections along the asymptotic Reeb orbit.

Let $\{\lambda_i\}_{i\in\mathbb{Z}}$ and $\{f_i\}_{i\in\mathbb{Z}}$ be the eigenvalues and corresponding eigenvectors of A, ordered such that $\cdots \leqslant \lambda_{-1} < 0 < \lambda_1 \leqslant \ldots$. Using these, we construct a family of sections of the relevant cotangent bundle:

$$\tilde{f}_j(s,t) = \beta(s) f_k(t) \otimes (ds - i\,dt),$$

where $\beta(s)$ is a cutoff function that localizes the section to the asymptotic region.

The key step is to define a finite-dimensional vector bundle E as the span of a suitable finite collection of these sections, $E = \operatorname{span}\{\tilde{f}_1, \ldots, \tilde{f}_\ell\}$. One can then show that the section defined by $\bar{\partial}$ is transverse to E. The domain of the Kuranishi chart, V, is then defined as the preimage of this bundle:

$$V := \bar{\partial}^{-1}(E).$$

By construction, V is a finite-dimensional manifold, and the pair (V, E) with the section $\bar{\partial}$ forms the local chart.

Finally, this local construction must be shown to be compatible with the gluing operations that define the boundary strata of the moduli space. We do this by showing the commutativity of the following diagram, which relates the charts for broken trajectories to the chart for the glued trajectory:

$$\begin{tikzcd}
\mathcal{O}_+ \oplus \mathcal{O}_- \arrow[r, "G_\#"'] \arrow[d] & \mathcal{O}_{+-} \arrow[d] \\
\mathcal{M}_+ \times \mathcal{M}_- \times [\mathbb{R}, \infty) \arrow[r, "G"'] \arrow[u, bend left, "{(0,0)}"] & V \arrow[u, bend right, "\mathcal{L}"']
\end{tikzcd}$$

This ensures that the charts defined for different strata fit together properly. $\square$

3 Mike Usher: Quantitative Symplectic Geometry

There were three lectures:

1. Day 1: Symplectic Embedding Obstructions and Constructions

   The question of which subsets of $\mathbb{R}^{2n}$ embed symplectically into which others has turned out to be quite rich and has led to the development of many techniques over the past 40 years. In my first lecture, I will explain proofs of classic results of Gromov that give obstructions to symplectic squeezing and packing, and will contrast this with cases where an explicit construction allows one to give a non-obvious positive answer to a symplectic embedding question.

2. Day 2: Capacities and Symplectic Homology

   The second lecture will formally introduce the notion of a symplectic capacity, and will discuss two examples of these: the Hofer-Zehnder capacity based on periodic orbits of Hamiltonian systems, and the Floer-Hofer-Wysocki capacity based on symplectic homology.

3. Day 3: Obstructing Embeddings Using Equivariant Symplectic Homology

   The third lecture will explain how S^1-equivariant symplectic homology supplies additional restrictions on symplectic embeddings, both via a sequence of capacities coming from spectral invariants associated to various homology classes, and via chain-level information that vanishes in homology but can in some cases be used to show that two known embeddings are not symplectically isotopic.

3.1 Symplectic Embedding Obstructions and Constructions

3.1.1 Introduction

Quantitative symplectic geometry is concerned with several fundamental questions regarding the existence and properties of symplectic embeddings. Important examples of such questions include:

1. Suppose $X(\vec{r})$ and $Y(\vec{s})$ are symplectic manifolds depending on parameters $\vec{r}$ and $\vec{s}$. For what values of these parameters do there exist symplectic embeddings $X(\vec{r}) \hookrightarrow Y(\vec{s})$? For instance, for which $a > 0$ and $s_j > 0$ does a symplectic embedding exist from the ball

$$X(a) = B^{2n}(a) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \mid \pi \sum_{j=1}^{n} (x_j^2 + y_j^2) \leqslant a \right\}$$

   into the ellipsoid

$$Y(\vec{s}) = E^{2n}(s_1, \ldots, s_n) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \mid \pi \sum_{j=1}^{n} \frac{x_j^2 + y_j^2}{s_j} \leqslant 1 \right\}?$$

2. If $X \subset \mathbb{R}^{2n}$ is a domain with a contact-type boundary, what can be said about the action of closed characteristics on ∂X? What is the connection between this question and the previous one?

3. For a Hamiltonian diffeomorphism $\phi : M \to M$, what is the asymptotic behavior of the number of fixed points of its iterates, $\#\mathrm{Fix}(\phi^k)$, and the Hofer norm, $||\phi^k||_{\mathrm{Hofer}}$, as $k \to \infty$?

For the remainder of this discussion, we will focus primarily on the first question, which led to the first result that got mathematicians interested in studying quantitative symplectic geometry.

Theorem 3.1 (Gromov's Non-Squeezing Theorem, 1985)**.** *Let the standard ball and cylinder in $\mathbb{R}^{2n}$ be defined as*

$$B^{2n}(a) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \mid \pi \sum_{j=1}^{n} (x_j^2 + y_j^2) \leqslant a \right\}$$

and

$$Z^{2n}(A) = \left\{ (\vec{x}, \vec{y}) \in \mathbb{R}^{2n} \mid \pi (x_1^2 + y_1^2) \leqslant A \right\} = B^2(A) \times \mathbb{R}^{2n-2}.$$

A symplectic embedding $B^{2n}(a) \hookrightarrow Z^{2n}(A)$ exists only if $a \leqslant A$.

Basically, this result states that it is impossible to deform a ball to fit into a cylinder of a smaller radius while preserving the symplectic structure.

3.1.2 Proof of Gromov's Non-Squeezing Theorem

We present a sketch of the proof. Suppose there exists a symplectic embedding $\phi : B^{2n}(a) \hookrightarrow Z^{2n}(A)$. For any $\epsilon > 0$, we wish to show that $a \leqslant A$; it suffices to show $a - \epsilon < A + \epsilon$.

Choose a length L such that the image of ϕ is contained in $B^2(A) \times (-L, L)^{2n-2}$. We can regard $B^2(A)$ as a subset of the 2-sphere $S^2(A+\epsilon)$ of area $A+\epsilon$. The map ϕ can then be viewed as embedding into the symplectic manifold $(M, \omega) = (S^2(A+\epsilon) \times (\mathbb{R}/2L\mathbb{Z})^{2n-2}, \omega_{\text{std}} \oplus \omega_{\text{std}})$.

The proof relies on two key facts:

Proposition 3.2 (Fact #1)**.** *For any ω-compatible almost complex structure J on M, there exists a J-holomorphic map $u : S^2 \to M$ such that $\phi(\vec{0}) \in Im(u)$ and $u_*[S^2] = [S^2 \times \{pt\}] \in H_2(M)$. This holds in particular for any J that agrees with $\phi_* J_0$ on the image $\phi(B^{2n}(a - \epsilon/2))$, where J_0 is the standard complex structure.*

Proposition 3.3 (Fact #2)**.** *For any J_0-holomorphic map $v : \Sigma \to B^{2n}(c)$, where Σ is a compact surface with boundary and $c \in (a-\epsilon, a-\epsilon/2)$, such that $v(\partial\Sigma) \subset \partial B^{2n}(c)$ and $\vec{0} \in Im(v)$, the area of the image is bounded below: $Area(v) \geqslant c$.*

Assuming these facts, the theorem follows. For a generic $c \in (a-\epsilon, a-\epsilon/2)$, let $\Sigma = u^{-1}(\phi(B^{2n}(c)))$. Define $v : \Sigma \to B^{2n}(c)$ by $v = \phi^{-1} \circ u|_\Sigma$. Then

$$a - \epsilon < c \leqslant \text{Area}(v) = \int_\Sigma v^*\omega_0 = \int_\Sigma (\phi^{-1} \circ u)^*\omega_0 = \int_\Sigma u^*\omega = \text{Area}(u|_\Sigma) \leqslant \text{Area}(u) = A + \epsilon.$$

Since this holds for any $\epsilon > 0$, we conclude that $a \leqslant A$.

Proof Sketch of Fact 1. For any ω-compatible J, consider the moduli space of curves passing through the specified point:

$$\mathcal{M}_J = \left\{ u : S^2 \to M \mid u_*[S^2] = [S^2 \times \{\text{pt}\}], u(\text{origin}) = \phi(\vec{0}) \right\}.$$

If $J = J_0$ is a standard complex structure, $\mathcal{M}_{J_0}$ consists of a single element. For contradiction, suppose $\mathcal{M}_{J_1} = \varnothing$ for some other compatible J_1. For a generic path of almost complex structures $\{J_t\}_{t\in[0,1]}$ from J_0 to J_1, the parameterized moduli space $\bigcup_{t\in[0,1]}\{t\} \times \mathcal{M}_{J_t}$ would be a compact 1-manifold whose boundary is $\mathcal{M}_{J_0} \cup \mathcal{M}_{J_1}$. But this implies its boundary consists of a single point, which is impossible as compact 1-manifolds have an even number of boundary points. ☐

The proof of Fact 2 is omitted.

3.1.3 4-Dimensional Packing Problem

Problem 3.4 (4-Dimensional Packing Problem)**.** *Given $k \in \mathbb{N}$ and $a > 0$, does there exist a symplectic embedding of k disjoint copies of the 4-ball of capacity a into the unit 4-ball?*

$$\coprod_k B^4(a) \hookrightarrow B^4(1)$$

By a result of McDuff and Polterovich, we have the identification $B^4(1) \cong \mathbb{CP}^2(1)$, the complex projective plane of area 1. The volume of $\coprod_k B^4(a)$ is $k\frac{a^2}{2}$, while $\text{Vol}(B^4(1)) = \frac{1}{2}$. The fraction of the total volume filled by the packing is ka^2. The question is whether this fraction can approach 1.

Theorem 3.5 (2-Ball Theorem)**.** *If $k = 2$, an embedding $\coprod_2 B^4(a) \hookrightarrow B^4(1)$ exists only if $a \leqslant \frac{1}{2}$.*

Proof. We present two distinct proof sketches.

1. Let $\phi_1, \phi_2 : B^4(a) \hookrightarrow \mathbb{CP}^2(1)$ be symplectic embeddings with disjoint images. One can construct an almost complex structure J that agrees with $\phi_{1*}J_0$ and $\phi_{2*}J_0$ on the respective images. There exists a J-holomorphic curve u passing through both $\phi_1(\vec{0})$ and $\phi_2(\vec{0})$ and representing the class of a line $[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$. The area of this curve must be the sum of its areas inside each ball and its area outside, giving
$$1 = \mathrm{Area}(u) \geqslant a + a \implies a \leqslant \frac{1}{2}.$$

2. By blowing up the centers of the two embedded balls, we obtain a symplectic manifold $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$. The symplectic form is such that the class of the proper transform of the line, $L = [\mathbb{CP}^1]$, has area 1, and the two exceptional divisors E_1, E_2 both have area a. The class $L - E_1 - E_2$ can be represented by a holomorphic sphere, which must have non-negative area. This implies the area inequality $1 - a - a \geqslant 0$, which gives $a \leqslant \frac{1}{2}$.

□

3.2 Capacities and Symplectic Homology

3.2.1 Symplectic Capacities

Let $\mathcal{C}$ be a collection of $2n$-dimensional symplectic manifolds closed under scaling of the symplectic form; that is, if $(M, \omega) \in \mathcal{C}$, then $(M, a\omega) \in \mathcal{C}$ for all $a > 0$.

Definition 3.6. *A **symplectic capacity** on $\mathcal{C}$ is a function $c : \mathcal{C} \to [0, \infty]$ satisfying:*

1. ***Monotonicity:** If there exists a symplectic embedding $(M, \omega) \hookrightarrow (M', \omega')$, then $c(M, \omega) \leqslant c(M', \omega')$.*
2. ***Conformality:** For any $a > 0$, $c(M, a\omega) = a \cdot c(M, \omega)$.*
3. ***Nontriviality:** $c(B^{2n}(1)) > 0$ and $c(Z^{2n}(1)) < \infty$.*

Remark 3.7. *A stronger version of nontriviality is **normalization**, which requires $c(B^{2n}(1)) = c(Z^{2n}(1)) = 1$.*

When $\mathcal{C}$ consists of subsets of $(\mathbb{R}^{2n}, \omega_0)$, the conformality axiom is often phrased in terms of scaling the domain: for a domain U, let $aU := \{\sqrt{a}\vec{z} \mid \vec{z} \in U\}$. Then $c(aU) = a \cdot c(U)$.

Let's look at some examples of capacities:

Example 3.8 (Gromov Capacity)**.** *The **Gromov capacity** is defined as*
$$c_B(M, \omega) := \sup\{a \mid \exists \text{ a symplectic embedding } B^{2n}(a) \hookrightarrow (M, \omega)\}.$$

One major source of symplectic capacities are Hamiltonian systems. Suppose (M, ω) is a symplectic manifold and $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ is a smooth Hamiltonian, denoted $(t, m) \mapsto H_t(m)$. The corresponding time-dependent Hamiltonian vector field X_{H_t}, given by $\omega(\cdot, X_{H_t}) = dH_t$, generates a Hamiltonian flow $\varphi_H^t : M \to M$ satisfying
$$\frac{d\varphi_H^t(m)}{dt} = X_{H_t}(\varphi_H^t(m)).$$
We can study symplectic capacities via filtered Floer homology or symplectic homology, which are built from the dynamics of such flows.

Example 3.9 (Hofer-Zehnder Capacity)**.** *The **Hofer-Zehnder capacity** is defined for compactly supported Hamiltonians. Let $\mathcal{H}(M)$ be the set of smooth Hamiltonians $H : M \to \mathbb{R}$ that are compactly supported and vanish on some non-empty open set. Then*
$$c_{HZ}(M, \omega) := \sup\{\max H \mid H \in \mathcal{H}(M) \text{ and all 1-periodic orbits of } X_H \text{ are constant}\}.$$

3.2.2 (Filtered) Floer Homology

For the remainder of this section, we assume (M,ω) is an exact symplectic manifold, $\omega = d\lambda$. For example, take the Euler vector field $E_X, M \subset \mathbb{R}^{2n}$, and $\lambda = \lambda_0 := \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$. For a 1-periodic Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, Floer homology, $\mathrm{HF}(H)$, is defined as the Morse homology of the action functional $\mathcal{A}_H : C^\infty(\mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$, given by

$$\mathcal{A}_H(\gamma) = -\int_{S^1} \gamma^*\lambda + \int_0^1 H(t,\gamma(t))\,dt.$$

The critical points of $\mathcal{A}_H$ are the 1-periodic orbits of the Hamiltonian vector field X_{H_t}:

$$\mathrm{Crit}(\mathcal{A}_H) = \{\gamma : \mathbb{R}/\mathbb{Z} \to M \mid \dot{\gamma}(t) = X_{H_t}(\gamma(t))\}.$$

There is a 1-to-1 correspondence between $\mathrm{Crit}(\mathcal{A}_H)$ and $\mathrm{Fix}(\varphi'_H)$ given by $\gamma \mapsto \gamma(0)$. The negative gradient flow lines of $\mathcal{A}_H$ are solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ to Floer's equation:

$$\frac{\partial u}{\partial s} + J_t(u(s,t))\left(\frac{\partial u}{\partial t} - X_{H_t}(u(s,t))\right) = 0.$$

The Floer chain complex, $\mathrm{CF}(H)$, is the vector space spanned by $\mathrm{Crit}(\mathcal{A}_H)$, graded by the Conley-Zehnder index μ_{CZ}. The boundary operator $\partial : \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$ is defined on a generator γ_- by

$$\partial\gamma_- = \sum_{\gamma_+} \#(\text{flow lines from } \gamma_- \text{ to } \gamma_+)\gamma_+,$$

where the sum is over generators γ_+ with index difference $\mathrm{ind}(\gamma_-) - \mathrm{ind}(\gamma_+) = 1$.

Proposition 3.10. *Under suitable compactness and transversality assumptions:*

- *The differential satisfies $\partial^2 = 0$, which gives the Floer homology $HF(H)$.*
- *For Hamiltonians H and H' satisfying the same asymptotic conditions, there exist continuation maps $CF(H) \leftrightarrows CF(H')$ inducing isomorphisms on homology.*

The filtered version can be given as follows: The action functional provides a natural filtration on the Floer complex. For any real number $t \in \mathbb{R}$, we define the **filtered Floer complex** as the subspace

$$CF^t(H) := \mathrm{span}\left\{\gamma \in \mathrm{Crit}(\mathcal{A}_H) \mid \mathcal{A}_H(\gamma) \leqslant t\right\}.$$

Since the action functional $\mathcal{A}_H$ strictly decreases along non-constant negative gradient flow lines, the Floer differential maps this subcomplex to itself, i.e., $\partial(CF^t(H)) \subset CF^t(H)$. Consequently, we may take the homology of this subcomplex, which defines the **filtered Floer homology**, $HF^t(H)$, for each $t \in \mathbb{R}$.

The natural inclusions of chain complexes $CF^s(H) \hookrightarrow CF^t(H)$ for $s \leqslant t$ induce homomorphisms on homology, $HF^s(H) \to HF^t(H)$. Furthermore, if H, H' are two Hamiltonians such that $H \geqslant H'$, there exist continuation chain maps $CF^t(H) \to CF^t(H')$ that respect the filtration level. These two structures are compatible, as expressed by the following commutative diagram for any $s \leqslant t$ and $H \geqslant H'$:

$$\begin{array}{ccc}
\mathrm{HF}^s(H) & \longrightarrow & \mathrm{HF}^s(H') \\
\downarrow & & \downarrow \\
\mathrm{HF}^t(H) & \longrightarrow & \mathrm{HF}^t(H')
\end{array}$$

3.2.3 Liouville Embeddings

Definition 3.11. *A **Liouville domain** (W,ω) is a compact manifold with boundary, equipped with a 1-form λ such that $\omega = d\lambda$ is symplectic and the associated Liouville vector field V_λ (defined by $\iota_{V_\lambda}\omega = \lambda$) points strictly outwards along ∂W.*

Example 3.12. *For $\lambda_0 = \frac{1}{2}\sum(x_j\, dy_j - y_j\, dx_j)$, we have $V_{\lambda_0} = \frac{1}{2}\sum(x_j \partial x_j + y_j \partial_{y_j})$, so we can take W to be the strongly star-shaped around the region in $\mathbb{R}^{2n}$.*

In this case, $\alpha := \lambda\,|_{\partial W}$ for $\alpha \in \Omega^1(\partial W)$ is a contact form on ∂W, and there exists a collar neighborhood U of ∂W such that $(U, \lambda) \approx ([1-\epsilon, 1] \times \partial W, r\alpha)$. From the completion, we have

$$\left(\hat{W}, \hat{\lambda}\right) = (W, \lambda) \bigcup_{\partial W} ([1, \infty) \times \partial W, r\alpha).$$

Example 3.13. *Let $\lambda_0 = \frac{1}{2}\sum_j(x_j\, dy_j - y_j\, dx_j)$ be the standard Liouville form on $\mathbb{R}^{2n}$. The corresponding Liouville vector field is the radial vector field $V_{\lambda_0} = \frac{1}{2}\sum_j(x_j \partial_{x_j} + y_j \partial_{y_j})$. Consequently, any strongly star-shaped domain in $\mathbb{R}^{2n}$ is a Liouville domain with respect to the restriction of λ_0.*

In this setting, the restriction $\alpha := \lambda|_{\partial W}$ is a contact form on the boundary ∂W. There exists a collar neighborhood U of ∂W in W on which the Liouville structure is standard, i.e., $(U, \lambda) \cong ([1-\epsilon, 1] \times \partial W, r\alpha)$. This allows for the construction of the completion of W, a non-compact manifold defined by attaching a cylindrical end:

$$(\hat{W}, \hat{\lambda}) := (W, \lambda) \cup_{\partial W} ([1, \infty) \times \partial W, r\alpha).$$

For a strongly star-shaped domain in $\mathbb{R}^{2n}$, the completion $\hat{W}$ is symplectomorphic to $\mathbb{R}^{2n}$. The relation $\hat{W} \simeq \mathbb{R}^{2n}$ is called the ***Liouville isomorphism****.*

The homology theory is defined using a specific class of Hamiltonians on the completion.

Definition 3.14. *Given a Liouville domain (W, λ), a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \hat{W} \to \mathbb{R}$ is W***-admissible** *if it satisfies two conditions:*

1. *H is non-negative on W.*
2. *On the cylindrical end $[1, \infty) \times \partial W$, the Hamiltonian is linear in the radial coordinate. That is, there exist constants $a > 0$ and $b \in \mathbb{R}$ such that for all $(r, y) \in [1, \infty) \times \partial W$,*

   $$H(t, (r, y)) = ar + b,$$

   where a is chosen not to be the period of any closed Reeb orbit of the contact form α.

Remark 3.15. *The condition on the slope a ensures that all 1-periodic orbits of the Hamiltonian flow are contained within W. The linear behavior on the end implies that the Hamiltonian vector field is purely rotational: $X_H = aR_\alpha$ on $(1, \infty) \times \partial W$, where R_α is the Reeb vector field.*

Let $\mathcal{H}_W$ be the set of all W-admissible Hamiltonians. This set is directed under the pointwise inequality $H \geqslant H'$. For any such pair, there exist continuation maps $HF^t(H) \to HF^t(H')$ for each filtration level $t \in \mathbb{R}$.

Definition 3.16. *For a Liouville domain W and any $t \in \mathbb{R}$, the* ***symplectic homology*** *$SH^t(W)$ is defined as the direct limit of the filtered Floer homology groups over the directed set of admissible Hamiltonians:*

$$SH^t(W) := \varinjlim_{H \in \mathcal{H}_W} HF^t(H).$$

The resulting homology groups form a filtered system, with maps $SH^s(W) \to SH^t(W)$ for all $s \leqslant t$. This structure is a powerful invariant of the Liouville domain:

Proposition 3.17. *The completed symplectic homology, $SH^\infty(W)$, depends only on the Liouville isomorphism type of the completion $\hat{W}$. For any strongly star-shaped domain $W \subset \mathbb{R}^{2n}$, $SH^\infty(W) = 0$.*

Symplectic homology is functorial with respect to Liouville embeddings.

Proposition 3.18. *If W' is a Liouville domain contained in the interior of W, there are induced* ***transfer maps***

$$SH^t(W) \to SH^t(W').$$

More generally:

Proposition 3.19. *A Liouville embedding $\varphi : W' \hookrightarrow \text{int}(W)$ (meaning $\varphi^*\lambda - \lambda' = df$ for some function f) induces transfer maps $\varphi_* : SH^t(W) \to SH^t(W')$.*

3.2.4 Floer-Hofer-Wysocki Capacity

We discuss the Floer-Hofer-Wysocki Capacity for a strongly star-shaped $W \subset \mathbb{R}^{2n}$. We begin with the following result:

Proposition 3.20. *For a strongly star-shaped domain $W \subset \mathbb{R}^{2n}$, for sufficiently small $\epsilon > 0$, the low-level filtered symplectic homology is isomorphic to the relative homology of the domain:*

$$SH^{\epsilon}_*(W) \cong H_{*+n}(W, \partial W).$$

In particular, $SH^{\epsilon}_n(W) \cong \mathbb{Q}$.

This motivates the following definition:

Definition 3.21. *The **Floer-Hofer-Wysocki capacity** of a strongly star-shaped Liouville domain $W \subset \mathbb{R}^{2n}$ is*

$$c_{FHW}(W) := \inf\{t \in \mathbb{R} \mid \text{the map } SH^{\epsilon}_n(W) \to SH^t_n(W) \text{ is zero for small } \epsilon > 0\}.$$

Theorem 3.22. *c_{FHW} defines a symplectic capacity.*

Proof. We verify that c_{FHW} satisfies the three axioms of a symplectic capacity.

- **Monotonicity:** Let $W' \hookrightarrow \text{int}(W)$ be a Liouville embedding. This induces transfer maps on symplectic homology which fit into the following commutative diagram for any $t \in \mathbb{R}$ and small $\epsilon > 0$:

$$\begin{tikzcd} \mathrm{SH}^{\epsilon}(W) \arrow[r, "\cong", "\varphi'"'] \arrow[d] & \mathrm{SH}^{\epsilon}(W') \arrow[d] \\ \mathrm{SH}^{t}(W) \arrow[r, "\varphi"'] & \mathrm{SH}^{t}(W') \end{tikzcd}$$

  By definition, if $t > c_{\text{FHW}}(W)$, the left vertical map is zero. By commutativity, the composition $SH^{\epsilon}_n(W) \to SH^{\epsilon}_n(W') \to SH^t_n(W')$ is also zero. Since the top horizontal map is an isomorphism (as both groups are isomorphic to the top relative homology of the domains), the right vertical map must be zero. This implies that $t \geqslant \inf\{t' \mid SH^{\epsilon}_n(W') \to SH^{t'}_n(W') \text{ is zero}\} = c_{\text{FHW}}(W')$. Therefore, $c_{\text{FHW}}(W) \geqslant c_{\text{FHW}}(W')$, establishing monotonicity.

- **Conformality:** The property $c_{\text{FHW}}(W, a\omega) = a \cdot c_{\text{FHW}}(W, \omega)$ follows from the behavior of the action functional under scaling of the symplectic form. A scaling of ω by a factor of a scales the action values by the same factor, and thus the filtration levels defining the capacity are scaled accordingly.

- **Nontriviality:** The conditions $c_{\text{FHW}}(B^{2n}(1)) > 0$ and $c_{\text{FHW}}(Z^{2n}(1)) < \infty$ hold by the construction of symplectic homology and its relation to the Reeb dynamics on the boundaries of these standard domains.

This completes the proof.

□

Example 3.23. *For the standard ellipsoid $W = E(a_1, \ldots, a_n) = \left\{\sum_{j=1}^n \frac{\pi(x_j^2+y_j^2)}{a_j} \leqslant 1\right\}$ with $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, the Floer-Hofer-Wysocki capacity is given by the smallest action value:*

$$c_{FHW}(W) = a_1.$$

3.3 Obstructing Embeddings Using Equivariant Symplectic Homology

3.3.1 Symplectic Homology

Suppose we have (W, λ) a Liouville domain, $\eta = \partial W$, a contact form $\alpha = \lambda_Y$, and C a scalar of Y such that $(C\lambda) \cong ((1-\epsilon, 1] \times Y, r\alpha)$. The completion is

$$\hat{W} := W \bigcap_Y ([1, \infty] \times Y, r\alpha).$$

Define

$$\mathcal{H}_W := \{W\text{-admissible Hamiltonians}\}$$

where $H : \mathbb{R}/\mathbb{Z} \times \hat{W} \to \mathbb{R}$ such that $H \mid_{\mathbb{R}/\mathbb{Z} \times H} \geqslant 0$ and for $r \geqslant 1, y \in Y$, $H(t, (r, y)) = -ar + b$ for $a > 0, b \in \mathbb{R}$ such that a is not the closed period of any Reeb orbit for α.

Suppose $H, H \in \mathcal{H}_W$ and $H \geqslant H'$. Then for $t \in \mathbb{R}$ autonomous, we have construction maps $\mathrm{HF}^t(H) \to \mathrm{HF}^t(H')$.

Definition 3.24.

$$SH^t(W) = \varinjlim_{H \in \mathcal{H}_W} HF^t(H).$$

A family of H's approaching the limit might look like the following: take $H \mid_W$ to be a small Morse function $W \to [0, \infty)$ that is C^2 small in the complement of the collar, with $H \mid_{\partial W} = 0$.

- On $C \cong (1-\epsilon, 1] \times Y$, we have $H(r, y) = -h(r)$, where h' increases rapidly from $\delta \approx 0$ to $a \gg 0$.
- On $\hat{W} \backslash W = (1, \infty) \times Y$, we have $H(r, y) = -a(r-1)$.

Note that the Hamiltonian vector field given by $H(r, y) = -h(r)$ is $X_H = -h'(r) R_\alpha$.

Recall that for $\gamma : \mathbb{R}/\mathbb{Z} \to M$,

$$\mathcal{A}_H(\gamma) = -\int_{\mathbb{R}/\mathbb{Z}} \gamma^* \lambda + \int_0^1 H(t, \gamma(t))\, dt.$$

For these H, Crit $\mathcal{A}_H$ consist of

- Constants at critical points of $H_{W \backslash C}$ with $\mathcal{A}_H = H(p) \approx 0$.
- In $C = [1-\epsilon, 1] \times Y$, the critical points are given by reparameterizations of closed orbits of R_α with period $h'(r)$, with $\mathcal{A}_H \approx h'(r)$.

If $t_0 <$ the minimal action of a Reeb orbit,

$$\begin{aligned} \mathrm{SH}^t_*(W) &= H_{*+n}(\text{Morse complex of } H \mid_W) \\ &= H_{*+n}(W, Y). \end{aligned}$$

Once t is bigger than the minimal period of a Reeb orbit, SH^t is affected by the Reeb orbits, any of which gives an S^1 family in $\mathrm{Crit}(\mathcal{A}_H)$. Morse-Bott perturbation splits these into two orbits different in index by 1, both with action approximately the period of the original orbit.

Example 3.25. *Consider the ellipsoid*

$$E(a_1, ..., a_n) = \left\{ \sum_j \frac{\pi(x_j^2 + y_j^2)}{a_j} \leqslant 1 \right\}.$$

with $a_1 < ... < a_n$. Assume that a_j are linearly independent over $\mathbb{Q}$. The Reeb orbits are circles in $x_j y_j$ planes (and their m-fold covers) with action ma_j. After Morse-Bott perturbation, there exists 1 orbit in each index starting at $n+1$ arranged in increasing order of action.

3.3.2 Positive Symplectic Homology

A useful variation of symplectic homology is obtained by quotienting out the low-action part of the complex, which isolates the dynamics on the boundary from the topology of the domain itself.

Definition 3.26. *For a Liouville domain W, the* ***positive symplectic homology****, denoted $SH^{+,t}(W)$, is defined for any $t \in \mathbb{R}$ as the direct limit*

$$SH^{+,t}(W) := \varinjlim_{H \in \mathcal{H}_W} H_* \left(\frac{HF^t(H)}{HF^\epsilon(H)} \right),$$

where the limit is taken over all W-admissible Hamiltonians H and for all sufficiently small $\epsilon > 0$.

Example 3.27. *For any strongly star-shaped domain $W \subset \mathbb{R}^{2n}$, the completed positive symplectic homology is concentrated in a single degree:*

$$SH_*^{+,\infty}(W) = \begin{cases} \mathbb{Q} & \text{if } * = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

A more powerful invariant arises from incorporating the natural S^1-action on the free loop space $C^\infty(S^1, \hat{W})$. This leads to the construction of **S^1-equivariant symplectic homology**. Rigorously, this is the filtered Morse homology of the action functional on the Borel construction of the loop space, $(\mathcal{L}\hat{W} \times ES^1)/S^1$.

When working with $\mathbb{Q}$ coefficients, an important feature of the equivariant theory is that its underlying chain complex is generated by one generator for each closed Reeb orbit, rather than the two generators that typically appear in the non-equivariant, Morse-Bott setting.

Example 3.28. *Let $CH_*^t(W)$ denote the filtered, positive, S^1-equivariant symplectic homology, i.e., $CH_*^t(W) := SH_*^{S^1,+,t}(W)$. For any strongly star-shaped domain in $\mathbb{R}^{2n}$, the completed version is given by*

$$CH_*^\infty(W) = \begin{cases} \mathbb{Q} & \text{if } * = n-1+2k \text{ for } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.3.3 The Gutt-Hutchings Capacity

Equivariant symplectic homology does not give rise to only a single capacity, but rather to an infinite sequence of them, indexed by the degree of the homology classes.

Definition 3.29. *For an integer $k \geqslant 1$ and a strongly star-shaped domain $W \subset \mathbb{R}^{2n}$, the k-th* ***Gutt-Hutchings capacity*** *is defined as*

$$c_k^{GH}(W) := \inf \left\{ t \in \mathbb{R} \mid \text{the map } CH_{n-1+2k}^t(W) \to CH_{n-1+2k}^\infty(W) \text{ is nonzero} \right\}.$$

Example 3.30. *For the ellipsoid $E(a_1, \ldots, a_n)$, the k-th Gutt-Hutchings capacity is the k-th smallest value in the ordered set of all Reeb orbit actions:*

$$c_k^{GH}(E(a_1, \ldots, a_n)) = k^{th} \text{ smallest number in the set } \{ma_j \mid m \geqslant 1, j = 1, \ldots, n\}.$$

Example 3.31. *For the polydisc $W = B^2(a_1) \times \cdots \times B^2(a_n)$, the capacities are given by integer multiples of the smallest radius:*

$$c_k^{GH}(B^2(a_1) \times \cdots \times B^2(a_n)) = k \cdot \min_j a_j.$$

3.3.4 Symplectic Banach-Mazur Distances

The capacities derived from equivariant symplectic homology can be applied to problems beyond embedding obstructions, such as measuring a notion of distance between symplectic domains.

Definition 3.32. *The* ***symplectic Banach-Mazur distance*** *between two Liouville domains X and Y is given by*

$$\delta(X,Y) := \inf\left\{\lambda \geqslant 1 \mid \exists \text{ a Liouville embedding } \varphi : X \hookrightarrow \mathit{int}(\lambda Y) \text{ such that } \lambda^{-1}Y \subset \varphi(X)\right\}.$$

A powerful strategy for obtaining lower bounds on this distance involves the transfer maps in equivariant symplectic homology. The existence of a Liouville embedding φ satisfying the "sandwich" condition $\lambda^{-1}Y \subset \varphi(X) \subset \lambda Y$ for a given λ implies the existence of a sequence of transfer maps on homology:

$$\mathrm{CH}_k^t(\lambda Y) \to \mathrm{CH}_k^t(X) \to \mathrm{CH}_k^t(\lambda^{-1}Y).$$

By the conformality property of the filtration with respect to scaling, this sequence of maps is equivalent to

$$\mathrm{CH}_k^{\lambda^{-1}t}(Y) \to \mathrm{CH}_k^t(X) \to \mathrm{CH}_k^{\lambda t}(Y).$$

Therefore, if the known algebraic structures of the homology groups $\mathrm{CH}_k^*(X)$ and $\mathrm{CH}_k^*(Y)$ preclude the existence of such a factorization of maps for a given λ, one can conclude that no such embedding exists. This provides a lower bound on the distance, $\delta(X,Y) > \lambda$.

RESEARCH TALKS

There were 10 research talks, each one hour long.

1. Symplectic Orbifold Gromov-Witten Invariants by Mark McLean (Stony Brook)
2. Spectral equivalence of nearby Lagrangians by Alex Pieloch (MIT)
3. Low-action holomorphic curves and invariant sets by Dan Cristofaro-Gardiner (University of Maryland)
4. High-dimensional families of holomorphic curves and three-dimensional energy surfaces by Rohil Prasad (University of California, Berkeley)
5. Deformation inequivalent symplectic structures and Donaldson's four-six question by Luya Wang (Stanford)
6. Taut foliations through a contact lens by Thomas Massoni (MIT)
7. Symplectic annular Khovanov homology and knot symmetry by Kristen Hendricks (Rutgers University)
8. How to construct symplectic homotopy theory by Vardan Oganesyan (University of California, Santa Cruz)
9. Homological mirror symmetry for Batyrev mirror pairs by Daniel Pomerleano (University of Massachusetts, Boston)
10. Derived moduli spaces of pseudo-holomorphic curves by John Pardon (Simons Center for Geometry and Physics)

4 Mark McLean: Symplectic Orbifold Gromov-Witten Invariants

Abstract: Chen and Ruan constructed symplectic orbifold Gromov-Witten invariants more than 20 years ago. In ongoing work with Alex Ritter, we show that moduli spaces of pseudo-holomorphic curves mapping to a symplectic orbifold admit global Kuranishi charts. This allows us to construct other types of Gromov-Witten invariants, such as K-theoretic counts. The construction relies on an orbifold embedding theorem of Ross and Thomas.

4.1 Introduction

The primary objective of this work is the construction of Gromov-Witten type invariants for symplectic orbifolds. This theory was previously developed over the field of rational numbers, $\mathbb{Q}$, by Chen and Ruan. The approach here uses global Kuranishi charts. This is part of a larger research project aimed at proving a version of the crepant resolution conjecture. Related work on the global quotient case is concurrently being developed by Mak, Seyfaddini, and Smith. This entire talk is joint work with Ritter.

Informally, an orbifold is like a manifold, but the charts look like V/Γ, where $V \subset \mathbb{R}^n$ is an open finite subset and $\Gamma \to \mathrm{GL}_n(\mathbb{R})$. For example, $\{\mathrm{pt}\}/\mathbb{Z}/2$ is an orbifold. We will not define an orbifold formally, but the following construction provides the important intuition:

Suppose a compact Lie group G acts on a smooth manifold M such that all stabilizer subgroups are finite. The resulting quotient space, denoted $[M/G]$, is a canonical example of an orbifold. The local structure is described by the Slice Theorem.

Theorem 4.1 (The Slice Theorem). *For each point $x \in M$, there exists a G_x-equivariant submanifold $S_x \subseteq M$ containing x and a G-equivariant neighborhood $U_x \subseteq M$ of x such that the map*

$$G \times_{G_x} S_x \to U_x$$

is a diffeomorphism. Here, G_x denotes the stabilizer subgroup of x.

Definition 4.2. *If the slice S_x admits a global G_x-equivariant coordinate system, then the pair (S_x, G_x) is an **orbifold chart** for $[M/G]$ centered at the point corresponding to the orbit of x.*

Theorem 4.3 (Pardon). *Every compact orbifold is diffeomorphic to a global quotient $[M/G]$ for some compact Lie group G acting on a smooth manifold M with finite stabilizers.*

Morphisms between orbifolds are defined in the framework of Hilsum-Skandalis maps.

Definition 4.4. *A **Hilsum-Skandalis morphism** between two orbifolds $[M_1/G_1]$ and $[M_2/G_2]$ is represented by a diagram*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & M_2 \\ M_1 & \end{tikzcd}$$

where P is a manifold equipped with a $G_1 \times G_2$ action, $\pi : P \to M_1$ is a G_1-equivariant principal G_2-bundle, and $f : P \to M_2$ is a G_2-equivariant map.

This framework allows for definitions of geometric structures on orbifolds, such as symplectic or complex structures.

Definition 4.5. *If $X = [M/G]$ is an orbifold, its **underlying coarse moduli space**, denoted $\underline{X}$, is the topological quotient space M/G.*

4.2 Twisted Nodal Curves

Let (X, ω, J) be a compact symplectic orbifold, where ω is a symplectic form and J is a compatible almost complex structure that tames ω. Let $\beta \in H_2(\underline{X}; \mathbb{Z})$ be a homology class. The objects of study are maps from certain singular domains into X.

Definition 4.6. *A **twisted nodal domain** Σ is a topological space obtained from a one-dimensional complex orbifold $\tilde{\Sigma}$ by an equivalence relation $\sim$. This relation identifies distinct pairs of smooth points $p \sim q$, subject to the following **balancing condition**:*

- *The point p admits an orbifold chart with a coordinate z centered at p where the stabilizer group, isomorphic to $\mathbb{Z}/k\mathbb{Z}$, acts via $(m, z) \mapsto e^{2\pi i m/k} z$.*
- *The point q admits an orbifold chart with a coordinate w centered at q where its stabilizer group, also $\mathbb{Z}/k\mathbb{Z}$, acts via $(m, w) \mapsto e^{-2\pi i m/k} w$.*

Remark 4.7. *Near a node where two points with stabilizer group $\mathbb{Z}/k\mathbb{Z}$ are identified, the local model for Σ is given by $\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}/(\mathbb{Z}/k\mathbb{Z})$, where the group action is $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$ for $\zeta = e^{2\pi i/k}$. This singularity can be smoothed to $\{(x, y) \in \mathbb{C}^2 \mid xy = t\}/(\mathbb{Z}/k\mathbb{Z})$.*

Definition 4.8. *A **marking** on a twisted nodal domain Σ is a collection of distinct smooth points $p_1, \dots, p_n$ that are disjoint from the nodes. This set must include all smooth points with non-trivial stabilizer groups.*

For a detailed exposition of the category of twisted curves, see Abramovich and Vistoli.

Definition 4.9. *A **twisted nodal curve** is a map $u : \Sigma \to X$, where Σ is a twisted nodal domain and u is a J-holomorphic Hilsum-Skandalis morphism satisfying:*

- *The map descends to a continuous map from the domain to the coarse moduli space, $\underline{u} : \Sigma \to \underline{X}$.*
- *For every point $\sigma \in \tilde{\Sigma}$, the induced map of stabilizer groups $G_\sigma \to G_{u(\sigma)}$ is injective.*

Remark 4.10.

- *Abramovich and Vistoli studied the case where the target orbifold is the point with a $\mathbb{Z}_2$ stabilizer, $X = [pt/\mathbb{Z}_2]$.*
- *For comparison, in the classical smooth, genus-zero case, a map $u : \Sigma \to X$ is analyzed by choosing a line bundle $L \to X$ and a framing F (a basis for $H^0(\Sigma, u^*L)$). This framing induces a map into projective space, $\phi_F : \Sigma \to \mathbb{P}^d$, where $d = \dim H^0(\Sigma, u^*L) - 1$. This technique restates the problem in a more well-understood framework of the moduli space of stable maps to projective space.*

4.3 Problems

Generalizing the established theory of Gromov-Witten invariants from smooth domains to the orbifold setting presents several significant challenges. For instance, in the smooth genus-zero case, one can embed the curve into a projective space $\mathbb{P}^d$ using a basis for the sections of a line bundle. This approach faces two primary obstacles in the orbifold context:

1. For curves of higher genus, the space of line bundles of a fixed degree forms its own moduli space, adding a layer of complexity.
2. Twisted nodal curves with non-trivial stabilizer groups generally do not admit maps to a standard projective space $\mathbb{CP}^d$.

To address the second challenge, we adopt an idea introduced by Ross and Thomas. Instead of mapping to standard projective spaces, the target space is replaced by a weighted projective space.

Definition 4.11. *A **weighted projective space** $\mathbb{P}(w_0, \dots, w_d)$ with weights $w_i \in \mathbb{Z}^+$ is the quotient space*

$$\mathbb{P}(w_0, \dots, w_d) = (\mathbb{C}^{d+1} \backslash \{0\})/\mathbb{C}^\times,$$

where the $\mathbb{C}^\times$-action is given by $t \cdot (z_0, \dots, z_d) = (t^{w_0} z_0, \dots, t^{w_d} z_d)$.

Let Y be a compact complex orbifold with only cyclic stabilizer groups (for instance, the domain $\tilde{\Sigma}$ of a twisted curve).

Definition 4.12. *A line bundle L on Y is **locally ample** if for every point $y \in Y$, the stabilizer group at y acts faithfully on the fiber L_y.*

Definition 4.13. *A line bundle L is* ***globally positive*** *if for some integer $N > 0$, the power $L^{\otimes N}$ is the pullback of an ample line bundle from the coarse moduli space $\underline{Y}$.*

Definition 4.14. *A line bundle L on Y is* ***orbi-ample*** *if it is both locally ample and globally positive.*

Using an orbi-ample line bundle, one can construct an embedding.

Definition 4.15. *Let L be a line bundle and let $n_i = \dim H^0(Y, L^{\otimes i})$. For a fixed integer $k > 0$, a* ***k-framing*** *of L is a tuple of bases*

$$(f_{ij})_{i \in \{k,\dots,2k\}, j \in \{1,\dots,n_i\}},$$

where for each i, the set $\{f_{i1}, \dots, f_{in_i}\}$ is a basis for the vector space of global sections $H^0(Y, L^{\otimes i})$.

Theorem 4.16 (Ross-Thomas Embedding)**.** *Let L be an orbi-ample line bundle on Y and let $F = (f_{ij})$ be a k-framing for sufficiently large k. Define the weighted projective space*

$$\mathbb{P}_k(L) = \mathbb{P}(\underbrace{k, \dots, k}_{n_k \text{ times}}, \underbrace{k+1, \dots, k+1}_{n_{k+1} \text{ times}}, \dots, \underbrace{2k, \dots, 2k}_{n_{2k} \text{ times}}).$$

Then the map $\phi_F : Y \to \mathbb{P}_k(L)$ defined by

$$y \mapsto [f_{ij}(s)]_{i,j}$$

for any non-zero section $s \in L_y$, is an embedding of orbifolds.

5 Dan Cristofaro-Gardiner: Low-Action Holomorphic Curves and Invariant Sets

Abstract: I will discuss a new compactness theorem for sequences of low-action punctured holomorphic curves of controlled topology, in any dimension, without imposing the typical assumption of uniformly bounded Hofer energy; in the limit, we extract a family of closed Reeb-invariant subsets. I will also explain why such sequences exist in abundance in low-dimensional symplectic dynamics, via the theory of embedded contact homology. This has various applications: the one I want to focus on in my talk is a generalization to higher genus surfaces and three-manifolds of the celebrated Le Calvez-Yoccoz theorem. All of this is joint with Rohil Prasad.

5.1 Introduction

5.1.1 A New Compactness Theorem

Consider a closed, oriented $(2n+1)$-manifold Y. We are interested in framed Hamiltonian structures.

Definition 5.1. *A **framed Hamiltonian structure** is a pair (λ, ω) where λ is a 1-form, ω is a closed 2-form, and the condition $\lambda \wedge \omega^n > 0$ holds.*

Example 5.2. *The following are examples of framed Hamiltonian structures:*

1. *Let λ be a contact form and $\omega = d\lambda$. Consider a symplectic automorphism $\phi : (M^{2n}, \omega) \to (M^{2n}, \omega)$. The mapping torus is given by*
$$Y = (M^{2n} \times [0,1])/\sim$$
where $(x,1) \sim (\phi(x), 0)$. The pair (dt, ω) defines a framed Hamiltonian structure on Y.
2. *Let $H : (M^{2n}, \omega) \to \mathbb{R}$ be a proper Hamiltonian function. If c is a regular value of H, then the level set $H^{-1}(c)$ admits a framed Hamiltonian structure.*
3. *Any non-singular, volume-preserving flow on a closed 3-manifold corresponds to a framed Hamiltonian structure.*

Given a framed Hamiltonian structure (λ, ω), there exists a vector field R satisfying $\omega(R, \cdot) = 0$ and $\lambda(R) = 1$.

Example 5.3. *In the contact case, where $\omega = d\lambda$, the vector field R is the Reeb vector field.*

Our objective is to find non-trivial (i.e., non-empty and proper) closed invariant sets of the flow generated by R.

Example 5.4. *Examples of such invariant sets include:*

- *A periodic orbit of the flow.*
- *Invariant tori, as arise in KAM theory.*
- *The closure of an orbit for a proper flow.*

Consider the symplectization $X = \mathbb{R} \times Y$ equipped with an almost complex structure $J : TX \to TX$ such that $J^2 = -1$. We require J to satisfy $J(\partial_s) = R$ and to preserve the kernel of λ, $\ker(\lambda)$, compatibly with ω. We are interested in sequences of proper J-holomorphic curves

$$u_k : C_k \to \mathbb{R} \times Y$$

where each C_k is a closed Riemann surface minus a finite number of punctures. For such a sequence, we define the limit set $L(u_*)$ as the collection of all closed subsets $K \subset (-1,1) \times Y$ for which there exists a sequence of shifts $s_k \in \mathbb{R}$ such that the translated curves $u_k(C_k) - (s_k, 0)$ converge to K within the slab $(-1,1) \times Y$. This can be interpreted as the set of subsequential limits of height-2 slices of the curves.

Proposition 5.5. *The limit set $L(u_*)$ is connected.*

Associated with each holomorphic curve u are two classical quantities:

- The **action**, defined as

$$\mathcal{A}(u) = \int_C u^*\omega$$

- The **Hofer energy**, defined as

$$\mathcal{E}(u) = \sup_{s\in\mathbb{R}} \int_{C\cap u^{-1}(\{s\}\times Y)} u^*\lambda$$

Theorem 5.6. *Let $u_k : C_k \to \mathbb{R}\times Y$ be a sequence of proper J-holomorphic curves such that the action vanishes in the limit, $\lim_{k\to\infty}\mathcal{A}(u_k) = 0$, and the Euler characteristic is uniformly bounded from below, $\inf_k \chi(C_k) > -\infty$. Then every element $K \in L(u_*)$ is of the form $K = (-1,1)\times\Lambda$ for some closed, non-empty, R-invariant set $\Lambda \subset Y$.*

The upshot is that a sequence satisfying the hypotheses of this theorem yields a connected family of of invariant sets.

Remark 5.7. *The main novelty of this theorem is the absence of any requirement for a uniform bound on the Hofer energy $\mathcal{E}(u_k)$.*

5.2 Dynamical Applications

The upshot is that the theorem widely applicable in low dimensions. In higher dimensions, related problems are more open. Of particular importance is the connection to generalizations of the Le Calvez–Yoccoz theorem.

Definition 5.8 (Birkhoff)**.** *A dynamical system is* ***minimal*** *if every orbit is dense in the phase space.*

One motivation for studying invariant sets is that if a dynamical system on Y is not minimal, then there exists a non-trivial, proper, closed invariant set $K \subset Y$. The existence of such a set provides a decomposition of the space.

Problem 5.9 (Ulam)**.** *Does there exist a minimal homeomorphism of $\mathbb{R}^n$ or of $\mathbb{R}^n\backslash\{p\}$ for some point p?*

Theorem 5.10 (Le Calvez, Yoccoz, 1997)**.** *A homeomorphism of $S^2\backslash\{p_1,\ldots,p_k\}$ is never minimal.*

This motivates the following definition.

Definition 5.11. *A system has the* ***(strong) Le Calvez–Yoccoz property*** *if for any non-trivial closed invariant set, the dynamics on its complement is never minimal.*

Theorem 5.12 (Cristofaro-Gardiner, Prasad)**.** *The following systems have the strong Le Calvez–Yoccoz property:*

1. *Any Hamiltonian diffeomorphism of a closed surface.*
2. *Any Reeb flow on a rational homology 3-sphere.*
3. *Any geodesic flow on a closed surface (considered as a flow on its unit tangent bundle).*

Remark 5.13. *No genericity assumptions are required for this result to hold.*

Corollary 5.14. *For the systems (1)–(3) listed above, the union of all non-trivial, proper, closed invariant sets is dense in the phase space.*

Corollary 5.15. *For any geodesic flow on a closed surface, there exists a dense set of points such that for each point in this set, there is more than one non-dense geodesic passing through it.*

5.3 Proof Ideas

5.3.1 Finding Low Action Curves of Controlled Topology

This component of the proof utilizes the theory of Embedded Contact Homology (ECH). For a closed, oriented 3-manifold (Y, λ) with a contact form, ECH(Y, λ) is the homology of a chain complex whose generators are certain finite sets of Reeb orbits and whose differential counts specific (mostly embedded) J-holomorphic curves in the symplectization.

Theorem 5.16. *The Embedded Contact Homology is isomorphic to a version of Heegaard Floer or Monopole homology:*

$$ECH(Y, \lambda) \cong HM(Y)$$

There exists a map U : ECH $\to$ ECH defined by counting holomorphic curves that pass through a generic marked point in the symplectization. The Weyl law for the ECH spectrum, combined with properties of the U-map, allows us to produce sequences of holomorphic curves with vanishing action.

Problem 5.17. *How do we bound the Euler characteristic, $\chi(C_k)$, for the sequence of curves C_k produced by this method?*

A priori, there is no bound on the genus of these curves.

Theorem 5.18 (Cristofaro-Gardiner, Prasad)**.** *The curves C_k in the sequence can be chosen such that their Euler characteristic is bounded from below:*

$$\chi(C_k) \geqslant -2.$$

5.3.2 Proving the Compactness Theorem

The central idea in this proof is a new estimate. This estimate provides a bound on the portion of a holomorphic curve C contained within a small ball, provided that the curve has low action. The bound is given in terms of the Euler characteristic $\chi(C)$.

6 Alex Pieloch: Spectral Equivalence of Nearby Lagrangians

Abstract: Fix a commutative ring spectrum R. In this talk, we will show that any nearby Lagrangian in a cotangent bundle of a closed manifold is equivalent in the wrapped Fukaya category with R-coefficients to an R-brane supported on the zero section. As an application, we impose topological restrictions on the embeddings of exact Lagrangian fillings of the standard Legendrian unknot in sub-critical Stein domains. This is joint work with Johan Asplund and Yash Deshmukh.

6.1 Introduction

The main result of this work is a generalization of a theorem by Abouzaid to the setting of arbitrary commutative ring spectra.

Theorem 6.1 (Abouzaid)**.** *Let $L \subset T^*Q$ be an exact compact Lagrangian submanifold (a nearby Lagrangian) equipped with a rank-1 local system. Then in the wrapped Fukaya category $\mathcal{W}(T^*Q;\mathbb{Z})$, L is equivalent to the zero section endowed with some rank-1 local system.*

Our goal is to extend this result by replacing integer coefficients with coefficients in a commutative ring spectrum R. A spectrum is an object from stable homotopy theory that generalizes the notion of a topological space, admitting a richer algebraic structure suitable for defining generalized homology theories.

Example 6.2. *The homology of a manifold M with coefficients in a ring spectrum R is given by the homotopy groups of the smash product:*

$$\pi_*(M \wedge R) = H_*(M;R).$$

For specific choices of R, we recover familiar theories:

- *If $R = HK$ for some ring K, then $\pi_*(M \wedge HK) = H_*(M;K)$, the ordinary singular homology.*
- *If $R = \mathrm{MO}$, the real cobordism spectrum, then $\pi_*(M \wedge \mathrm{MO}) = \Omega_*^{\mathrm{MO}}(M)$, the bordism groups of M.*

Our main theorem is stated as follows:

Theorem 6.3. *Let $L \subset T^*Q$ be a nearby Lagrangian equipped with an R-brane. In the wrapped Fukaya category $\mathcal{W}(T^*Q;R)$, L is equivalent to an R-brane supported on the zero section.*

This result has applications in constraining the topology of Lagrangian fillings.

Theorem 6.4. *Let X be a subcritical Weinstein domain with $c_1(X) = 0$ and $c_2(X) = 0$. Let $\Lambda \subseteq \partial X$ be a Legendrian unknot with its standard filling C. If $L \subseteq X$ is any exact Lagrangian filling of Λ, then L is homotopic to C relative to Λ in the space X/X_{n-2}, which is homotopy equivalent to a wedge of spheres $\bigvee S^{n-1}$.*

Before we prove this, we define the wrapped Fukaya category with spectral coefficients.

Definition 6.5. *Let X be a Liouville domain and R be a commutative ring spectrum.*

1. *The **wrapped Fukaya category** $\mathcal{W}(X;\mathbb{Q})$ has objects which are exact Lagrangians with vanishing Maslov class, equipped with rank-1 local systems. The morphism spaces are chain complexes built from Floer intersection theory.*
2. *The category $\mathcal{W}(X;R)$ has objects which are exact Lagrangians with vanishing Maslov class, equipped with R-branes. The morphism spaces are R-module spectra constructed from Floer theory.*

6.2 R-Branes and their Properties

The notion of an R-brane generalizes that of a local system. It is defined in terms of an R-orientation.

Definition 6.6. *A vector bundle $\mathcal{E} \to \mathcal{B}$ is R-**orientable** if the classifying map composed with the map to the classifying space of the general linear group of R is null-homotopic:*

$$\mathcal{B} \xrightarrow{\mathcal{E}} \mathrm{BO} \to \mathrm{BGL}_1(S) \to \mathrm{BGL}_1(R).$$

Example 6.7. *Let $R = H\mathbb{Z}$. The unit map induces a map* $\mathrm{BGL}_1(H\mathbb{Z}) \simeq B\mathrm{Aut}(\mathbb{Z}) \simeq B(\mathbb{Z}/2)$. *An $H\mathbb{Z}$-orientation is then a null-homotopy of the composite map $\mathcal{B} \to \mathrm{BO} \to B(\mathbb{Z}/2)$, which is equivalent to a spin structure on the vector bundle.*

Now, assume X is a Weinstein domain such that its tangent bundle is symplectically trivializable, i.e., $TX \cong \mathbb{C} \otimes \mathbb{R}^n$. For a Lagrangian $L \subset X$, let $\mathrm{GL}_L : L \to \mathcal{U}/\mathrm{O}$ be the classifying map of its tangent bundle viewed in the Lagrangian Grassmannian.

Definition 6.8. *An R-**brane** on a Lagrangian L is a choice of null-homotopy for the composite map:*

$$L \xrightarrow{\mathrm{GL}_L} \mathcal{U}/\mathrm{O} \xrightarrow{Bott} B^2(\mathbb{Z} \times \mathrm{BO}) \to B^2\mathrm{BGL}_1(S) \to B^2\mathrm{BGL}_1(R).$$

Remark 6.9.

1. *The set of homotopy classes of R-branes on L is given by $[L, B\,\mathrm{GL}_1(R)]$. For specific choices of R, this recovers the classical notion of a rank-1 local system.*
2. *For the Eilenberg-MacLane spectrum $R = H\mathbb{Z}$, we have $[L, B\,\mathrm{GL}_1(H\mathbb{Z})] \cong [L, B(\mathbb{Z}/2)]$. These correspond to rank-1 local systems with fiber $\mathbb{Z}$ and monodromy in $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2$.*
3. *Let $\mathcal{M}_L = \{(D, \partial D) \to (X, L) \mid \bar{\partial} u = 0, based\}$ be the moduli space of based pseudo-holomorphic disks with boundary on L. A key consistency condition in Floer theory requires the compatibility of orientations on this moduli space with the brane structure on L, often visualized via a diagram ensuring two maps are homotopic. The following diagram illustrates this relationship:*

$$\begin{tikzcd}
\mathcal{M}_L \arrow[r, "T\mathcal{M}_L"] \arrow[d] & BO \arrow[r] & BGL_1(R) \\
\Omega_L \arrow[r, "\Omega GL"] \arrow[urr, bend right, "*"'] & BGL_1(R) \arrow[u, "Bott"'] &
\end{tikzcd}$$

The morphism spectra in $\mathcal{W}(X; R)$ have several key properties.

Proposition 6.10. *Let L, K be objects in $\mathcal{W}(X, R)$.*

1. *The endomorphism spectrum of L is given by $\mathrm{HW}(L, L; R) \simeq L \wedge R$.*
2. *If L is compact, the homotopy groups of the morphism spectra are abelian groups. In particular, $\pi_*(\mathrm{HW}(L, L; R)) \cong H_*(L; R)$.*
3. *Change of Coefficients: If S is a module spectrum over R, then $\mathcal{W}(X; S) \simeq \mathcal{W}(X; R) \wedge_R S$.*

From here, assume R is a connective spectrum, $\pi_0(R) = K$ is a discrete ring, and the Hurewicz map $\mathrm{Hw} : R \to HK$ is the identity on π_0.

Proposition 6.11. *Let M, M' be connected R-module spectra.*

1. *If $\pi_*(M \wedge_R HK)$ is K in degree 0 and vanishes otherwise, then $M \simeq R$.*
2. *Given a map $f : M \to M'$, if the induced map on homology $\pi_*(f) \wedge_R HK$ is an equivalence, then f is an equivalence.*

Sketch of Proof of Main Theorem. The proof relies on showing that the natural map from the zero section Q to a nearby Lagrangian L is an equivalence in $\mathcal{W}(T^*Q, R)$. Using the previous proposition, it suffices to show this after applying the functor $(-) \wedge_R HK$. This reduces the problem to the known case of the Fukaya category with coefficients in the ring $K = \pi_0(R)$, where Abouzaid's argument applies. Let F be the fiber of the map from the zero section to L. One shows that $\mathrm{HW}(F, F; R) \simeq \Omega Q \wedge R$ and $\mathrm{HW}(F, L; R) \simeq R$. A commutative diagram involving the category's product structure

$$\begin{tikzcd}
\mathrm{HW}(F,F) \wedge_R \mathrm{HW}(F,L) \arrow[r, "\mu^2"] \arrow[d] & \mathrm{HW}(F,L) \arrow[d] \\
(\Omega Q \wedge R) \wedge_R R \arrow[r] & R
\end{tikzcd}$$

implies that the necessary structures on L are induced by a map $Q \to B\,\mathrm{GL}_1(R)$, which corresponds to a rank-1 local system on Q. All such systems can be realized, allowing us to build an equivalence. $\square$

6.3 Proof

We now prove the theorem restricting Lagrangian fillings of the Legendrian unknot.

Proposition 6.12. *Let L and C be two exact Lagrangian fillings of the Legendrian unknot $\Lambda \subset \partial X$. It suffices to show that the class of the glued sphere $[L \cup_\Lambda C]$ vanishes in the spin bordism group $\tilde{\Omega}_n^{\mathrm{spin}}(X) = \tilde{H}_n(X, M\mathrm{Spin})$.*

Proof. The condition $[L \cup_\Lambda C] = 0 \in \tilde{\Omega}_n^{\mathrm{spin}}(X)$ implies that the sphere $S^n = L \cup_\Lambda C$ is null-bordant in X. By the Pontryagin-Thom construction, this is equivalent to the corresponding map $S^n \to X$ being null-homotopic after composing with the projection $X \to X/X_{n-2}$. For a subcritical domain X, the space X/X_{n-2} is a wedge of $(n-1)$-spheres. The homotopy class is detected by $\pi_n(X/X_{n-2})$, and the vanishing of the bordism class ensures the homotopy class is trivial, implying L is homotopic to C relative to Λ. $\square$

Let's prove the application of the theorem from earlier. Recall that X is a subcritical Weinstein domain if $c_1(X) = 0 = a(X)$, $\dim_{\mathbb{C}} X \geqslant 4$, $\Lambda \subset \partial X$ the unknown with standard filling C, and $L \subset X$ is an exact filling of L (where $L = \mathbb{D}^n$).

Proposition 6.13. *It suffices to show that $[L \cup_\Lambda C] = 0 \in \tilde{\Omega}_n^{spin}(X) = \tilde{H}_X(X, \mathcal{M}Spin)$*

Proof. It suffices to show that $L \cup_\Lambda C \cong S^n$ implies that its image in X/X_{n-2} is based null. We have a sequence

$$\mathbb{Z}/2\mathbb{Z} \cong \pi_n(S^{n-1}) \to \tilde{\Omega}_n^{\mathrm{spin}}(S^{n-1}) \xrightarrow{\cong} \Omega_1^{\mathrm{spin}}(\mathrm{pt}) \cong \mathbb{Z}/2\mathbb{Z}$$

which is an isomorphism by the Pontryagin-Thom construction. If $X/X_{n-2} \simeq S^{n-1}$, the claim holds. If $X/X_{n-2} \simeq \bigvee S^{n-1}$, then the claim holds by the Hilton-Milnor theorem. $\square$

Let $\hat{X} = X \cup_\Lambda H^n$ be the Weinstein domain obtained by attaching a standard handle along Λ. Let $\hat{L} = L \subset \hat{X}$ and let $\hat{C} \subset H^n$ be the core disk of the attached handle.

Proposition 6.14. *In the spin bordism group of $\hat{X}$, we have the equality of classes $[\hat{L}] = [\hat{C}] \in \tilde{\Omega}_n^{\mathrm{spin}}(\hat{X})$.*

Proof. Take a map $f : \hat{X} \to X$ such that the induced map on bordism sends $\hat{f}([\hat{L}]) = [L \cup_\Lambda C]$. We have $f([\hat{C}]) = 0 \in \Omega_n(B^{2n})$.

The obstruction to constructing an MSpin-brane lies in cohomology. We can assume $\pi_1(L) = 0$, $w_2(L) = 0$, and $H^3(L; \mathbb{Z}/2\mathbb{Z}) = 0$. Additionally, we have

$$\hat{X} = (\text{subcritical handles}) \cup T^*S^n \implies \mathcal{W}(\hat{X}, R) \cong \mathcal{W}(T^*S^n, R).$$

In this category, $\hat{L}$ and $\hat{C}$ are nearby Lagrangians and thus $\hat{L} \cong \hat{C}$ in $\mathcal{W}(\hat{X}, R)$. We conclude by considering the map

$$\mathrm{HW}(L, L, R) \to H_n(\hat{X}; R) = \Omega_n^{\mathrm{spin}}(\hat{X}).$$

$\square$

7 Rohil Prasad: High-Dimensional Families of Holomorphic Curves and Three-Dimensional Energy Surfaces

Abstract: Let H be any smooth function on $\mathbb{R}^4$. I'll discuss some recent dynamical theorems for the Hamiltonian flow on level sets of H ("energy surfaces"). The results are proved using holomorphic curves and neck stretching. One important tool is the compactness theorem from Dan's talk.

7.1 Basics

Let's establish our setting. Consider the standard symplectic manifold $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function, referred to as the Hamiltonian. The associated Hamiltonian vector field, denoted X_H, is defined by the relation $dH(\cdot) = \omega(X_H, \cdot)$.

Lemma 7.1. *The flow of the Hamiltonian vector field X_H preserves the Hamiltonian H.*

Corollary 7.2. *The flow of X_H preserves the level sets of H.*

Remark 7.3. *Historically, the level sets of a Hamiltonian function were termed* ***energy surfaces****.*

Definition 7.4. *Let $s \in \mathbb{R}$ be a regular value of H. The level set $Y_s := H^{-1}(s)$ is a smooth $(2n-1)$-dimensional manifold. An energy surface that arises in this way is called a* ***regular energy surface****.*

7.2 Invariant Sets

A basic question in Hamiltonian dynamics is about the structure of the flow on a given energy surface. The central problem in this area was asked by Herman.

Problem 7.5 (Herman, ICM 1998)**.** *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian and let Y be a compact, regular energy surface. Does Y necessarily contain a proper, closed, X_H-invariant subset?*

Significant progress has been made on this problem. For the case $n = 2$, the question was answered in the affirmative.

Theorem 7.6 (Fish-Hofer, 2018)**.** *For $n = 2$, the answer to Problem 7.5 is yes.*

Prior results established the existence of closed orbits, which are the simplest type of closed invariant subsets, under certain geometric assumptions.

Theorem 7.7 (Weinstein, Rabinowitz, Viterbo)**.** *If a compact, regular energy surface Y is of contact type, then it contains a closed orbit of X_H.*

However, closed orbits do not always exist, motivating the search for more general invariant structures.

Theorem 7.8. *Examples of compact, regular energy surfaces Y with no closed orbits have been constructed:*

- *For $n \geqslant 3$ by Ginzburg, Gurel, and Herman.*
- *For $n \geqslant 2$ with a C^2-smooth Hamiltonian by Ginzburg and Gurel.*

Recent work provides a more complete answer to Herman's problem, showing the existence of a family of invariant sets.

Theorem 7.9 (Prasad, 2024; Theorem A)**.** *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ and let Y be a compact, regular energy surface. There exists an infinite family of distinct, proper, closed, X_H-invariant subsets whose union is dense in Y.*

Let $\mathrm{Reg}_c(H)$ denote the set of regular values s of H for which the level set $H^{-1}(s)$ is compact. A related result concerns the dynamics on the quotient space after identifying a closed orbit.

Theorem 7.10 (Prasad, 2024; Theorem B)**.** *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$. For almost every $s \in Reg_c(H)$, the energy surface $H^{-1}(s)$ has the following property: for any closed orbit $\Lambda \subset H^{-1}(s)$, the induced flow on the quotient space $H^{-1}(s)/\Lambda$ is not minimal.*

Remark 7.11. *The property described in Theorem 7.10 is related to the Le Calvez-Yoccoz property and implies the dense existence of invariant sets.*

7.3 Closed Orbits and Closed Holomorphic Curves

The study of closed orbits has a long history, with many results on their existence.

Theorem 7.12 (Hofer-Zehnder, 1987)**.** *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$. For almost every $s \in Reg_c(H)$, the energy surface $H^{-1}(s)$ contains a closed orbit.*

Further results provide lower bounds on the number of closed orbits.

Theorem 7.13. *For any $H : \mathbb{R}^{2n} \to \mathbb{R}$, almost every $s \in Reg_c(H)$ is such that $H^{-1}(s)$ contains at least two closed orbits. This bound is sharp.*

Proof. Consider the Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$ given by

$$H(x_1, y_1, x_2, y_2) = \frac{x_1^2 + y_1^2}{a} + \frac{x_2^2 + y_2^2}{b},$$

where the ratio a/b is irrational. Each regular energy surface for this Hamiltonian is a torus on which the flow is a linear irrational flow, and it contains exactly two closed orbits, corresponding to the circles $\{x_2 = y_2 = 0\}$ and $\{x_1 = y_1 = 0\}$.

□

Under generic conditions, a much stronger statement holds in dimension four.

Theorem 7.14. *Let $H : \mathbb{R}^4 \to \mathbb{R}$. For a C^∞-generic set of Hamiltonians H, almost every compact, regular energy surface contains infinitely many closed orbits.*

Sketch of Proof. The result follows from a strictly stronger version of Theorem 7.10 combined with known results concerning the dynamics of generic Hamiltonians. □

The proofs of these modern results rely on constructing families of pseudoholomorphic curves, using techniques introduced by Gromov and further developed by Taubes in the context of Seiberg-Witten theory.

Theorem 7.15 (Taubes)**.** *Let $(\mathbb{CP}^2, \omega)$ be the complex projective plane with its standard symplectic form. For any compatible almost complex structure J and for any sufficiently large integer d, there exists a set $S \subset \mathbb{CP}^2$ with $\#S \approx d^2$ such that there exists a closed, J-holomorphic curve $u : C \to \mathbb{CP}^2$ satisfying:*

1. *$S \subset u(C)$,*
2. *$\int_C u^*\omega = d$,*
3. *$\chi(C) \sim -d^2$, where $\chi(C)$ is the Euler characteristic of the domain curve C.*

7.4 Theorem A Proof Idea

The proof of Theorem 7.9 involves a neck-stretching argument applied to a sequence of holomorphic curves. We consider a symplectic manifold constructed by gluing a long neck $[-k, k] \times Y$ to a symplectic cap, modeled on $\mathbb{CP}^2 \backslash B^4$. Let this manifold be W_k. We then study a sequence of holomorphic curves $u_k : C_k \to W_k$ of degree d.

As $k \to \infty$, the sequence of curves can be analyzed by considering its limit in the neck region. This leads to the notion of a stretched limit set.

Definition 7.16. *The* ***stretched limit set*** *of the sequence $\{u_k\}$ is a subset $\chi(\{u_k\}) \subset \mathcal{P}((-1, 1) \times Y) \times [-1, 1]$. We say $(\Xi, s) \in \chi(\{u_k\})$ if there exists a sequence of points $\{z_k\}$ in the domains C_k and a sequence of shifts $\{s_k\} \subset [-k, k]$ such that:*

1. *The sequence of translated curves $u_k(\cdot)$ in the neighborhood of z_k, viewed in the coordinates of the neck, converges to a limit curve set $\Xi \subset (-1,1) \times Y$.*
2. *The normalized height converges: $s_k/k \to s$.*

Let $\mathcal{U}_{d,k}$ be the set of degree d curves in W_k. The main result concerning the stretched limit set is the following proposition:

Proposition 7.17. *Let $\{u_k\}$ be a sequence with $u_k \in \mathcal{U}_{d,k}$.*

1. *For almost every $s \in [-1,1]$, if $(\Xi, s) \in \chi(\{u_k\})$, then the limit set Ξ has the structure of a trivial cylinder $\Xi = (-1,1) \times \Lambda$, where $\Lambda \subset Y$ is an X_H-invariant set.*
2. *For all but approximately d^2 heights $s \in [-1,1]$, every $(\Xi, s) \in \chi(\{u_k\})$ is such that Ξ is ϵ-almost invariant, where $\epsilon \to 0$ as $d \to \infty$.*

8 Luya Wang: Deformation Inequivalent Symplectic Structures and Donaldson's Four-Six Question

Abstract: Studying symplectic structures up to deformation equivalences is a fundamental question in symplectic geometry. Donaldson asked: given two homeomorphic closed symplectic four-manifolds, are they diffeomorphic if and only if their stabilized symplectic six-manifolds, obtained by taking products with $\mathbb{CP}^1$ with the standard symplectic form, are deformation equivalent? I will discuss joint work with Amanda Hirschi on showing how deformation inequivalent symplectic forms remain deformation inequivalent when stabilized, under certain algebraic conditions. This gives the first counterexamples to one direction of Donaldson's "four-six" question and the related Stabilizing Conjecture by Ruan. In the other direction, I will also discuss more supporting evidence via Gromov-Witten invariants.

8.1 Introduction

We begin by defining the central notion of equivalence used throughout this work.

Definition 8.1. *Two symplectic manifolds (X_1, ω_1) and (X_2, ω_2) are said to be **deformation equivalent**, denoted $(X_1, \omega_1) \simeq (X_2, \omega_2)$, if there exists a diffeomorphism $\varphi : X_1 \to X_2$ such that the pullback form $\varphi^*\omega_2$ is in the same path-component as ω_1 in the space of symplectic forms on X_1. We denote this path-connectedness by $\varphi^*\omega_2 \rightsquigarrow \omega_1$.*

The primary motivation for our investigation is a question posed by Donaldson, which connects the diffeomorphism problem in dimension four to a symplectic equivalence problem in dimension six.

Problem 8.2 (Donaldson's Four-Six Question)**.** *Let (X_1^4, ω_1) and (X_2^4, ω_2) be two closed, simply-connected, and homeomorphic symplectic four-manifolds. Is it true that X_1 is diffeomorphic to X_2 if and only if their stabilized counterparts are deformation equivalent? That is,*

$$(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std}),$$

where S^2 is equipped with a standard area form ω_{std}.

This question is related to the exotic nature of four-dimensional topology. While Freedman's work classifies topological four-manifolds, the classification of smooth structures remains largely open. The h-cobordism theorem, which provides a powerful tool for classifying manifolds in higher dimensions, fails in dimension four.

Theorem 8.3 (Wall, 1964)**.** *Two closed, simply-connected, homeomorphic 4-manifolds are h-cobordant.*

Theorem 8.4 (Smale, 1962)**.** *For $n \geqslant 5$, if two closed, simply-connected n-manifolds are h-cobordant, then they are diffeomorphic.*

The failure of the h-cobordism theorem in dimension four leads to the existence of exotic smooth structures. Donaldson's question can be seen as an attempt to understand this failure through the lens of symplectic geometry. The historical context for this problem includes:

- **Ruan (1994):** Provided examples of homeomorphic but not diffeomorphic Kähler surfaces, namely the blow-up of the complex plane at nine points, $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, and the Barlow surface.
- **Ruan, Tian (1997):** Formulated the Stabilizing Conjecture for simply-connected elliptic surfaces, which is closely related to Donaldson's question.
- **Ionel, Parker (1999):** Constructed exotic smooth structures on elliptic surfaces $E(n)$ using knot surgery techniques.
- **Smith (2000):** Constructed, for any integer $n \geqslant 2$, n distinct symplectic structures on a fixed simply-connected four-manifold Z^4 that could be distinguished by their first Chern classes. This demonstrated that Donaldson's question cannot be simplified by replacing the six-manifold stabilization with a product with $\mathbb{CP}^2$.

This work provides a negative answer to one direction of Donaldson's question. Specifically, we construct examples of diffeomorphic four-manifolds with symplectic structures that become deformation inequivalent after stabilization.

Theorem 8.5 (Hirschi, Wang, 2023)**.** *There exist infinitely many pairs of symplectic four-manifolds (X, ω_1) and (X, ω_2) such that the underlying smooth manifold X is the same, but their stabilizations are not deformation equivalent:*

$$(X \times S^2, \omega_1 \oplus \omega_{std}) \not\simeq (X \times S^2, \omega_2 \oplus \omega_{std}).$$

Conversely, we provide evidence for the other direction of the question, showing that the deformation equivalence of stabilized manifolds imposes strong constraints on the Gromov-Witten invariants of the original four-manifolds.

Theorem 8.6 (Hirschi, Wang, 2023)**.** *Let (X_1, ω_1) and (X_2, ω_2) be closed, simply-connected symplectic 4-manifolds. If their stabilizations are deformation equivalent,*

$$(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std}),$$

then the Gromov-Witten invariants of X_1 and X_2 are equal, i.e., $\mathrm{GW}(X_1) = \mathrm{GW}(X_2)$.

A direct consequence relates this to Seiberg-Witten theory via the Taubes-Bryan-Pandharipande theorem.

Corollary 8.7. *If (X_1, ω_1) and (X_2, ω_2) satisfy the hypotheses of Theorem 8.6 and have $b_2^+ \geqslant 2$, then their Seiberg-Witten invariants are equal, i.e.,* $\mathrm{SW}(X_1) = \mathrm{SW}(X_2)$.

Our approach to proving Theorem 8.5 relies on a new invariant derived from the first Chern class. The invariant is the orbit of $c_1(TX, J)$ under the action of the group of cohomology equivalences, where J is any almost complex structure tamed by the symplectic form ω.

8.2 Proof of Theorem 8.5

The strategy to prove Theorem 8.5 is to construct a smooth four-manifold X admitting two symplectic forms ω_1 and ω_2 whose first Chern classes are inequivalent in a carefully defined sense, and then show that this inequivalence persists after stabilization.

The key invariant is the orbit of the first Chern class under a group of automorphisms of the cohomology ring. Let (X, ω) be a symplectic manifold. We can choose an almost complex structure J tamed by ω, and consider its first Chern class $c_1(\omega) := c_1(TX, J) \in H^2(X; \mathbb{Z})$.

Definition 8.8. *Let X and Y be smooth manifolds. We define $G_{X,Y}$ to be the group of ring isomorphisms $\psi^* : H^*(X \times Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$ that respect the product structure in the following sense: ψ^* must induce an automorphism on $H^*(X; \mathbb{Z})$ via the map $\alpha \mapsto \mathrm{pr}_{X*}(\psi^*(\alpha \otimes 1))$, where $\mathrm{pr}_X : X \times Y \to X$ is the projection.*

The proof of Theorem 8.5 proceeds in three main steps:

1. Find a smooth four-manifold X that admits two symplectic forms, ω_1 and ω_2, such that their first Chern classes, $c_1(\omega_1)$ and $c_1(\omega_2)$, lie in different orbits under the action of the group of cohomology self-equivalences of X.
2. Show that if $c_1(\omega_1)$ and $c_1(\omega_2)$ lie in different orbits of cohomology equivalences on X, then the stabilized Chern classes, $c_1(\omega_1 \oplus \omega_{\mathrm{std}})$ and $c_1(\omega_2 \oplus \omega_{\mathrm{std}})$, lie in different orbits under the action of G_{X,S^2}.
3. Show that any diffeomorphism of the product manifold $X \times S^2$ induces a cohomology automorphism that lies in the group G_{X,S^2}.

Steps (2) and (3) establish that the orbit of the stabilized first Chern class under G_{X,S^2} is a symplectic deformation invariant. Step (1) then provides the necessary examples to produce counterexamples to Donaldson's question.

8.2.1 Lifting Inequivalence of Chern Classes

Let's prove the assertion in step (2). We use the Künneth isomorphism $H^*(X \times S^2; \mathbb{Z}) \cong H^*(X; \mathbb{Z})[h]/\langle h^2 \rangle$, where h is the generator of $H^2(S^2; \mathbb{Z})$. The first Chern class of the stabilized manifold is $c_1(\omega \oplus \omega_{\text{std}}) = c_1(\omega) \otimes 1 + 1 \otimes c_1(\omega_{\text{std}}) = c_1(\omega) + 2h$.

Proof of Step (2). Suppose, for the sake of contradiction, that there exists $\psi^* \in G_{X,S^2}$ such that $\psi^* c_1(\omega_2 \oplus \omega_{\text{std}}) = c_1(\omega_1 \oplus \omega_{\text{std}})$. By the definition of G_{X,S^2}, the action of ψ^* on the generator h must be of the form $\psi^* h = \pm h + \alpha$ for some $\alpha \in H^2(X; \mathbb{Z})$. The condition $\psi^*(h^2) = 0$ implies $(\pm h + \alpha)^2 = 0$. Expanding this gives $h^2 \pm 2\alpha h + \alpha^2 = 0$. Since $h^2 = 0$ and $\alpha^2 \in H^4(X)$, while $\alpha h \in H^4(X \times S^2)$ with the h-component, this implies $2\alpha = 0$. For integer coefficients, this means α is a 2-torsion class. Let us assume for simplicity that $\psi^* h = h + \alpha$. Then the initial assumption becomes:

$$\begin{aligned} c_1(\omega_1) + 2h &= \psi^*(c_1(\omega_2) + 2h) \\ &= \psi^* c_1(\omega_2) + 2\psi^* h \\ &= \psi^* c_1(\omega_2) + 2(h + \alpha) \\ &= \psi^* c_1(\omega_2) + 2h + 2\alpha. \end{aligned}$$

This simplifies to $c_1(\omega_1) = \psi^* c_1(\omega_2) + 2\alpha$. Let $\hat{\psi}^*$ be the automorphism on $H^*(X; \mathbb{Z})$ induced by ψ^*. We have thus found an automorphism relating $c_1(\omega_1)$ and $c_1(\omega_2)$ up to a torsion class, which contradicts the initial assumption that they lie in different orbits. We can also show that $\hat{\psi}^*$ is indeed a cohomology equivalence on X, and for the examples we construct, this is sufficient to establish a contradiction. □

8.2.2 Construction of the Counterexample

For steps (1) and (3), we rely on specific constructions. The required manifold X can be constructed using fiber sums. Let $E(1)$ be the rational elliptic surface. Consider the manifold $Z := \mathbb{T}^4 \# 5E(1)$, where the sum is a fiber sum. Smith constructed symplectic forms on such manifolds with distinct Chern classes. For our purposes, we can take X to be a related manifold where we can find a basis of homology classes with certain intersection properties. Smith's work provides a key result:

Theorem 8.9 (Smith)**.** *For certain symplectic manifolds (Z, ω), the first Chern class $c_1(TZ, \omega)$ is a prime (indivisible) class in $H^2(Z; \mathbb{Z})$.*

By carefully choosing the manifold and applying techniques from rational blowdowns and knot surgery, one can construct symplectic forms ω_1, ω_2 on a single manifold X whose Chern classes lie in different orbits of the automorphism group of $H^*(X; \mathbb{Z})$, satisfying step (1). A further argument shows that for these manifolds, the condition in step (3) holds, completing the proof of Theorem 8.5.

8.3 Proof of Theorem 8.6

We now turn to the proof of Theorem 8.6, which states that the deformation equivalence of stabilized manifolds implies the equality of Gromov-Witten invariants of the original four-manifolds.

Let (X_0, ω_0) and (X_1, ω_1) be simply-connected symplectic four-manifolds. Suppose that their stabilizations are deformation equivalent:

$$(X_0 \times S^2, \omega_0 \oplus \omega_{\text{std}}) \simeq (X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}}).$$

This implies there is a diffeomorphism $\Phi : X_0 \times S^2 \to X_1 \times S^2$ such that $\Phi^*(\omega_1 \oplus \omega_{\text{std}})$ is deformation equivalent to $\omega_0 \oplus \omega_{\text{std}}$. The invariance of Gromov-Witten theory under deformation equivalence and diffeomorphisms implies that for any genus g, number of markings n, and homology class $B \in H_2(X_0 \times S^2; \mathbb{Z})$,

$$\mathrm{GW}_{g,n,B}^{X_0 \times S^2, \omega_0 \oplus \omega_{\text{std}}} = \mathrm{GW}_{g,n,\Phi_* B}^{X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}}}.$$

Our goal is to relate these six-dimensional invariants back to the four-dimensional invariants of X_0 and X_1. This is achieved using a product formula for Gromov-Witten invariants.

Theorem 8.10 (Hirschi-Swaminathan Product Formula)**.** *Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds with $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$ torsion-free. Let $B = (B_X, B_Y) \in H_2(X) \oplus H_2(Y) \cong H_2(X \times Y)$. Then for cohomology classes $\alpha_i \in H^*(X)$ and $\beta_i \in H^*(Y)$, the following equality holds in $H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$:*

$$\mathrm{GW}^{X\times Y,\omega_X\oplus\omega_Y}_{g,n,B}(\alpha_1 \otimes \beta_1, \ldots, \alpha_n \otimes \beta_n) = \mathrm{GW}^{X,\omega_X}_{g,n,B_X}(\alpha_1, \ldots, \alpha_n) \cdot \mathrm{GW}^{Y,\omega_Y}_{g,n,B_Y}(\beta_1, \ldots, \beta_n).$$

To use this formula to extract information about $\mathrm{GW}(X_0)$, we need to choose classes β_i on S^2 such that the corresponding Gromov-Witten invariant of S^2 is non-zero.

Problem 8.11. *For which parameters (g, n, d) and classes $\beta_1, \ldots, \beta_n \in H^*(S^2;\mathbb{Q})$ is the Gromov-Witten invariant $\mathrm{GW}^{S^2}_{g,n,d}(\beta_1, \ldots, \beta_n)$ non-zero?*

Lemma 8.12. *For any odd genus g, number of markings n, and degree d, if we choose all $\beta_i = 1 \in H^0(S^2)$, the point-evaluator of the invariant is non-zero. Specifically,*

$$\mathrm{GW}^{S^2}_{g,n,d}(1^{\otimes n})([pt]) = 2^g.$$

With this non-vanishing result, we can apply the product formula. Assume there exists a homeomorphism $\phi : X_0 \to X_1$ induced by the six-dimensional diffeomorphism Φ. Let the induced map on cohomology be $\tilde{\phi}^*$. We can choose the diffeomorphism Φ such that its action on cohomology is of the form $\Phi^* = \phi^* \otimes \mathrm{id}$. Let $A \in H_2(X_0)$ and $d \in \mathbb{Z}_{>0}$. Applying the product formula to both sides of the invariant equality gives:

$$\begin{aligned}
&\mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\alpha_1, \ldots, \alpha_n) \cdot \mathrm{GW}^{S^2}_{g,n,d}(1, \ldots, 1) \\
&= \mathrm{GW}^{X_0\times S^2}_{g,n,(A,d)}(\alpha_1 \otimes 1, \ldots, \alpha_n \otimes 1) \\
&= \mathrm{GW}^{X_1\times S^2}_{g,n,\Phi_*(A,d)}(\Phi^{*-1}(\alpha_1 \otimes 1), \ldots, \Phi^{*-1}(\alpha_n \otimes 1)) \\
&= \mathrm{GW}^{X_1\times S^2}_{g,n,(\phi_* A,d)}((\phi^*)^{-1}\alpha_1 \otimes 1, \ldots, (\phi^*)^{-1}\alpha_n \otimes 1) \\
&= \mathrm{GW}^{X_1,\omega_1}_{g,n,\phi_* A}((\phi^*)^{-1}\alpha_1, \ldots, (\phi^*)^{-1}\alpha_n) \cdot \mathrm{GW}^{S^2}_{g,n,d}(1, \ldots, 1).
\end{aligned}$$

Since GW^{S^2} is non-zero, we can cancel it from both sides, yielding

$$\mathrm{GW}^{X_0,\omega_0}_{g,n,A}(\alpha_1, \ldots, \alpha_n) = \mathrm{GW}^{X_1,\omega_1}_{g,n,\phi_* A}((\phi^*)^{-1}\alpha_1, \ldots, (\phi^*)^{-1}\alpha_n).$$

This establishes an isomorphism between the Gromov-Witten theories of (X_0, ω_0) and (X_1, ω_1), proving Theorem 8.6.

9 Thomas Massoni: Taut Foliations Through a Contact Lens

Abstract: In the late '90s, Eliashberg and Thurston established a remarkable connection between foliations and contact structures in dimension three: any co-oriented, aspherical foliation on a closed, oriented 3-manifold can be approximated by both positive and negative contact structures. Additionally, if the foliation is taut then its contact approximations are tight. In this talk, I will present a converse result on constructing taut foliations from suitable pairs of contact structures. While taut foliations are rather rigid objects, this viewpoint reveals some degree of flexibility and offers a new perspective on the L-space conjecture.

9.1 Introduction

Let M be a closed, oriented, connected 3-manifold. Informally, a **foliation** $\mathcal{F}$ of M is a decomposition of the manifold into a collection of disjoint, injectively immersed 2-dimensional submanifolds, known as the leaves of the foliation. When the foliation is smooth, its tangent bundle $T\mathcal{F}$ is a 2-plane field that can be expressed locally as the kernel of a 1-form α. This plane field is integrable, a condition captured by the Frobenius Integrability Theorem.

Theorem 9.1 (Frobenius Integrability Theorem)**.** *A smooth 2-plane field $\xi = \ker(\alpha)$ is integrable (i.e., tangent to a foliation) if and only if the 1-form α satisfies the condition*

$$\alpha \wedge d\alpha = 0.$$

In contrast, a plane field that is "maximally non-integrable" gives rise to a contact structure.

Definition 9.2. *A **contact structure** is a completely non-integrable 2-plane field ξ. Such a field can be described as $\xi = \ker(\alpha)$, where the 1-form α, called a contact form, satisfies*

$$\alpha \wedge d\alpha \neq 0$$

at every point on M.

Locally, every contact structure is equivalent to a standard model, as described by Darboux's Theorem.

Theorem 9.3 (Darboux)**.** *For any contact form α on a 3-manifold M, there exist local coordinates (x, y, z) such that α takes the form $\alpha = dz + x\,dy$.*

The theory of foliations and contact structures are linked through the study of 2-plane fields. The h-principle from differential geometry tells us about their connection.

Theorem 9.4. *Every 2-plane field on a closed 3-manifold is homotopic to an integrable plane field (i.e., one that is tangent to a foliation).*

Proof. This is a direct consequence of the h-principle for foliations. □

A similar result holds for contact structures.

Theorem 9.5 (Eliashberg)**.** *Every 2-plane field on a closed 3-manifold is homotopic to a contact structure.*

Among all foliations, a particularly important class is that of taut foliations.

Definition 9.6. *A co-oriented foliation $\mathcal{F}$ is said to be **taut** if it satisfies the following two conditions:*

1. *For every point $p \in M$, there exists a closed loop γ passing through p that is everywhere transverse to the leaves of $\mathcal{F}$.*
2. *There exists a closed 2-form ω (a volume form for the leaves) such that ω restricts to a positive area form on each leaf, i.e., $\omega|_{T\mathcal{F}} > 0$.*

The corresponding notion of "well-behaved" in contact geometry is tightness.

Definition 9.7. *A contact structure is called **tight** if it is not overtwisted. An overtwisted contact structure is one that contains an embedded disk (an overtwisted disk) whose boundary is tangent to the contact planes.*

This leads to several existence questions in 3-manifold topology.

Problem 9.8. *Which 3-manifolds M admit a taut foliation? A Reebless foliation? A tight contact structure?*

A classical result provides a partial answer for manifolds with positive first Betti number.

Theorem 9.9. *If the first Betti number $b_1(M) > 0$, then M admits a taut foliation.*

Theorem 9.10 (Eliashberg, Thurston)**.** *If a 3-manifold M admits a taut foliation, then it also admits a tight contact structure.*

Problem 9.11. *The situation is more subtle when $b_1(M) = 0$, that is, for rational homology 3-spheres ($\mathbb{Q}HS^3$). Which rational homology spheres admit a taut foliation?*

This question is at the heart of the L-space conjecture, which connects taut foliations, Heegaard Floer homology, and the algebraic properties of the fundamental group.

Conjecture 9.12 (L-Space Conjecture)**.** *Let M be an irreducible rational homology 3-sphere. The following are equivalent:*

1. *M admits a taut foliation.*
2. *M is not an L-space (i.e., its reduced Heegaard Floer homology, $\widehat{HF}(M)$, is non-trivial).*
3. *The fundamental group $\pi_1(M)$ is left-orderable.*

Significant progress has been made on this conjecture.

Theorem 9.13 (Ozsváth, Szabó)**.** *If an irreducible rational homology sphere M admits a co-orientable taut foliation, then it is not an L-space. That is,* (1) $\implies$ (2)*.*

However, the full conjecture remains open, and there is considerable skepticism regarding its validity in full generality.

9.2 From Foliations to Contact Structures

The basic result connecting foliations and contact structures is the following theorem of Eliashberg and Thurston.

Theorem 9.14 (Eliashberg, Thurston, 1998)**.** *Let $\mathcal{F}$ be a co-oriented, C^2-smooth foliation on a closed, oriented 3-manifold M.*

1. *If $\mathcal{F}$ has no spherical leaves, then its tangent plane field $T\mathcal{F}$ can be C^0-approximated by both a positive contact structure ξ_+ and a negative contact structure ξ_-.*
2. *If, in addition, $\mathcal{F}$ is taut, then the approximating contact structures ξ_+ and ξ_- can be chosen to be tight.*

Sketch of Proof for (2). Consider the manifold $[-1,1] \times M$. Since $\mathcal{F}$ is taut, there exists a closed 2-form ω on M such that its restriction to the leaves is an area form, $\omega|_{T\mathcal{F}} > 0$. Let α be a 1-form such that $T\mathcal{F} = \ker(\alpha)$. We can construct a 2-form on the product manifold, $\Omega = d(t\alpha) + \omega$, where t is the coordinate on $[-1,1]$. For small perturbations, this form can be made symplectic.

The boundary of this manifold is $(-M, \xi_-) \sqcup (M, \xi_+)$, where $\xi_\pm$ are contact structures that are C^0-close to $T\mathcal{F}$. This setup provides a weak symplectic filling of the boundary components. By a theorem of Gromov and Eliashberg on the properties of symplectic fillings, the contact structure ξ_+ must be tight. A similar argument applies to ξ_-. $\square$

Recent advancements have extended this theorem to foliations with less regularity.

Definition 9.15. *A **C^0-foliation with smooth leaves** is a topological foliation whose leaves are smoothly immersed submanifolds and whose tangent plane field $T\mathcal{F}$ is continuous.*

Theorem 9.16 (Bowden, Kazez, Roberts). *The Eliashberg-Thurston approximation theorem holds for C^0-foliations with smooth leaves.*

The Eliashberg-Thurston theorem can be thought of as a procedure that transforms a suitable foliation $\mathcal{F}$ into a pair of contact structures (ξ_-, ξ_+).

9.3 From Positive Constant Pairs to Foliations

We now explore the converse direction: constructing a foliation from a pair of contact structures.

Definition 9.17. *A pair (ξ_-, ξ_+) is a* ***positive contact pair*** *if ξ_+ is a positive contact structure, ξ_- is a negative contact structure, and there exists a vector field Z that is positively transverse to both ξ_- and ξ_+.*

Given a fixed positive contact pair (ξ_+, ξ_-) and a transverse vector field Z, we can construct a foliation under certain conditions.

Theorem 9.18 (Massoni, 2024). *Let (ξ_-, ξ_+) be a positive contact pair on M. Assume that either ξ_- and ξ_+ are transverse as plane fields, or that at least one of them is tight. Then there exists a foliation $\mathcal{F}$ that is transverse to the vector field Z.*

Definition 9.19. *A positive contact pair (ξ_-, ξ_+) is called* ***strongly tight*** *if there exists a volume-preserving vector field Z that is transverse to both ξ_- and ξ_+.*

This leads to a characterization of manifolds admitting taut foliations.

Corollary 9.20. *A closed 3-manifold M (where $M \neq S^1 \times S^2$) admits a taut foliation if and only if it admits a strongly tight contact pair.*

The proof of the theorem involves constructing a limiting plane field from the contact pair.

Let X be a vector field contained in the intersection $\xi_- \cap \xi_+$ which vanishes precisely along the set where the planes coincide:

$$\Delta = \{p \in M \mid \xi_-(p) = \xi_+(p)\}.$$

Let ϕ_X^t be the flow generated by X. Consider the family of plane fields $\xi_\pm^t = (\phi_X^t)_* \xi_\pm$. The core of the argument relies on the following propositions.

Proposition 9.21. *As $t \to \infty$, the plane fields ξ_-^t and ξ_+^t converge to a common continuous plane field η.*

Proposition 9.22. *The map $(\xi_-, \xi_+) \mapsto \eta$ that sends a contact pair to its limiting plane field is continuous with respect to the C^0 topology.*

Proposition 9.23. *For a generic choice of the pair (ξ_-, ξ_+), the resulting plane field η is integrable.*

Remark 9.24. *The limiting plane field η is, in general, only continuous (C^0) and may not be uniquely integrable. It is also related to a more complex object known as a branching foliation, which we will not define.*

Proposition 9.25. *Assume that there is no immersed disk $\overline{D} \to M$ that is tangent to η and has its boundary $\partial\overline{D}$ tangent to the vector field X. Then η is tangent to a branching foliation.*

Proposition 9.26. *The plane field η can be approximated by integrable plane fields.*

9.4 Future Directions

The idea of constructing foliations from contact pairs opens several avenues for future research. While strongly tight pairs are difficult to construct and analyze, they motivate several interesting conjectures and problems.

Conjecture 9.27 (Massoni). *If (ξ_-, ξ_+) is a positive pair and both ξ_- and ξ_+ are tight, then M admits a Reebless foliation.*

A central goal is to formulate an intrinsic condition on the isotopy classes of ξ_- and ξ_+ that guarantees the existence of a foliation.

Problem 9.28. *Consider a contact pair (ξ_-, ξ_+) on M, not necessarily positive. Assume that:*

1. *Both ξ_- and ξ_+ are tight.*
2. *They are homotopic as 2-plane fields.*
3. *Their contact invariants in Heegaard Floer homology satisfy $\langle c(\xi_+), c(\xi_-)\rangle = 1$.*

Does M necessarily admit a Reebless foliation?

The third condition is motivated by the following result concerning the contact invariants of Eliashberg-Thurston approximations.

Theorem 9.29 (Lin)**.** *If $\mathcal{F}$ is a taut foliation and $\xi_\pm$ are its tight contact approximations, then the pairing of their contact class invariants $c(\xi_\pm) \in \widehat{HF}(-M)$ is $\langle c(\xi_+), c(\xi_-)\rangle = 1$.*

This machinery has applications in Dehn surgery theory.

Theorem 9.30. *Let $\mathcal{F}$ be a taut foliation on a 3-manifold M (with $M \neq S^1 \times S^2$), and let K be a transverse, framed knot in M. Then there exists a constant $s_0 > 0$ such that for every rational number $s \in \mathbb{Q}$ with $|s| \geqslant s_0$, the manifold $M_K(s)$ obtained by s-Dehn surgery on K admits a taut foliation. Furthermore, this foliation is transverse to the core of the surgery torus.*

10 Kristen Hendricks: Symplectic Annular Khovanov Homology and Knot Symmetry

Abstract: Khovanov homology is a combinatorially-defined invariant which has proved to contain a wealth of geometric information. In 2006 Seidel and Smith introduced a candidate Lagrangian Floer analog of the theory, which has been shown by Abouzaid and Smith to be isomorphic to the original theory over fields of characteristic zero. The relationship between the theories is still unknown over other fields. In 2010 Seidel and Smith showed there is a spectral sequence relating the symplectic Khovanov homology of a two-periodic knot to the symplectic Khovanov homology of its quotient; in contrast, in 2018 Stoffregen and Zhang used the Khovanov homotopy type to show that there is a spectral sequence from the combinatorial Khovanov homology of a two-periodic knot to the annular Khovanov homology of its quotient. (An alternate proof of this result was subsequently given by Borodzik, Politarczyk, and Silvero.) These results necessarily use coefficients in the field of two elements. This inspired investigations of Mak and Seidel into an annular version of symplectic Khovanov homology, which they defined over characteristic zero. In this talk we introduce a new, conceptually straightforward, formulation of symplectic annular Khovanov homology, defined over any field. Using this theory, we show how to recover the Stoffregen-Zhang spectral sequence on the symplectic side. We further give an analog of recent results of Lipshitz and Sarkar for the Khovanov homology of strongly invertible knots. This is work in progress with Cheuk Yu Mak and Sriram Raghunath.

10.1 (Symplectic) Khovanov Homology

Khovanov homology assigns to a link $L \subseteq S^3$ a bigraded vector space $\mathrm{Kh}(L)$ over a field $\mathbb{F}$, whose graded Euler characteristic is the Jones polynomial. It is connected to other invariants in low-dimensional topology, such as Heegaard Floer homology.

Theorem 10.1 (Ozsváth-Szabó)**.** *There exists a spectral sequence from the reduced Khovanov homology of a link L to the Heegaard Floer homology of its branched double cover:*

$$\widehat{\mathrm{Kh}}(L;\mathbb{F}_2) \implies \widehat{HF}(\Sigma(\overline{L})).$$

The construction of symplectic Khovanov homology is analogous to the setup of Lagrangian Floer homology. We briefly recall the essential components of this setup.

- Let (M,ω) be an exact symplectic manifold, $\omega = d\lambda$, which is convex at infinity.
- Let L_0, L_1 be two exact, compact Lagrangian submanifolds satisfying $\lambda|_{L_i} = df_i$ for some functions $f_i : L_i \to \mathbb{R}$.
- This data (M, L_0, L_1) gives rise to the Lagrangian Floer cochain complex $\mathrm{CF}(L_0, L_1) = (\mathbb{F}\langle L_0 \pitchfork L_1\rangle, \partial)$, whose homology is the Floer homology $\mathrm{HF}(L_0, L_1)$.

To define symplectic Khovanov homology, let $p(z) = \prod_{i=1}^{n}(z - k_i)$ be a polynomial with distinct real roots. Consider the surface S in $\mathbb{C}^3$ defined by the equation:

$$S := \left\{(u, v, z) \in \mathbb{C}^3 \mid u^2 + v^2 + p(z) = 0\right\}.$$

Within the n-fold symmetric product $\mathrm{Sym}^n(S)$, we define two totally real submanifolds Σ_A and Σ_B. These submanifolds are constructed from collections of "thimbles" attached to the critical points of $p(z)$. This construction takes place within a resolution of singularities of $\mathrm{Sym}^n(S)$, namely a space $\mathcal{Y}_n$ which is a subset of the Hilbert scheme of n points on S, $\mathrm{Hilb}^n(S)$. The space $\mathcal{Y}_n$ is defined via the Hilbert–Chow map $\mathrm{HC} : \mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ as

$$\mathcal{Y}_n := \mathrm{HC}^{-1}\left(\{\{(u_1, v_1, z_1), \ldots, (u_n, v_n, z_n)\} \mid z_i = z_j \implies (u_i, v_i) = (u_j, v_j)\}\right).$$

The submanifolds $\Sigma_A, \Sigma_B \subset \mathcal{Y}_n$ are Lagrangian.

Definition 10.2. *The **symplectic Khovanov homology** of a link L, denoted $\mathrm{Kh}_{symp}(L)$, is defined as the Lagrangian Floer homology of the pair (Σ_A, Σ_B):*

$$\mathrm{Kh}_{symp}(L) := HF(\Sigma_A, \Sigma_B).$$

Theorem 10.3 (Abouzaid-Smith). *Over a field of characteristic zero, there is an isomorphism of bigraded vector spaces:*

$$\mathrm{Kh}(L) \cong \mathrm{Kh}_{symp}(L).$$

The group $O(2)$ acts on the (u, v)-plane, which induces a symplectic action on the triple $(\mathcal{Y}_n, \Sigma_A, \Sigma_B)$. The existence of such symmetries leads to powerful results via equivariant Floer theory.

Example 10.4 (Smith, Borel). *Let τ be an involution on a topological space X, and let X^{Fix} denote the fixed-point set of τ. We have a spectral sequence relating the homology of X to the homology of its fixed-point set. In the context of Borel homology, this takes the form:*

$$\begin{aligned} H^*(X; \mathbb{F}_2) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies & H^*_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{F}_2) \\ & \cong H^*(X^{\mathrm{Fix}}; \mathbb{F}_2) \otimes \mathbb{F}_2[\theta, \theta^{-1}], \end{aligned}$$

where $H^*_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{F}_2)$ is the $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology of X and $H^*(\mathrm{B}\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2) \cong \mathbb{F}_2[\theta]$.

The terms in the spectral sequence are related as follows:

$$\mathbb{F}_2[\theta, \theta^{-1}] = H^*(\mathrm{B}\mathbb{Z}/2\mathbb{Z}; \mathbb{F}_2)$$

and

$$H_{\mathbb{Z}/2\mathbb{Z}}(X; \mathbb{F}_2) = \mathit{Ext}_{\mathbb{F}_2[\mathbb{Z}_2]}(C_*(X), \mathbb{F}_2)$$

Example 10.5 (Seidel-Smith, 2010). *This principle extends to Lagrangian Floer homology. Let τ be a symplectic involution on M such that $\tau(L_i) = L_i$ for $i = 0, 1$. Then there is a spectral sequence relating the Floer homology of (M, L_0, L_1) to that of the fixed-point sets $(M^{\mathrm{Fix}}, L_0^{\mathrm{Fix}}, L_1^{\mathrm{Fix}})$:*

$$HF(M, L_0, L_1) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies HF(M^{\mathrm{Fix}}, L_0^{\mathrm{Fix}}, L_1^{\mathrm{Fix}}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

An important symmetry is the *intrinsic* symmetry given by the involution $(u, v, z) \mapsto (u, -v, z)$. Manolescu studied its consequences.

Theorem 10.6 (Manolescu). *The intrinsic symmetry $(u, v, z) \mapsto (u, -v, z)$ induces a spectral sequence:*

$$\mathrm{Kh}_{symp}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies g\widehat{HF}(\Sigma(\overline{L})) \otimes H_*(S^1) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

An *extrinsic* symmetry arises when the link L itself is symmetric, for example, a two-periodic link. Let L be a two-periodic link with quotient $\overline{L}$. This periodicity corresponds to an involution $\tau : (u, v, z) \mapsto (u, v, -z)$ on the surface S. The quotient surface is $\overline{S} = S/\tau$.

Theorem 10.7. *The involution τ on S induces an involution on $\mathrm{Hilb}^n(S)$ and $\mathrm{Sym}^n(S)$, and the spaces for the quotient link $\overline{L}$ embed into the fixed-point sets. This relationship is summarized by the following commutative diagram:*

$$\begin{tikzcd} \mathrm{Hilb}^{n/2}(\overline{S}) \arrow[r, hook] \arrow[d, "\mathrm{HC}"'] & \mathrm{Hilb}^n(S)^\tau \\ \mathrm{Sym}^{n/2}(\overline{S}) \arrow[r, hook] & \mathrm{Sym}^n(S)^\tau \arrow[u, "\mathrm{HC}"'] \end{tikzcd}$$

Theorem 10.8 (Seidel-Smith). *For a two-periodic link L with quotient $\overline{L}$, there is a spectral sequence:*

$$\mathrm{Kh}_{symp}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies \mathrm{Kh}_{symp}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

This result has a combinatorial counterpart involving annular Khovanov homology.

Theorem 10.9 (Stoffregen-Zhang, 2018; Borodzik-Politarczyk-Silvero). *For a two-periodic link L with quotient $\overline{L}$, there is a spectral sequence:*

$$\mathrm{Kh}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies \mathrm{AKh}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

10.2 Annular (Symplectic) Khovanov Homology

Annular Khovanov homology, $\mathrm{AKh}(L)$, is a refinement of Khovanov homology for links in the solid torus, introduced by Asaeda, Przytycki, and Sikora, and further developed by Roberts. It can be understood as the associated graded object of a filtration on the Khovanov complex. Mak and Seidel later defined a symplectic version over characteristic zero.

Theorem 10.10 (Mak-Seidel, 2019)**.** *Over a field of characteristic zero, there is an isomorphism*

$$\mathrm{AKh}^{HH}_{symp}(L) \cong \mathrm{AKh}(L),$$

where AKh^{HH}_{symp} *is defined using Hochschild homology.*

We present a new, more direct definition of symplectic annular Khovanov homology, $\mathrm{AKh}_{\mathrm{symp}}$. This theory is constructed by replacing the ambient space $\mathrm{Hilb}^n(S)$ with $\mathrm{Hilb}^n(S \backslash D)$, where D is a divisor over $z = 0$. This geometric modification corresponds to working with an annulus instead of a disk.

Theorem 10.11 (Hendricks-Mak-Raghunath)**.** *The resulting homology theory,* $\mathrm{AKh}_{symp}(L)$*, is an invariant of the link* L*.*

Conjecture 10.12. *Over a field of characteristic zero, this newly defined* $\mathrm{AKh}_{symp}(L)$ *is isomorphic to the Hochschild homology version* $\mathrm{AKh}^{HH}_{symp}(L)$*.*

This new framework allows us to analyze the effect of symmetries in the annular setting.

- For the intrinsic symmetry $(u, v, z) \mapsto (u, -v, z)$, we obtain a spectral sequence

$$\mathrm{AKh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies \widehat{CFK}(\Sigma(mL), \tilde{A}) \otimes H_*(S^1) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

- For the extrinsic symmetry $(u, v, z) \mapsto (u, v, -z)$ of a two-periodic link, we obtain

$$\mathrm{AKh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies \mathrm{AKh}_{\mathrm{symp}}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

  This provides a symplectic analogue of the Stoffregen–Zhang spectral sequence.

- For a strongly invertible link, with symmetry $(u, v, z) \mapsto (-u, -v, -z)$, we consider the induced action on the original (non-annular) symplectic Khovanov homology. The fixed-point set of the action $\{(u, v, z), (-u, -v, -z)\}$ relates to the annular theory for the quotient. This yields a spectral sequence

$$\mathrm{Kh}_{\mathrm{symp}}(L) \otimes \mathbb{F}_2[\theta, \theta^{-1}] \implies \mathrm{AKh}_{\mathrm{symp}}(\overline{L}) \otimes \mathbb{F}_2[\theta, \theta^{-1}].$$

10.3 Summary of Results

We can summarize the results in the following table:

Table 1: Symmetries and Induced Spectral Sequences

Action on (u, v, z)	**Spectral Sequence Consequence**	**Combinatorial Analog**
$(u, -v, z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $g\widehat{\mathrm{HF}}(\Sigma(mL))$	Ozsvath-Szabo
$(u, v, -z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $\mathrm{Kh}_{\mathrm{symp}}(\overline{L})$	None
$(u, -v, z)$	$\mathrm{AKh}_{\mathrm{symp}}(L)$ to $\widehat{\mathrm{HF}}\left(\Sigma(mL), \hat{A}\right)$	Roberts
$(u, v, -z)$	$\mathrm{AKh}_{\mathrm{symp}}(L)$ to $\mathrm{AKh}_{\mathrm{symp}}(\overline{L})$	Zhang
$(-u, -v, -z)$	$\mathrm{Kh}_{\mathrm{symp}}(L)$ to $\mathrm{AKh}_{\mathrm{symp}}(\overline{L})$	Szabó-Ozsváth; Borodzik-Politarczyk-Silvero

11 Vardan Oganesyan: How to Construct Symplectic Homotopy Theory

Abstract: In 1968 Dold and Thom proved that singular homology groups of X are isomorphic to homotopy groups of infinite symmetric product of X. In 1990-2000 Morel, Suslin, and Voevodsky used a similar definition to define motivic cohomology groups of algebraic varieties. Moreover, they defined homotopy theory for algebraic varieties. Motivated by these results, we construct homotopy theory for symplectic manifolds. In particular, we define some new homology groups for symplectic manifolds and prove that these homology groups have all required properties. We will not discuss details, but we will show that these new homology groups appear in a very natural way. If time permits, we will also discuss some possible applications.

11.1 Introduction to the Dold-Thom Construction

Let (X, e) be a pointed topological space. The n-th symmetric product of X, denoted $\mathrm{SP}^n(X)$, is defined as the quotient space

$$\mathrm{SP}^n(X) = X^n/S_n,$$

where S_n is the symmetric group on n letters acting by permutation of coordinates.

We can define a sequence of inclusions by stabilizing with the basepoint $e \in X$:

$$\mathrm{SP}^n(X) \hookrightarrow \mathrm{SP}^{n+1}(X)$$

given by the map $\{p_1, \ldots, p_n\} \mapsto \{p_1, \ldots, p_n, e\}$. This yields a filtration

$$\mathrm{SP}^0(X) \hookrightarrow \mathrm{SP}^1(X) \hookrightarrow \cdots \hookrightarrow \mathrm{SP}^n(X) \hookrightarrow \ldots$$

where $\mathrm{SP}^0(X)$ is a point corresponding to the empty set.

Definition 11.1. *The* ***infinite symmetric product*** *of X is the direct limit of this sequence:*

$$\mathrm{SP}(X) = \bigcup_{n \geqslant 0} \mathrm{SP}^n(X).$$

The space $\mathrm{SP}(X)$ is an abelian semigroup with the operation given by the union of finite sets:

$$\{p_1, \ldots, p_n\} + \{q_1, \ldots, q_k\} = \{p_1, \ldots, p_n, q_1, \ldots, q_k\}.$$

An element in $\mathrm{SP}(X)$ can be viewed as a formal finite sum of points in X.

Let Δ^n be the standard topological n-simplex. The set of continuous maps $\mathrm{Map}(\Delta^n, \mathrm{SP}(X))$ forms a semigroup. We can turn this into an abelian group via the Grothendieck group construction, which we denote by $C_n(X)$:

$$C_n(X) = \mathrm{Map}(\Delta^n, \mathrm{SP}(X))^+.$$

The elements of $C_n(X)$ are formal differences of maps $f - g$. The collection of these groups forms a chain complex $C_*(X)$ with a boundary operator $d : C_n(X) \to C_{n-1}(X)$ defined by the alternating sum of face maps $\partial_k : C_n(X) \to C_{n-1}(X)$:

$$d = \sum_{k=0}^{n} (-1)^k \partial_k,$$

where ∂_k is induced by pre-composition with the standard inclusion of the k-th face $\Delta^{n-1} \hookrightarrow \Delta^n$. It is a standard result that this operator satisfies $d^2 = 0$.

11.2 The Dold-Thom Theorems

The homology of the chain complex constructed above recovers the singular homology of the original space. This remarkable result is the content of the Dold-Thom theorem.

Theorem 11.2 (Dold, Thom, 1968). *The homology groups of the chain complex $(C_*(X), d)$ are naturally isomorphic to the singular homology groups of X with integer coefficients. Furthermore, these homology groups are the homotopy groups of the infinite symmetric product space:*

$$H_*(C_*(X), d) \cong \pi_*(\mathrm{SP}(X)) \cong H_*^{sing}(X; \mathbb{Z}).$$

This framework was adapted by Morel, Suslin, and Voevodsky to the setting of algebraic geometry to define motivic cohomology.

Theorem 11.3 (Voevodsky, Suslin, Morel, 1990s). *Let X be a smooth algebraic variety over a field. Let Δ^n_{alg} be the algebraic n-simplex, defined as $\{(z_0, \ldots, z_n) \in \mathbb{A}^{n+1} \mid \sum z_i = 1\}$. Consider the abelian group of algebraic maps*

$$C_n^{alg}(X) = \mathrm{Map}_{alg}(\Delta^n_{alg}, \mathrm{SP}(X))^+.$$

*The homology of the resulting algebraic chain complex $(C_*^{alg}(X), d)$ defines the **Suslin homology** of X, denoted $H_*^{sus}(X)$.*

Remark 11.4. *The Suslin homology groups can contain rich geometric information. For example:*

$$H_0^{sus}(T^2) = \mathbb{Z} \times T^2$$
$$H_0^{sus}(\Sigma_g) = \mathbb{Z} \times \mathrm{Jac}(\Sigma_g)$$

where T^2 is the two-torus and Σ_g is a compact Riemann surface of genus g, with $\mathrm{Jac}(\Sigma_g)$ *its Jacobian variety.*

11.3 A Framework for Symplectic Homotopy Theory

We aim to construct a similar theory for symplectic manifolds. A key step is to define the appropriate category and the corresponding notion of "maps" into the symmetric product.

11.3.1 Categories and Sheaves

Let X be a symplectic manifold. Similar to the algebraic case, for an open set $\mathcal{U}$ in a variety Y, we can consider the set of maps from $\mathcal{U}$ into $\mathrm{SP}(X)$ to define a presheaf. Sheafification then yields a sheaf on Y.

$$\mathcal{U} \mapsto C_*(\mathcal{U}, X)$$

To adapt this to the symplectic setting, we must define an appropriate category of maps. We consider several possibilities:

1. Morphisms are **symplectic embeddings**.
2. Morphisms are **generalized Lagrangian correspondences**.
3. Morphisms are **J-holomorphic maps**.

The second option, involving Lagrangian correspondences, appears to be the most powerful but is also the most technically challenging.

Problem 11.5. *What is the symplectic analogue of the simplex Δ^n? What is the correct notion of a map from a symplectic manifold Y to* $\mathrm{SP}(X)$*?*

To address this, we introduce the notion of a symplectic correspondence. Let (Y, ω_Y) and (X, ω_X) be symplectic manifolds. A **symplectic correspondence** from an open set $\mathcal{U} \subseteq Y$ to $\mathrm{SP}^n(X)$ is a symplectic embedding $\mathcal{U} \hookrightarrow X^n/S_n$. A more concrete description involves a collection of maps $\{f_1, \ldots, f_n\}$ from $\mathcal{U}$ to

X satisfying a certain condition on the symplectic forms. For instance, we can consider unordered n-tuples of maps $f_i : \mathcal{U} \to X$ such that

$$\sum_{i=1}^{n} f_i^* \omega_X = \omega_{\mathcal{U}}.$$

This defines a presheaf. After sheafification, the global sections form a set we denote $\mathrm{SCor}_n(Y, X)$. This set is our analogue of $\mathrm{Map}(Y, \mathrm{SP}^n(X))$.

To construct a group structure, we introduce isotropic correspondences. An **isotropic correspondence** $\mathcal{U} \to X^k$ is a map whose graph is an isotropic submanifold. We denote the set of such correspondences by $\mathrm{ICor}_k(Y, X)$.

Definition 11.6. *Let $F_1 \in \mathrm{SCor}_n(Y, X)$ and $F_2 \in \mathrm{SCor}_{n+k}(Y, X)$. We say F_1 and F_2 are equivalent, denoted $F_1 \sim F_2$, if there exists an isotropic correspondence $G \in \mathrm{ICor}_k(Y, X)$ such that $F_2 = F_1 + G$.*

This equivalence relation allows us to define the space of symplectic correspondences as

$$\mathrm{SCor}(Y, X) = \left(\bigsqcup_{n \geqslant 0} \mathrm{SCor}_n(Y, X) \right) / \sim,$$

which is the desired symplectic analogue of $\mathrm{Map}(Y, \mathrm{SP}(X))$.

11.3.2 A Symplectic Chain Complex

Fix a symplectic manifold M with two disjoint Lagrangian submanifolds p_0 and p_1, which we will call a "segment." An example is $M = T^*[0, 1]$. We define the n-chains from Y to X as

$$\mathrm{SC}_n(Y, X) = \mathrm{SCor}(Y \times M^n, X).$$

Face maps $\partial_{k,\epsilon} : \mathrm{SC}_n(Y, X) \to \mathrm{SC}_{n-1}(Y, X)$ are defined by restriction to the faces $Y \times M^{k-1} \times \{p_\epsilon\} \times M^{n-k}$ for $\epsilon \in \{0, 1\}$ and $k = 1, \ldots, n$. The boundary operator is the alternating sum of these face maps:

$$d = \sum_{k=1}^{n} (-1)^k (\partial_{k,1} - \partial_{k,0}).$$

One can verify that $d^2 = 0$.

Definition 11.7. *The homology of this chain complex defines the* ***embedded homology*** *groups, denoted* $\mathrm{EH}_*(Y, X)$.

$$H(\mathrm{SC}_*(Y, X); d) = \mathrm{EH}_*(Y, X).$$

It can be shown that any standard symplectic embedding from Y to X defines a non-trivial class in $\mathrm{SCor}(Y, X)$. For example, if Y is contractible, then $\mathrm{SCor}(Y, X)$ is non-empty.

11.4 Properties and Applications

11.4.1 Homotopy Invariance

The notion of homotopy translates naturally into this framework. Let M be our segment with boundaries p_0, p_1.

Definition 11.8. *Two correspondences $F_0, F_1 \in \mathrm{SCor}(Y, X)$ are M**-homotopic** if there exists a correspondence $H \in \mathrm{SCor}(Y \times M, X)$ such that*

$$H|_{Y \times \{p_0\}} = F_0 \quad \textit{and} \quad H|_{Y \times \{p_1\}} = F_1.$$

This definition leads to the expected properties for the resulting homology theory.

Proposition 11.9. *The embedded homology groups $\mathrm{EH}_*(Y, X)$ have the following properties:*

1. *M-homotopy is an equivalence relation on* $\mathrm{SCor}(Y, X)$.
2. *If $\phi_t : Y \to Y$ is a Hamiltonian isotopy, then the induced correspondences ϕ_0^* and ϕ_1^* are M-homotopic.*
3. *The assignment $(Y, X) \mapsto \mathrm{EH}_*(Y, X)$ is functorial with respect to composition of correspondences.*
4. *The groups $\mathrm{EH}_*(Y, X)$ are homotopy invariant in the sense that if $F : Y \to X$ and $G : Y' \to X$ are homotopic correspondences, they induce the same map on homology.*
5. *The theory admits long exact sequences, analogous to those in singular homology.*

11.4.2 Triangulated Persistence Categories and Further Directions

The category whose objects are symplectic manifolds and whose morphisms are $\mathrm{Mor}(Y, X) = \mathrm{SCor}(Y, X)$ forms an additive category. Following ideas of Biran, Cornea, and Zhang, one can construct a triangulated persistence category on chain complexes over this category.

A variant of this theory can be defined using J-holomorphic curves, leading to groups we denote $\mathrm{JH}_*(Y, X)$. This connects our construction to enumerative symplectic geometry.

Theorem 11.10. *Let $M = \mathbb{CP}^1 \backslash \{0, \infty\}$. If X is a Kähler manifold such that $\mathrm{JH}_0(pt, X) = 0$, then X is a projective algebraic variety.*

12 Daniel Pomerleano: Homological Mirror Symmetry for Batyrev Mirror Pairs

Abstract: I will survey a recent proof of a version of Kontsevich's homological mirror symmetry conjecture for a large class of mirror pairs of Calabi-Yau hypersurfaces in toric varieties. These mirror pairs were constructed by Batyrev from dual reflexive polytopes. The theorem holds in characteristic zero and in all but finitely many positive characteristics. This is joint work with Ganatra, Hanlon, Hicks, and Sheridan.

12.1 Introduction and Setup

We begin by establishing the geometric setting. Let K be a field. We fix a lattice $M \cong \mathbb{Z}^n$ of rank $n \geqslant 4$, and let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated real vector space. Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope. The dual polytope is denoted $\Delta^* \subset M^*_{\mathbb{R}}$, where $M^*_{\mathbb{R}}$ is the dual vector space.

From this data, we construct two fans:

1. Let $\overline{\Sigma} \subset M^*_{\mathbb{R}}$ be the fan whose one-dimensional cones (rays) are generated by the vertices of Δ^*. This is the normal fan of the polytope Δ.
2. Let $\Sigma^* \subset M_{\mathbb{R}}$ be the fan whose rays are generated by the vertices of Δ. This is the normal fan of Δ^*.

These fans give rise to toric varieties $\overline{Y} := Y_{\overline{\Sigma}}$ and $Y^* := Y_{\Sigma^*}$, respectively. Throughout, we make the following assumptions:

- The fan Σ^* is smooth, which implies that the toric variety Y^* is smooth.
- Let $P := \Delta^* \cap M^*$ denote the set of integer lattice points in the dual polytope Δ^*. Let $P^0 \subset P$ be the subset of lattice points which lie on a face of codimension $\geqslant 2$.

12.1.1 The B-Side: Complex Geometry

The B-side of the mirror correspondence is constructed from the smooth toric variety Y^*. Let $\mathcal{L}_{\Delta^*} \to Y^*$ be the line bundle associated with the polytope Δ^*. We define a family of hypersurfaces within Y^* using a superpotential W_r.

Let $\Lambda_K := \left\{\sum_{i=0}^{\infty} a_i T^{b_i} \mid a_i \in K, b_i \in \mathbb{R}, \lim_{i\to\infty} b_i = \infty\right\}$ be the Novikov ring over the field K. The superpotential is a section of $\mathcal{L}_{\Delta^*}$ given by

$$W_r = -z^0 + \sum_{p\in P} r_p z^p,$$

where the coefficients $(r_p)_{p\in P}$ are elements of Λ_K^P. The Calabi-Yau hypersurface on the B-side is the zero locus of this section:

$$X_r^* := \{W_r = 0\} \subset Y^*.$$

The derived category of coherent sheaves on this hypersurface, $\mathcal{D}^b\mathrm{Coh}(X_r^*)$, is the categorical object of interest on the B-side.

12.1.2 The A-Side: Symplectic Geometry

On the A-side, we start with the toric variety $\overline{Y} = Y_{\overline{\Sigma}}$, which is not assumed to be smooth. Let $A := \Delta \cap M$ be the set of integer lattice points in Δ. The global sections of the line bundle $\mathcal{L}_{\Delta}$ over $\overline{Y}$ include monomials z^α for each $\alpha \in A$.

We consider a family of hypersurfaces in $\overline{Y}$ defined by the equation

$$\overline{X}_t = \left\{-tz^0 + \sum_{\alpha\in A\setminus\{0\}} z^\alpha = 0\right\} \subset \overline{Y},$$

where t is a parameter. To obtain a smooth model, we take a refinement Σ of the fan $\overline{\Sigma}$ such that the corresponding toric variety $Y := Y_\Sigma$ is smooth away from a subset of high codimension. The A-side Calabi-Yau manifold, X_t, is the proper transform of $\overline{X}_t$ in Y.

The symplectic geometry of X_t is determined by a Kähler class $[\omega]$ on Y, which we restrict to X_t. We consider Kähler classes of the form

$$[\omega] = \sum_{p \in P} \ell_p \mathrm{PD}([D_p^Y]), \qquad \ell_p \in \mathbb{R}^{>0},$$

where D_p^Y are the toric divisors of Y corresponding to the rays of Σ, and $\mathrm{PD}(\cdot)$ denotes the Poincaré dual.

The categorical object on the A-side is a variant of the Fukaya category, denoted $\mathrm{Fuk}(X_t, D; \Lambda)$, where $D = \cup_{p \in P}(X_t \cap D_p^Y)$ is the toric boundary divisor in X_t. The objects of this category are compact exact Lagrangian submanifolds in the complement $X_t \backslash D$. The Floer cochains are defined over the Novikov ring Λ, where holomorphic disks u with boundary on a Lagrangian are weighted by the factor $T^{\sum \ell_p (u \cdot D_p)}$.

12.2 Main Result

We can now state the main theorem, which establishes an equivalence between the A-side and B-side categories.

Theorem 12.1. *Suppose that the toric divisors D_p are connected. Away from a finite set of "bad" characteristics for the field K, there exist coefficients $b(\Lambda) = (b(\Lambda)_p)_{p \in P} \in \Lambda^P$ with $\mathrm{val}(b(\Lambda)_p) = \ell_p$ for each $p \in P$, and an equivalence of triangulated categories:*

$$\mathrm{Fuk}(X_t, D; \Lambda) \cong \mathcal{D}^b\mathrm{Coh}(X^*_{b(\Lambda)}).$$

Remark 12.2. *In characteristic 0, Homological Mirror Symmetry is known to imply Givental's Hodge-theoretic mirror symmetry, which relates the Gromov-Witten invariants of X_t to the period integrals of the mirror family X^*_r.*

This result naturally leads to the following question concerning the implications of HMS in positive characteristic.

Problem 12.3. *Is there an analogue of the Gromov-Witten implications of Homological Mirror Symmetry when working over a field of positive characteristic?*

12.3 Strategy of Proof

The proof strategy builds upon the groundbreaking work of Seidel (in the case of the quartic surface in $\mathbb{P}^3$) and proceeds in two main steps:

1. **Open Equivalence:** Establish an equivalence $\mathrm{Fuk}(X_t \backslash D) \cong \mathcal{D}^b\mathrm{Coh}(\partial Y^*)$, where ∂Y^* is the toric boundary divisor in Y^* cut out by the coordinate z^0.
2. **Deformation Theory:** Employ a deformation theory argument to extend the equivalence from the open subvarieties to the compact Calabi-Yau hypersurfaces X_t and $X^*_{b(\Lambda)}$.

The remainder of this discussion will focus on the key ideas behind Step 1.

Let $H := X_t \backslash D \subset (\mathbb{C}^\times)^n$. The central object of study is the wrapped Fukaya category $\mathcal{W}(H)$, which allows for certain non-compact Lagrangians. This category relates to the wrapped Fukaya category of the ambient torus, $\mathcal{W}((\mathbb{C}^\times)^n, H)$. An important result by Gammage and Shende provides the toric part of the correspondence.

Theorem 12.4 (Gammage, Shende)**.** *There are equivalences of categories, indicated by the following diagram:*

$$\begin{tikzcd} \mathcal{W}H) \arrow[r, phantom, "\cong"] \arrow[d, "\cup"] & \mathcal{D}^b Coh(\partial Y^*) \arrow[d] \\ \mathcal{W}(\mathbb{C}^\times)^n, H) \arrow[r, phantom, "\cong"] & \mathcal{D}^b Coh(Y^*) \end{tikzcd}$$

Further insight comes from a result of Abouzaid for toric varieties, which connects a "tropical" Fukaya category to the derived category of the Picard group.

$$\mathcal{F}_{\text{trop}}((\mathbb{C}^\times)^n, H) \simeq \text{Pic}^{dg}(Y^*)$$

These equivalences can be assembled into a larger commutative diagram that illustrates the interplay between the different categorical constructions.

Theorem 12.5. *The following diagram of categories and functors commutes:*

$$\begin{tikzcd} \mathcal{W}(H) \arrow[rr, "GS", "\cong"'] & & \mathcal{D}Coh(\partial Y^\times) \\ \mathcal{W}\left((C^\times)^n, H\right) \arrow[u, "(\cup)^*"'] \arrow[rr, "KGPS", "\cong"'] & & \mathcal{D}^b Coh(Y^\times) \arrow[u, "C^\times"'] \\ \mathcal{F}_{trop}\left((\mathbb{C}^\times)^n, H\right) \arrow[uu, bend left=60, "\subset \textit{ with the fiber}"] \arrow[rr, phantom, "\overset{A}{\cong}"] & & Pic^{dg}(Y^\times) \end{tikzcd}$$

A key tool that was unavailable in the original work of Seidel and Sheridan, is the isomorphism between Symplectic Cohomology and Hochschild Cohomology provided by the closed-open string map:

$$\text{CO} : \text{SH}^*(X_t \backslash D) \xrightarrow{\cong} \text{HH}^*(\mathcal{W}(X_t \backslash D)) \xrightarrow{\cong} \text{HH}^*(\text{Fuk}(X_t \backslash D)).$$

This is particularly powerful because the Symplectic Cohomology group $\text{SH}^*(X_t \backslash D)$ can be computed directly from the topology of the pair (X_t, D).

13 John Pardon: Derived Moduli Spaces of Pseudo-Holomorphic Curves

Abstract: We will present the derived representability approach to working with moduli spaces of pseudo-holomorphic curves.

13.1 Introduction

We are interested in the properties and structures of moduli spaces of solutions to elliptic partial differential equations, which in turn lead to the definition of enumerative invariants. These properties can be broadly categorized as follows:

1. **Global Topological Property**: Compactness of the moduli space, as established by Uhlenbeck and Gromov.
2. **Local Structure**: Regularity of the moduli space. This property can be understood in two different, though related, ways:
   (a) **Classical Regularity**: The moduli space $\mathcal{M}$ is locally isomorphic to $\mathbb{R}^n$ (or, more generally, spaces like $\mathbb{R} \times \mathbb{R}^m_{\geqslant 0}$). Achieving classical regularity typically requires transversality, which is often established by choosing "generic" data for the partial differential equation.
   (b) **Derived Regularity**: The moduli space $\mathcal{M}$ is locally isomorphic to the zero set of a smooth function on $\mathbb{R}^n$ (or a similar space). Derived regularity holds in much wider generality. However, describing the precise structure on $\mathcal{M}$ that encodes such a chart is technically demanding, as seen in the work of Fukaya, Ono, Oh, Ohta, Li, Tian, Ruan, Seibert, and others.

Our primary goal is to associate a well-behaved moduli space with every relevant moduli problem, from which an enumerative invariant can be extracted. We will focus on the construction of the moduli space itself.

Problem 13.1. *Why is the notion of derived regularity significantly more complicated than its classical counterpart?*

The essential answer is that derived regularity is fundamentally a homological or "derived" structure, not simply a set-theoretic one.

Example 13.2. *Consider a proper submersion $Q \to B$, and let E, F be vector bundles over Q. Let $L : C^\infty(Q, E) \to C^\infty(Q, F)$ be a vertical elliptic operator. We are interested in the pushforward $\pi_* L$, which is a 2-term complex of vector bundles on B. The cohomology of this complex at a point $b \in B$ is given by $\ker L_b$ and cokerL_b, where L_b is the restriction of L to the fiber Q_b.*

This pushforward $\pi_* L$ is unique up to a contractible choice in the 2-category of 2-term vector bundles on B. A contractible choice relates two complexes $(V \xrightarrow{d} W)$ and $(V' \xrightarrow{d'} W')$ via maps $f, g : V \to V'$, $f, g : W \to W'$, and a homotopy $h : W \to V'$ such that $d'h + hd = f - g$. This relationship is illustrated by the following diagram:

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & W \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, "h"'] \\ V' \arrow[r, "d'"'] & W' \end{tikzcd}$$

To formalize this, we work within the framework of derived algebraic geometry. Let Sm be the category of smooth manifolds. We introduce DSm, the ∞-category of derived smooth manifolds. There is a functor $Sm \to DSm$ that freely adjoints finite limits while preserving finite products.

Concretely, a derived smooth manifold can be thought of as a formal symbol $\lim_K p$ for some finite diagram

$p : K \to Sm$. For instance, the derived intersection of $y = x^2$ and $y = 0$ in $\mathbb{R}$ is given by the limit:

$$\lim \left(\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd} \right) \in \mathrm{DSm}$$

Furthermore, the mapping space between two derived smooth manifolds is given by:

$$\mathrm{Hom}_{DSm}(\lim_K p, \lim_L q) = \text{sheafification on } \lim_K |p| \left(\lim_L \mathrm{colim}_{K_\Delta} \mathrm{Hom}_{Sm}(p_\Delta, q) \right)$$

Example 13.3. *Let τ be the derived point defined in the example above:*

$$\tau := \lim \left(\begin{tikzcd} & \mathbb{R} \arrow[d, "x \mapsto x^2"] \\ * \arrow[r, "0"] & \mathbb{R} \end{tikzcd} \right)$$

A map from τ to a smooth manifold $M \in Sm$ corresponds to a point $p \in M$ together with a tangent vector $v \in T_pM$.

Remark 13.4. *The category of topological manifolds is a full subcategory of the ∞-category of derived topological spaces, $D(\text{Topological Manifolds}) \subset D(Top)$.*

Crucially, every derived smooth manifold $X \in DSm$ has an associated tangent complex $TX \in \mathrm{Perf}^{\geqslant 0}(X)$, which is a perfect complex of quasi-coherent sheaves on X concentrated in non-negative degrees.

13.2 Representability

We can now define families of pseudo-holomorphic curves in the derived setting.

Definition 13.5. *Let C be a Riemann surface and (X, J) be an almost complex manifold. A **family** of ψ-holomorphic maps from C to X parameterized by a derived smooth manifold Z is a map $u : Z \times C \to X$ together with an isomorphism between the zero section and the anti-holomorphic derivative of u:*

$$(D_C u)^{0,1} \colon Z \times C \to TX \otimes_{\mathbb{C}} \overline{T^*C}$$

This isomorphism must hold in the appropriate derived sense.

The central result is that the moduli problem for such maps is representable in the category of derived smooth manifolds.

Theorem 13.6 (Pardon, Steffens)**.** *There exists a derived smooth manifold $Hol(C, X)$ and a natural bijection for any $Z \in DSm$:*

$$\{Maps\ Z \to Hol(C, X)\} \xrightarrow{\sim} \{Families\ of\ \psi\text{-}holomorphic\ maps\ C \to X\ parameterized\ by\ Z\}.$$

Remark 13.7. *For a classical ψ-holomorphic map $u : C \to X$, which corresponds to a point in the underlying classical space of $Hol(C, X)$, we have a canonical isomorphism for the cohomology of the tangent complex:*

$$H^k(T_u Hol(C, X)) = \begin{cases} \ker D_u & k = 0 \\ coker D_u & k = 1 \end{cases}$$

where D_u is the linearized Cauchy-Riemann operator:

$$D_u : C^\infty(C, u^*TX) \to C^\infty(C, u^*TX \otimes_{\mathbb{C}} \overline{T^*C})$$

Example 13.8. *If $cokerD_u = 0$ for a map u, then the tangent complex $T_u Hol(C,X)$ has cohomology supported only in degree 0. This implies that $Hol(C,X)$ is a smooth manifold in a neighborhood of u. The existence of this derived moduli space allows one to define enumerative invariants via a map L from the classical bordism ring to the derived bordism ring:*

$$\begin{array}{ccccc} \{\textit{Compact smooth mfds} \to A\}/\textit{bordism} & = & \Omega_+(A) & & \\ \Big\downarrow L & & \Big\downarrow & & \\ \{\textit{Compact derived smooth mfds} \to A\}/\textit{bordism} & = & \Omega_+^{der}(A) & \ni & [Hol(C,X)] \end{array}$$

The proof of the representability theorem relies on the following key proposition, which allows one to extend maps from derived parameter spaces to classical ones.

Proposition 13.9. *For any $Z \in DSm$ and any map $u : Z \times C \to X$ (not necessarily ψ-holomorphic), there exists a map from Z to a smooth manifold Q and an isomorphism $u \cong v|_Z$, where $v : Q \times C \to X$ is a map from a classical parameter space.*

This type of result has applications beyond the theory of pseudo-holomorphic curves. For instance, it can be used to extend classical results about stacks to the derived setting.

Theorem 13.10 (Zung)**.** *If X is a smooth stack with a submersive atlas and a proper diagonal, then X is locally isomorphic to a quotient stack $[M/G]$, where G is a compact Lie group acting on a manifold M.*

Using the proposition above, one can show that the same local structure theorem holds for any derived smooth stack.