2024 Beijing Summer Workshop in Mathematics and Mathematical Physics

Integrable Systems and Algebraic Geometry

Dedicated to the Memory of Igor Krichever

Summer 2024

This series of summer workshops is organized by the Beijing Institute of Mathematical Sciences and Applications (BIMSA). It aims at introducing young researchers to some of the active research areas in Mathematics and Mathematical Physics via a series of short lecture courses taught by some of the world's best mathematicians, combined with research talks given by world-renowned experts in the field. The theme for the inaugural workshop in Summer 2024 is Integrable Systems and Algebraic Geometry.

Courses taught by: Pavel Etingof, Samuel Grushevsky, Nikita Nekrasov, and Andrei Okounkov.

Talks given by: Mikhail Bershtein, Alexander Bobenko, Alexei Borodin, Thomas Bothner, Alexander Braverman, Ivan Cherednik, Anton Dzhamay, Sergei Lando, Henry Liu, Andrei Marshakov, Grigori Olshanski, Senya Shlosman, Stanislav Smirnov, Anton Zabrodin, Alexander Veselov, Paul Wiegmann, Da-Jun Zhang, and Youjin Zhang.

This is an unofficial set of notes scribed by Gary Hu, who is responsible for all mistakes. If you find any errors, please report them to: gh7@williams.edu

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Part I

Courses

There were four courses, each spanning five days with sessions lasting 90 minutes per day.

Week 1: Monday, June 24 to Friday, June 28

- Samuel Grushevsky: The Integrable Systems Approach To The Schottky Problem And Related Questions

  We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties. Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators, we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy, and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation. We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture, and related characterizations for Prym varieties.

- Pavel Etingof: The Hitchin System and its Quantization

  Let G be a simple complex Lie group. I will review the classical Hitchin integrable system on the cotangent bundle to the moduli space $\mathrm{Bun}_G(X)$ of principal G-bundles on a smooth complex projective curve X (possibly with punctures), as well as its quantization by Beilinson and Drinfeld using the loop group LG. I will explain how this system unifies many important integrable systems, such as Toda, Calogero–Moser, and Gaudin systems. Then I'll discuss operators (for the dual group $G^\vee$), which parameterize the (algebraic) spectrum of the quantum Hitchin system. Finally, I will discuss the analytic problem of defining and computing the spectrum of the quantum Hitchin system on the Hilbert space $L^2(\mathrm{Bun}_G(X))$, and will show that (modulo some conjectures, known in genus 0 and 1) this spectrum is discrete and parameterized by operators with real monodromy. Moreover, we will see that the quantum Hitchin system commutes with certain mutually commuting compact integral operators $H_{x,V}$ called Hecke operators (depending on a point $x \in X$ and a representation V of $G^\vee$), whose eigenvalues on the quantum Hitchin eigenfunction ψ_L corresponding to a real operators L are real analytic solutions $\beta(x, \bar{x})$ of certain differential equations $\partial\beta = 0$, $\bar{\partial}\beta = 0$ associated to L and V. This constitutes the analytic Langlands correspondence, developed in my papers with E. Frenkel and Kazhdan following previous work by Braverman–Kazhdan, Kontsevich, Langlands, Nekrasov, Teschner, and others. I will review the

analytic Langlands correspondence and explain how it is connected with arithmetic and geometric Langlands correspondence.

Week 2: Sunday, June 30 to Friday, July 5 (minus Tuesday, July 2 which is Igor Krichever's Day)

- Andrei Okounkov: From Elliptic Genera to Elliptic Quantum Groups

  This course will be an example-based introduction to elliptic cohomology, Krichever elliptic genera, rigidity, and related topics. We will work our way towards the geometric construction of elliptic quantum groups.

- Nikita Nekrasov: Integrable Many-Body Systems and Gauge Theories

  Elliptic Calogero-Moser and Toda systems, Gaudin and other spin chains are algebraic integrable systems which have intimate connections to gauge theories in two, three, and four dimensions. I will explain two such connections: first, classical, through Hamiltonian reduction and second, quantum, through dualities of supersymmetric gauge theories.

1 Samuel Grushevsky: The Integrable Systems Approach To The Schottky Problem And Related Questions

Abstract

We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties. Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators, we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy, and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation. We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture, and related characterizations for Prym varieties.

Contents

1.1 Motivation

There are two pieces of motivation, which we tie together:

1.1.1 Motivation #1: Solving differential equations

Our first motivation comes from solving differential equations. While this may initially seem like a basic approach, it becomes more sophisticated as we proceed. The theory of linear differential equations is well-established and familiar to most: for instance, linear equations such as $u'+2u=0$ or $u''''+\cdots+5u=0$ are solvable with standard techniques. However, the situation becomes more complicated with nonlinear differential equations, where the coefficients are functions of x.

Introductory courses on ordinary differential equations often cover nonlinear equations in a limited way, but the topic is far richer and more complex, leaving room for deeper exploration.

What does it mean to solve for u? In the case of linear differential equations, solutions often take the form of elementary functions such as exponentials, which are easy to solve using established methods. But for more complex equations, especially those involving nonlinearities, it is unlikely that solutions can be expressed in terms of elementary functions. Consider, for example, special functions like the Gamma function or hypergeometric functions. Of course, these functions appear in other situations as well and not just as solutions of differential equations, but we will use them as an example for motivation.

In some cases, the best way to describe a function is to state that it satisfies a given differential equation, without necessarily seeking an explicit expression. However, in this course, "solving" will mean more than just this minimal characterization. We want to gain additional insights of the function, so our goal shifts towards constructing the function explicitly.

We are also interested in systems of differential equations. Among these, we will focus on completely integrable systems. Such systems are characterized by the property that, even if explicit solutions cannot be determined, all integrals of motion are known in some form. This naturally leads to the study of algebro-geometric solutions, a class of solutions determined from algebraic geometry and constructed using specific geometric data. The foundational work in this area, and a central focus of this course, lies in the groundbreaking contributions of Igor Krichever from the 1970s, which established the basis for much of the modern study in this field.

1.1.2 Motivation #2: Curves

The second motivation arises from the study of curves. In these lectures, we define a curve as a complex projective curve that is algebraic, compact, connected, and reduced. Equivalently, it is a compact Riemann surface.

Curves are associated with a wealth of geometric data, including the Jacobian of a curve, denoted $\mathrm{Jac}(C)$. A curve can be embedded into its Jacobian via the map $C \hookrightarrow \mathrm{Jac}(C)$, where the Jacobian is a principally polarized abelian variety. This is a concept we will discuss in more detail later.

The construction $C \mapsto \mathrm{Jac}(C)$ induces an embedding $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$, where $\mathcal{M}_g$ represents the moduli space of curves of genus g, and $\mathcal{A}_g$ represents the moduli space of principally polarized abelian varieties.

This construction is of great interest because $\mathcal{M}_g$, the moduli space of curves, is inherently a geometric object, which can be studied through complex or algebraic geometry. On the other hand, $\mathcal{A}_g$, the moduli space of abelian varieties, has a more arithmetic flavor. Since all abelian varieties are quotients of complex vector spaces, the associated data is more arithmetic in nature.

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1.1.3 Linking Them Together

The main problem we are concerned with, and the most celebrated result in this area, is the Schottky problem. This 150 year old problem asks the following question:

Problem 1.1 (Schottky Problem, 1888)**.** *Which abelian varieties are Jacobians of curves? That is, describe*

$$Jac(\mathcal{M}_g) \subset \mathcal{A}_g.$$

A weaker version of the problem asks:

Problem 1.2 (Weaker Version)**.** *Given an embedding $C \hookrightarrow A$, is $A = Jac(C)$? That is, given a curve embedded into an abelian variety, is the abelian variety the Jacobian of a curve?*

This weaker version is much simpler because we are provided with much more data to work with: not only do we have the abelian variety, but we also have the specific curve embedded within it. The Schottky problem, on the other hand, is

more challenging because it asks: Given an abelian variety, can we find a curve such that this abelian variety is the Jacobian of that curve?

The connection between the two motivations lies in the idea of solving differential equations via functions originating from curves. Then, these differential equations will solve the Schottky problem. The core idea is: starting from a curve, we'll be able to construct some solutions to some differential equations. The fact that we can do this characterizes curves among all abelian varieties. This means we can try to do similar things starting from an abelian variety and we will succeed if and only if this abelian variety is Jacobian.

The main result, which is one of the most celebrated in this subject, is Krichever proof of the **Welter's trisecant conjecture** characterizing abelian varieties as Jacobians among abelian varieties by the property of the Kummer variety having certain sequences. We will not prove this fully, but rather a slightly weaker version. We are not yet in a position to understand what these words mean, and it will take us a while to get into that position.

1.2 Commuting Differential Operators

1.2.1 Formal Eigenfunctions of One Differential Operator

Let L be the general differential operator in one variable $x \in C$, defined as

$$\sum_{i=0}^{n} u_i(x) \frac{d^i}{dx_i}.$$

We are interested in finding the eigenfunctions of the equation:

$$L \cdot \Psi(x) = \text{constant} \cdot \Psi(x).$$

i.e., solving for a formal solution that can be expressed as a power series.

Exercise 1.3. *Show that it is "enough" to consider* $L = \frac{d^n}{dx^n} + \sum_{i=0}^{n-2} u_i(x) \frac{d^i}{dx^i}$.

Theorem 1.4. *For all* $x_0 \in \mathbb{C}$*, there exists a unique formal solution* Ψ *of* $L\Psi = k^n \Psi$ *of the form*

$$\Psi(x) = \sum_{s=0}^{\infty} \xi_s(x) k^{-s} e^{k(x-x_0)}$$

such that $\xi_0(x) = 1$ *and for all* $s > 0, \xi_s(x_0) = 0$.

Remark 1.5. *We call this a solution that is normalized at* x.

Note that no restriction is placed on k *- the theorem holds for any* k*. This means that, instead of solving for a single function, we can solve for all eigenfunctions simultaneously for all possible values of* k.

This observation highlights a key difference between this scenario and the finite-dimensional case in linear algebra. In finite-dimensional spaces, there is a finite

set of eigenvalues, each associated with a finite-dimensional eigenspace of solutions. However, in this infinite-dimensional context, the eigenvalue can be deformed, and the solutions will deform smoothly as the eigenvalue is varied. This is crucial to the whole story.

We denote the solution as $\Psi(x, k, x_0)$. The proof is elementary:

Proof. Take $\Psi = e^{k(x-x_0)}\left(1 + \xi_1(x)k^{-1} + \xi_2(x)k^{-2} + \dots\right)$. Differentiating, we get:

$$\frac{d^i}{dx^i}\left(\xi_s(x)k^{-s}e^{k(x-x_0)}\right) = k^{-s}\left(\frac{d^i\xi_s}{dx^i} + \frac{d^{i-1}}{dx^{i-1}} + \dots\right)e^{k(x-x_0)}.$$

Thus, at each order k^{-s}, we simply need to solve for the next ξ_s.

Exercise 1.6. *Finish the proof.*

And we are done.

□

Corollary 1.7. *Any formal solution of $L\Psi = k^n\Psi$ has the form*

$$\Psi(x, k) = \Psi(x, k, x_0) \cdot A(k, x_0).$$

Once we have the solution that is normalized at x, this corollary tells us that we can find the rest of the solutions.

Exercise 1.8. *Prove the corollary.*

1.2.2 Formal Eigenfunctions of Multiple Differential Operators

Consider

$$L_1 = \frac{d^n}{dx^n} + \sum_{i=0}^{n-2} u_i(x)\frac{d^i}{dx^i}$$

and another differential operator of the same form

$$L_2 = \frac{d^m}{dx^m} + \sum_{j=0}^{m-2} v_j(x)\frac{d^j}{dx^j}.$$

We assume that m and n are coprime, e.g., $n = 2$, $m = 3$.

In finite-dimensional linear algebra, consider the case of two linear operators. When do they share a common eigenfunction? Specifically, if we have two matrices, we must be careful: do we want them to have one common eigenfunction, or do we want them to have all eigenfunctions in common? This becomes an even more interesting question in infinite-dimensional spaces. Ideally, we want

the operators to share all eigenfunctions, meaning that the operators must commute.

Therefore, the next question to ask is:

Problem 1.9. *When is the commutator*

$$[L_1, L_2] = 0?$$

Exercise 1.10. *Write everything down explicitly for $n = 2, m = 3$.*

Theorem 1.11. *Assume we are given L_1 and $\Psi(x, k, x_0)$. Then for L_2, we have*

$$[L_1, L_2,] = 0 \implies \frac{L_2\Psi(x, k, x_0)}{\Psi(x, k, x_0)} = A(k),$$

where $A(k)$ is independent of x_0.

Proof. If $[L_1, L_2] = 0$, then $L_2\Psi(x, k, x_0)$ is also a k^n-eigenfunction of L_1:

$$L_1L_2\Psi(x, k, x_0) = L_2L_1\Psi(x, k, x_0) = L_2 \cdot k^n\Psi(x, k, x_0).$$

By the corollary, we have $L_2\Psi(x, k, x_0) = \Psi(x, k, x_0) \cdot A(k, x_0)$. For all x_0', we have $L_2\Psi(x, k, x_0') = \Psi(x, k, x_0') \cdot A(k, x_0')$. We want $A(k, x_0') = A(k, x_0)$, so:

$$\begin{aligned}\Psi(x, k, x_0)e^{k(x_0-x_0')} &= \left(e^{k(x-x_0)} + \xi_1(x)k^{-1}e^{k(x-x_0)}\right) \\ &= e^{k(x-x_0')}\left(1 + \xi_1(x)k^{-1} + \dots\right)\end{aligned}$$

which does not equal $\Psi(x, k, x_0')$. But we can conclude that:

$$\Psi(x, k, x_0)e^{k(x_0-x_0')} = \Psi(x, k, x_0) \cdot B(k, x_0').$$

Comparing $A(k, x_0')$ and $B(k, x_0')$ gives us $A(k, x_0) = A(k, x_0')$.

Exercise 1.12. *Finish this.*

The other direction: if

$$\frac{L_2\Psi(x, k, x_0)}{\Psi(x, k, x_0)} = A(k),$$

then $[L_1, L_2] = 0$, or equivalently,

$$L_1L_2\Psi(x, k, x_0) = L_1A(k)\Psi(x, k, x_0) = A(k) \cdot k^n\Psi(x, k, x_0)$$

for all x_0. Thus, $[L_1, L_2]$ has an infinite-dimensional kernel for all x_0, which implies $[L_1, L_2] = 0$.

□

We did one differential operator previously, and two differential operators above. What about three?

Corollary 1.13. *If* $[L_1, L_2] = [L_1, L_3] = 0$*, then* $[L_2, L_3] = 0$*.*

Proof. For all $\Psi(x, k, x_0)$ (eigenfunction for L_1),

$$L_2\Psi(x, k, x_0) = A_2(k)$$

and

$$L_3\Psi(x, k, x_0) = A_3(k)\Psi(x, k, x_0)$$

which imply

$$[L_2, L_3]\Psi(x, k, x_0) = 0.$$

So $[L_2, L_3]$ has an ∞-kernel for all k, x_0, implying $[L_2, L_3] = 0$ and we are done.

□

Theorem 1.14 (Burchnall, Chaundy, 1923)**.** *If* $[L_1, L_2] = 0$*, then there exists a polynomial* $Q(\alpha, \beta)$ *such that* $Q(L_1, L_2) = 0$*.*

Proof. Let $\mathcal{L}(E)$ be the space of eigenfunctions $L_1\Psi = E\Psi$ (with finite-dimensional eigenspaces for all E). Then $L_2 : \mathcal{L}(E) \to \mathcal{L}(E)$, and let $Q_E(\alpha)$ be the characteristic polynomial of L_2.

Claim: $Q_E(\alpha)$ depends polynomially on E. Thus, $Q(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]$. Consequently, $Q(L_1, L_2)|_{\mathcal{L}(E)} = 0$ for all E, implying $Q(L_1, L_2) = 0$.

□

Exercise 1.15. *Two parts:*

1. *Show that if* $[L_1, L_2] = 0$*, then* $L_2 = \frac{d^m}{dx^m}$*.*
2. *For* $[L_1, L_2] = 0, \dim L_2 \leq m + 1$*.*

The reason for focusing on explicit computations is that they play a crucial role in understanding Krichever's proof that characterizes Jacobians by their classes. These computations are an essential part of the proof.

1.3 Curves and Their Jacobians

1.3.1 Fundamental Properties

Now, let's switch to the other side of the story, focusing on curves. While these two perspectives will ultimately converge, for now, we can treat them as distinct. Consider the algebraic curve $\{Q(\alpha, \beta) = 0\} \subset \mathbb{C}^2$, which may posess singularities. For the purpose of this discussion, we will ignore the singularities, although they present important challenges in many contexts.

Let C be a complex compact genus g curve. Let's look at a couple (equivalent) definitions of the Jacobian.

If we look at the fundamental group of C, $\pi(C)$ has generators $A_1, ..., A_g$, $B_1, \ldots, B_g$, subject to the relation

$$\pi_1(C) = \langle A_1, \ldots, B_g \rangle / \prod_{i=1}^{g} [A_i, B_i] = 1.$$

Exercise 1.16. *Convince yourself that this is true.*

The homology group $H_1(C, \mathbb{C})$ is given by the quotient $\pi_1(C)/[\pi_1(C), \pi_1(C)]$, which is isomorphic to $\mathbb{Z}^{2g}$, generated by $A_1, \ldots, A_g, B_1, \ldots, B_g$. Additionally, there is a symplectic pairing on $H^1(C, \mathbb{C})$, defined by the following relations:

$$\langle A_i, A_j \rangle = 0, \quad \langle B_i, B_j \rangle = 0, \quad \langle A_i, B_j \rangle = \delta_{ij}.$$

Exercise 1.17. *Think about this via Poincaré duality, Hodge theory,...*

1.3.2 Analytic Definition of the Jacobian and Period Matrix

The analytic definition is as follows:

Definition 1.18.

$$Jac(C) = H^{1,0}(C, \mathbb{C})/H^*(C, \mathbb{Z}).$$

We can also give the explicit definition:

Definition 1.19. *Given $A_1, \ldots, A_g$, $B_1, \ldots, B_g$, there exists a unique basis $\omega_1, \ldots, \omega_g \in H^{1,0}(C, \mathbb{C})$ such that*

$$\int_{A_j} \omega_i = \delta ij.$$

*Then the **period matrix** τ of C is*

$$\tau_{ij} = \int_{B_j} \omega_i \in \mathbb{C}.$$

Theorem 1.20 (Riemann's Bilinear Relations)**.** *τ is a symmetric matrix and Im $\tau \in Mat_{g\times g}^{symmetric}(\mathbb{R})$ is positive definite.*

Remark 1.21. *$\mathbb{H}_g$ is the Siegel upper half space that is a subset of $Mat_{g\times g}$ satisfying these conditions.*

Viewing it from this perspective, the Jacobian of C is given by $\mathrm{Jac}(C) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$, where $\tau \in \mathbb{H}_g$ is a period matrix.

1.3.3 Moduli Spaces and Complex Tori

Let $\mathcal{M}_g$ denote the moduli space of genus g curves, and $\mathcal{A}_g$ the moduli space of complex g-dimensional tori, which can be expressed as $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ for $\tau \in \mathbb{H}_g$. Here, "moduli" refers to isomorphism classes of objects, considered up to biholomorphism. The key idea is the existence of a map from the Siegel upper half-space $\mathbb{H}_g$ to the moduli space of abelian varieties, described by the quotient $\mathcal{A}_g = \mathbb{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$, where $\mathrm{Sp}(2g, \mathbb{Z})$ is the symplectic group acting on the lattice. This map represents a quotient of a g-dimensional complex vector space by a rank $2g$ integral lattice.

By Riemann's bilinear relations, we have

$$(\mathbb{Z}^g + \tau\mathbb{Z}^g) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g,$$

which implies that $\mathbb{Z}^g + \tau\mathbb{Z}^g$ defines a non-degenerate lattice in $\mathbb{C}^g$. An isomorphism between these lattices corresponds to a linear map that can be lifted to $\mathbb{C}^g \to \mathbb{C}^g$, mapping one lattice to another. Such isomorphisms are represented by elements of $\mathrm{GL}(2g, \mathbb{Z})$, and the additional structure of principal polarization ensures that these transformations are symplectic matrices. While we will not explore this in further detail now, we will discuss this later.

Definition 1.22. *A **complex torus** is a quotient $\mathbb{C}^g/\Lambda$, where $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}^g$. A complex torus is called an **abelian variety** if there exists an embedding of it into $\mathbb{CP}^N$.*

Thus $\mathbb{C}^g/\Lambda$ is isomorphic to an abelian variety $A \subset \mathbb{CP}^n$. Why is $\mathbb{CP}^n$ useful? From a differential geometry perspective, it is endowed with the Fubini-Study metric. From the viewpoint of algebraic geometry, we consider the line bundle $\mathcal{O}(1)$. We can take the metric/bundle and restrict it to A, which gives a positive/ample line bundle L on A.

Definition 1.23. *A **complex principally polarized abelian variety** (A, L) is a projective variety A with an ample line bundle L such that $H^0(A, L) = 1$, and there is a group structure $A \times A \to A$.*

Exercise 1.24. *Prove that these two definitions of a principally polarized abelian variety are equivalent.*

Remark 1.25. *For the exercise, the definition is slightly imprecise. Instead of focusing directly on the line bundle itself, we should consider its Chern class. This is important because, in the context of an abelian variety, translation by a group element can modify the line bundle. Specifically, in a torus, adding a point to itself results in a translation of the line bundle, which produces a new line bundle. To avoid this, we focus on the first Chern class c_1 of the line bundle, which remains invariant under such translations.*

Let's relate everything to the setup:

Theorem 1.26 (Torelli)**.**

$$\begin{array}{ccc} \mathcal{M}_g & \hookrightarrow & \mathcal{A}_g \\ \downarrow & & \downarrow \\ C & \mapsto & J(C) \end{array}$$

is an embedding. Equivalently, given a principally polarized abelian variety that is a Jacobian, it is the Jacobian of a unique curve.

1.4 Principal Polarizations and Theta Functions

1.4.1 Principal Polarizations

Let X be an algebraic variety.

Definition 1.27. *A **polarization** of X is $c_1(L)$, where L is an ample line bundle on X.*

If X is smooth, is $c_1 \in H^{1,1}_{>0}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})$. It's easy to see that the homology of an abelian variety is just generated by H^1.

Definition 1.28. *A polarization is **principle** if* $\dim H^0(X,L) = 1$.

A physicist would likely start thinking about Ansatz - we will not do that here.

Now, let's define a principal polarization on a lattice τ

1.4.2 Theta Functions and the Theta Divisor

Definition 1.29. *For all $\tau \in \mathbb{H}_g = \{\tau \in Mat_{g\times g}(\mathbb{C}) : \tau^T = \tau, Im\ \tau > 0\}$ and for all $z \in \mathbb{C}^g$, we define the **theta function** as*

$$\theta(\tau, z) := \sum_{n\in\mathbb{Z}^g} \exp\left(\pi_i n^T(\tau n + 2z)\right)$$

The fact that τ is symmetric makes it nice, and the fact Im $\tau > 0$ makes sure the expression converges.

Exercise 1.30. *For all $m_1, m_2 \in \mathbb{Z}^g$, show that the **theta divisor***

$$\theta(z, z + m_1 + m_2\tau) = c\theta(\tau, z)$$

where c is a non-zero factor depending on m_1, m_2.

Corollary 1.31. *The $\Theta_\tau \subset A_\tau (= \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g)$ defined by $\Theta_\tau := \{z \in A_i : \theta(\tau, z) = 0\}$ is well-defined.*

It's easy to see that this is a well-defined holomorphic function on $\mathbb{C}^g$. But this corollary tells us that the zero locus is independent under translations by the lattice.

1.4.3 Principal Polarization and Moduli of Abelian Varieties

Proposition 1.32. *Θ_τ defines a principal polarization on A_τ.*

Since Θ_τ is a codimension-1 subset of A_τ, it is a divisor on A_τ. It also has a homology class in $H^{1,1}$ because it is a complex subvariety of dimension 1. This subvariety is well-defined, so it corresponds to an integral class, and we claim that it is an ample class. Furthermore, we claim that the dimension of the space of sections is 1.

Remark 1.33. *This is a bit imprecise. When we refer to polarization, we should think of three things simultaneously: an ample line bundle, its Chern class, and the space of sections. This is imprecise because for any $v \in A_\tau$, we can translate Θ_τ by v (consider $t_v\Theta_\tau := z : \Theta(\tau, z+v) = 0$).*

One final point: A_g is the moduli (stack) of principal abelian varieties and is also given by $\mathbb{H}_g/\mathrm{Sp}(2g,\mathbb{Z})$. The dimension of A_g is $\dim A_g = \frac{g(g+1)}{2}$.

1.5 Jacobians

1.5.1 Divisors, The Picard Group, and Jacobians

Definition 1.34. *A **divisor** on C is an expression of the form $\sum_{i=1}^N m_i p_i$ with $m_i \in \mathbb{Z}$ and $p_i \in C$, satisfying the condition that $deg \sum m_i p_i \to (\sum m_i) \in \mathbb{C}$. Let $f : C \to \mathbb{P}^1$ be a map such that $f \neq 0, \infty$, and*

$$\begin{aligned} div(f) &= \sum (mult_z f) \text{ zeroes } z \text{ of } f - \sum (mult_z f) \text{ poles of } f \\ &= \sum_{p \in C} ord_{\,p} f \cdot p. \end{aligned}$$

.

Definition 1.35. *The **Picard group** is*

$$Pic(C) := Div(C)/\text{divisor functions}.$$

$\mathrm{Div}(C)$ is an abelian group, and Pic is a subgroup (proof is easy since $\mathrm{div}(f\cdot g) = \mathrm{div}f + \mathrm{div}g$). This admits a map $\mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z}$, and the divisor always has degree 0 otherwise this map would not exist which is similar to the statement that the number of zeroes of a function is the same as the number of poles of a function counted with multiplicity. Note that it's fine if we have a constant map, for example.

Definition 1.36. *A **divisor** on C is $\sum_{i=1}^N m_i p_i$ with $m_i \in \mathbb{Z}, p_i \in C$ satisfying $deg \sum m_i p_i \to (\sum m_i) \in \mathbb{C}, f : C \to \mathbb{P}^1$, and $f \neq 0, \infty$ with*

$$\begin{aligned} div(f) &= \sum (mult_z f) \text{ zeroes } z \text{ of } f - \sum (mult_z f) \text{ poles of } f \\ &= \sum_{p \in C} ord_{\,p} f \cdot p. \end{aligned}$$

.

Definition 1.37. *The **Picard group** is*

$$Pic(C) := Div(C)/\textit{divisor functions}.$$

$\mathrm{Div}(C)$ is an abelian group, and Pic is a subgroup (proof is easy since $\mathrm{div}(f \cdot g) = \mathrm{div}f + \mathrm{div}g$). This admits a map $\mathrm{Pic}(C) \overset{\deg}{\to} \mathbb{Z}$, and the divisor always has degree 0 otherwise this map would not exist which is similar to the statement that the number of zeroes of a function is the same as the number of poles of a function counted with multiplicity. Note that it's fine if we have a constant map, for example.

We can define $\mathrm{Pic}^d(C) \subset \mathrm{Pic}(C) =$ degree d divisors.

We can define $\mathrm{Pic}^d(C) \subset \mathrm{Pic}(C)$ as the set of divisors of degree d.

Proposition 1.38. *$Pic^d(C)$ consists of line bundles on C of degree d, up to linear equivalence.*

Definition 1.39. *The **Jacobian** of C is given by*

$$Jac(C) = Pic^{g-1}(C).$$

1.5.2 Principal Polarization and the Structure of the Jacobian

Note that this is also approximately $\mathrm{Pic}^d(C)$, where $D \in \mathrm{Pic}^{g-1-d}(C)$. The purpose of this definition is to enable the construction of the principal polarization.

Definition 1.40. *The **principal polarization** on $Pic^{g-1}(C)$ is*

$$\Theta := \{p_1 + ... + p_{g-1}\} \subset Pic^{g-1}(C).$$

One should convince oneself that $\dim(\Theta) = g - 1$, $\dim(\mathrm{Jac}(C)) = g$, and that Θ is ample and has one section.

Remark 1.41. *This is not the "best" definition of the Jacobian, since it is not clear that the Jacobian is a group: There exists a natural map $Pic^0(C) \times Pic^0(C) \to Pic^0(C)$ defined by $D_1 \times D_2 \mapsto D_1 + D_2$, but there does not exist a natural map $Pic^{g-1}(C) \times Pic^{g-1}(C) \to Pic^{g-1}(C)$. However, we can take $D_1 \times D_2 = D_1 + D_2 - K_C$, where K_C is the canonical divisor with $deg(K_C) = 2g - 2$. Moreover, $Pic^{g-1}(C)$ has a natural involution given by $D \mapsto K_C - D$, which is responsible for Serre duality. This serves as a warning that although we would like to think of the Jacobian as $Pic^{g-1}(C)$ because the theta divisor can be seen as the polarization inside it in a natural way, we are losing the natural group structure.*

Theorem 1.42 (Riemann's Theta Singularity Theorem)**.** *Let C be a smooth projective curve of genus g, and let $L \in Pic^{g-1}(C)$, where $Pic^{g-1}(C) \cong Jac(C)$ denotes the space of line bundles of degree $g-1$ on C, identified with the Jacobian variety of C. Then, for any $L \in Pic^{g-1}(C)$, the following identity holds:*

$$mult_L\Theta = \dim H^0(C, L),$$

where $mult_L\Theta$ denotes the multiplicity of the theta divisor Θ at the point corresponding to the line bundle L, and $H^0(C, L)$ is the space of global sections of L.

Recall that if L is a line bundle on C corresponding to a divisor $D \in \mathrm{Div}^d(C)$, then

$$H^0(C, L) = \{\text{meromorphic functions } f \text{ on } C \text{ such that } \mathrm{div}(F) + D \geq 0\}.$$

On Riemann surfaces, a point is the same as a divisor, which is also the same as a line bundle. Then, we can trivialize a line bundle outside a finite collection of points, which is how we can perceive them as sections. We won't go too far into this direction.

Exercise 1.43.

1. *For $g = 1$, $C \approx Jac(C)$. What does Riemann's Theta singularity theorem say in this case? Is it true that $\mathcal{M}_1 = \mathcal{A}_1$?*
2. *For $g = 2$, how is C related to the Jacobian? And what does the Riemann's Theta singularity theorem say in this case?*
3. *For $g = 3$, what are the singularities of theta divisors on Jacobians?*

1.6 The Schottky Problem In Genera 4 and 5

1.6.1 The Map $\mathcal{J}$

Consider the map

$$\mathcal{J} : \mathcal{M}_g \to \mathcal{A}_g$$

which assigns to a curve C its Jacobian, i.e., the abelian variety $\mathrm{Jac}(C)$.

The following fundamental result characterizes the injectivity of this map:

Theorem 1.44 (Torelli)**.** *The map $\mathcal{J}$ is injective (on the coarse spaces).*

In other words, given an abelian variety and the promise that it is a Jacobian, we can determine which curve it is the Jacobian of.

Remark 1.45. *There are many proofs of this result, but we will not cover them here. This theorem is a cornerstone of many classification problems in moduli space theory. It is also an example of a result from Hodge theory, since the Jacobian is a classifying space for weight 1 Hodge structures.*

The dimensions of the moduli spaces $\mathcal{M}_g$ and $\mathcal{A}_g$ provide important context for the Schottky problem. For $g > 1$, we know that:

$$\dim(\mathcal{M}_g) = 3g - 3 \quad \text{and} \quad \dim(\mathcal{A}_g) = \frac{g(g+1)}{2}.$$

For the case of genus 2 and genus 3, we have the following proposition:

Proposition 1.46. *The maps $J : \mathcal{M}_2 \to \mathcal{A}_2$ and $J : \mathcal{M}_3 \to \mathcal{A}_3$ are dominant (i.e., their images are dense).*

This implies that most 2-dimensional and 3-dimensional varieties are Jacobians of curves, except for a small codimension subset. However, this phenomenon is specific to $g = 2$ and $g = 3$, and can be confirmed by plugging into the dimension formulas and comparing.

1.6.2 The Schottky Theorem for Genus 4

The complete solution to the Schottky problem for genus 4 is given by the following theorem, which provides a characterization of Jacobians of curves in terms of theta functions:

Theorem 1.47 (Schottky, 1880s; Igusa, 1970s)**.** *A_τ (with $\tau \in \mathbb{H}_4$) is a Jacobian (with $\mathcal{A}_\tau \in \mathcal{J}(\mathcal{M}_4)$) if and only if*

$$\sum \theta^{16} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, 0) = \left(\sum \theta^8 \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, 0) \right)$$

where

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, 0) = const_{\epsilon,\theta} \left(\tau, \frac{\tau\epsilon + \delta}{2} \right)$$

*and $\epsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^4$ are the **theta constants**.*

These theta constants are simply the values of the theta function at various 2-torsion points. A 2-torsion point means that when we multiply it by two, we get the lattice factor, i.e., it is a point on the abelian variety A_τ such that multiplying it by two results in the origin.

This gives us one equation, which we can expect because $\dim(\mathcal{A}_4) = 10$ and $\dim(\mathcal{M}_4) = 9$. This solves the Schottky problem completely for $g = 4$.

1.6.3 The Schottky Problem For Genus > 5

There is no such explicit characterization of $\mathcal{J}(\mathcal{M}_g) \subset \mathcal{A}_g$ for all $g \geq 5$.

Theorem 1.48 (Grushevsky, Salvati Manni)**.** *The Schottky-Igusa equation does not generalize to $\mathcal{J}(\mathcal{M}_5)$ in the obvious way.*

This disproves a conjecture of Belavin, Knizhnik, Morozov, D'Hoker, and Phong that the cosmological constant for $E_8 \times E_8$ and $SO(32)$ type superstring theories theories are the same. However, we do have the following result:

Theorem 1.49 (Grushevky, Farkas, Salvati Manni)**.** *There exists an explicit set of equations in the theta constants $F_1, \ldots, F_N$ such that $\mathcal{J}(\mathcal{M}_g)$ is an irreducible component of $\{F_1 = ... = F_N = 0\} \subset \mathcal{A}_g$.*

This provides a "weak" solution to the Schottky problem, first discovered by Andreotti and Mayer. While we do not fully characterize the Jacobian locus, we describe it up to additional components. There is a whole theory of the explicit Andreotti-Mayer locus, but it is quite complex and does not suffice to fully characterize the locus.

1.7 Theta Functions

1.7.1 Theta Functions on Jacobians

Let's consider the Theta function, which resides within the Jacobian, as a function of the curve. To do this, we first need to understand the Abel-Jacobi map $C \hookrightarrow \mathrm{Pic}^1(C), p \mapsto 1 \cdot p$.

Now, suppose we neglect the difference between Pic^1 and Pic^{g-1}, which of course we cannot do. We have a Theta function on Pic^{g-1}, where the divisor Θ is defined. However, this divisor can also be interpreted as a function, implying we have a function on the Jacobian, which we can restrict to the curve. The goal is to determine which function on the curve arises from this restriction.

To make sense of this function, we need to make sure that the degrees match up, as we are trying to map a point from Pic^1 into Pic^{g-1}. Recall that $\Theta := \{\theta(\tau, z) = 0\}$, and let's consider a divisor of dimension g. Fix points $p_1, \ldots, p_g \in \mathbb{C}$ and let $A_\tau = \mathrm{Jac}(C)$ be the Jacobian. Consider the function $\theta(\tau, p_1 + p_2 + ...p_g - x)$, where $x \in C$ and τ is the period matrix of c. The key observation is $p_1 + p_2 + ... + p_g - x = 0$ if $x = p_i$ for all $i = 1$. Thus, we know g zeroes of this function on the curve.

Proposition 1.50.

$$div\ \theta(\tau, p_1 + ... + p_g - x) = p_1 + ...p_g \leftrightarrow \theta(p_1 + ... + p_g - x) = 0$$

if and only if $x = p_i$.

Exercise 1.51. *Show that $\theta(\tau, z) = \theta(\tau, -z)$ for all $\tau \in \mathbb{H}_g, z \in \mathbb{C}^g$. What is Θ in genus 1? (Can you see modularity of $\theta(\tau, z)$ with respect to τ for $g = 1$?)*

1.7.2 Meromorphic Functions

Any (nonconstant) meromorphic function $C \to \mathbb{P}^1$ can be expressed in terms of theta functions (up to a constant factor):

$$\mathrm{div}(f) = \sum_{i=1}^{N} m_i x_i,$$

where $m_i \in \mathbb{Z}$

Proposition 1.52.

$$f(x) = const \prod_{i=1}^{N} \theta^{m_i}(p_1 + ... + p_{g-1} + x_i - x)$$

for any chosen points $p_1, \ldots, p_{g-1} \in C$.

Proof. The divisor of the function $\theta^{m_i}(p_1 + ... + p_{g-1} + x_i - x)$ is

$$\operatorname{div}(\theta^{m_i}(p_1 + ... + p_{g-1} + x_i - x)) = m_i \cdot (p_1 + ... + p_{g-1} + x_i).$$

Since the degree of f at 0 and ∞ is the same as $\sum m_i = 0$,

$$\sum_{i=1}^{N} m_i(p_1 + ... + p_{g-1}) + \sum_{i=1}^{N} m_i x_i = \left(\sum_{i=1}^{N} m_i\right)(p_1 + ... + p_{g-1}) + \sum m_i x_i.$$

Thus, the divisor of $f(x)$ is consistent with the degree condition.

□

1.8 Weil Reducibility

1.8.1 The Weil Reducibility Theorem

Now, we move to the topic of Weil reducibility. The following theorem states that for any curve C and for all $p, q, r, s \in C$, the intersection of two theta divisors behaves in a specific way:

Theorem 1.53 (Weil Reducibility)**.** *For all C and for all $p, q, r, s \in C$, the following inclusion holds:*

$$\Theta \cap \Theta_{p-q} \subset \Theta_{p-r} \cup \Theta_{s-q}$$

where the notation Θ_{a-b} refers to the translate of $a - b$ by Θ.

Here, Θ is a divisor. Take any point in Pic^{g-1} and add to it a divisor $p-q$ (which is also a divisor of degree $g-1$). Then, on the left hand side, we have two $g-1$ dimensional gadgets in an abelian variety and we have two large codimension-1 objects that intersect. For Jacobians, this intersection is irreducible, so the intersection will have codimension 2.

It is important to notice that the left hand side does not contain r or s anywhere, and this theorem holds for all $r, s \in C$. We're defining a two parameter family of unions of two theta divisors so that it always contains the intersection. Typically, the intersection has two components: one in Θ_{p-r} and the other in Θ_{s-q}.

Proof. Suppose $D = [L] \in \Theta \cap \Theta_{p-q}$. We have $D \in \Theta \leftrightarrow h^0(C, D) \geq 1$ by the Riemann Theta singularity theorem. Similarly, $D \in \Theta_{p-q} \leftrightarrow h^0(C, L+p-q) \geq 1$, so these are meromorphic functions on C with poles at D and a simple pole at p vanishing at q.

We have two cases:

1. If $h^0(C, D+p) = 1 \leftrightarrow$ exists a unique (up to constant factor) $s \in H^0(C, D+p) \implies s$ must vanish at q because $H^0(C, D+P) \supset H^0(C, D+p-q) \neq 0$. Then $h^0(C, D+p) \supseteq H^0(C, D) \neq 0 \implies s$ has no pole at p. Thus, $s \in H^0(C, D-q) \subseteq H^0(C, D+s-q)$ so $\text{mult}_D \Theta_{s-q} \geq 1$, ie. $D \in \Theta_{s-q}$.
2. If $h^0(C, D+p) \geq 2$. Suppose we have $s_1, s_2 \in H^0(C, D+p)$ not linearly dependent. But then there exists a linear combination $\alpha s_1 + \beta s_2$ (there exists $\alpha, \beta \in \mathbb{C}$) such that $(\alpha s_1 + \beta s_2)(r) = 0$ where $\left(\frac{\alpha}{\beta} = -\frac{s_2(r)}{s_1(r)}\right)$. If $s_1(r) = 0$, choose $\alpha = 1, \beta = 0$. Then $\alpha_1 + \beta s_2 \in H^0(c, D+p-r)$, ie. $D \in \Theta_{p-r}$.

This finishes the proof.

□

Let's try to understand what Weil reducibility means analytically for $z \in \mathbb{C}^g$ (or in Jac$C$): $\theta(z) = 0$ and $\theta(z+p-q) = 0$, then it follows that

$$\theta(z+p-r) \cdot \theta(z+s-q) = 0.$$

It would be lovely if we can write this as one equation. One way to do this is to show the existence of functions $F(z)$ and $G(z)$ such that

$$F(z)\theta(z) + G(z)\theta(z+p-q) = \theta(z+p-r) \cdot theta(z+s-q) = 0.$$

Observe that

$$\theta(z+p-r) \cdot \theta(z+s-q) \subset H^0(\text{Jac}C, 2\Theta_{p+s-r-q}).$$

We want the equation to be an equality of sections of line bundles on the Jacobian, so

$$G(z) \in H^0(z\Theta_{p+s-r-q} - \Theta_{p-q}) = H^0(2\Theta_{s-r} - \Theta_{p-q}),$$

where $2\Theta_{p+s-r-q} = \Theta^{\otimes 2} \otimes \mathcal{O}(p+s-r-q)$ and $-\Theta_{s-r} = (\Theta \otimes \mathcal{O}(s-r))^{-1}$. For constants $A, B \in \mathbb{C}$, we have

$$A \cdot \theta(z+p+s-r-q)\theta(z) + B \cdot \theta(z+s-r)\theta(z+p-q) = \theta(z+p-r)\theta(z+s-q)$$

for all $z \in \mathbb{C}^g$. To prove the equivalence of this equation to the previous one, a Koszul cohomology computation is required, though we omit the details of this computation here.

1.8.2 The Isogeny Theorem and Theta Constants

Given a Jacobian and an arbitrary quadruple of points on the curve, one observes that such a functional equation is satisfied by the theta function. Where does this equation take place? It involves three distinct sections of the bundle 2Θ. Since the bundle Θ itself has only one section and 2Θ is the square of Θ, we should determine the number sections of 2Θ. Our goal now is to extract a nice geometric statement from this equation.

Proposition 1.54. *For any principally polarized abelian variety (A, Θ),*

$$H^0(A, 2\Theta) = 2^g$$

*and a basis of $H^0(A, 2\Theta)$ is given by the **Theta constants of the 7th order**:*

$$\{\Theta[\epsilon](\tau, z) := \theta(2\tau, 2z + 2\epsilon)\}$$

for $\epsilon \in (\mathbb{Z}/2)^g$.

Proposition 1.55 (Isogeny Theorem/Riemann's Bilinear Relations)**.**

$$\theta(\tau, x + y) \cdot \theta(\tau, x - y) = \sum_{\epsilon \in (\mathbb{Z}/2)} \Theta[\epsilon](x) \cdot \Theta[\epsilon](y)$$

This is known as the isogeny theorem because there is a very interesting finite linear map of abelian varieties $A \times A \xrightarrow{\pi} A \times A, (x, y) \mapsto (x + y, x - y)$. Furthermore, this map is surjective and the kernel has two torsion points, namely $x = y$ and $2x = 0$. One might ask why $\pi^*(\Theta \boxtimes \Theta) = 2\Theta \boxtimes 2\Theta$. There are many ways to prove this, but straightforward computation works as well.

1.9 Kummer Maps

1.9.1 The Basics

Proposition 1.56. *The **Kummer map** $Kum : A \overset{|2\Theta|}{\to} \mathbb{P}^{2g-1}, z \mapsto \{\Theta[\epsilon](\tau, z)\}_{\epsilon \in (\mathbb{Z}/2)}$ defines an embedding $Kum : A/\pm 1 \hookrightarrow \mathbb{P}^{2^g - 1}$.*

This result is not trivial and requires advanced techniques, which we will not develop here. Let us examine the behavior of the Kummer map for different genera:

- For $g = 1$, the map becomes a double cover of the elliptic curve branched at two torsion points, yielding the map elliptic curve$/\pm 1 \to \mathbb{P}^1$.
- For $g = 2$, we get a surface$/\pm 1 \to \mathbb{P}^3$, which is called a **Kummer surface**. The Kummer surfaces has 16 double points. It is possible to write an explicit equation for the image and study the moduli of abelian surfaces via this map, a result that has been explored in many texts.

- For $g = 3$, the geometry becomes incredibly complicated because we have a threefold embedded in $\mathbb{P}^7$. There are many papers on this, but the maps are extremely complicated and we will not talk about this here.

For simplicity, define

$$\mathrm{Kum}(x) \cdot \mathrm{Kum}(y) := \sum_{\epsilon \in (\mathbb{Z}/2)} \Theta[\epsilon](x) \cdot \Theta[\epsilon](y),$$

precisely the expression in the Isogeny theorem. Now, let us revisit the explicit equation arising from Weil reducibility and rewrite it as follows:

$$A\theta(z)\theta(z++r-s-q) + B\theta(z+p-q)\theta(z+r-s) + C(z+p-s)\theta(z+r-q) = 0$$

where $A, B, C \in \mathbb{C}$. This is simply a rearrangement, and the inclusion of the constant C is valid since we are merely scaling the equation. Using our new notation, we can express this as:

$$\begin{aligned} &A \cdot \mathrm{Kum}\left(z + \frac{p+r-s-q}{2}\right) \cdot \mathrm{Kum}\left(\frac{p+r-s-q}{2}\right) \\ &+B \cdot \mathrm{Kum}\left(z + \frac{p+r-s-q}{2}\right) \cdot \mathrm{Kum}\left(\frac{p+s-r-q}{2}\right) \\ &+C \cdot \mathrm{Kum}\left(z + \frac{p+r-s-q}{2}\right) \cdot \mathrm{Kum}\left(\frac{p+q-r-s}{2}\right) \\ &= 0. \end{aligned}$$

We factor this expression as:

$$\mathrm{Kum}\left(z + \frac{p+r-s-q}{2}\right) \cdot (*) = 0,$$

which holds for all z, where $(\star)$ is independent of z. Since $z + \frac{p+r-s-q}{2}$ forms a basis of sections of 2Θ, this equation is equivalent to saying $\star = 0$. Thus, we obtain the following equation:

$$A{\cdot}\mathrm{Kum}\left(\frac{p+r-s-q}{2}\right) + B{\cdot}\mathrm{Kum}\left(\frac{p+s-r-q}{2}\right) + C{\cdot}\mathrm{Kum}\left(\frac{p+q-r-s}{2}\right) = 0.$$

1.9.2 The Major Results

This is an equation in $\mathbb{C}^{2^g}$ and leads us to very important the identityL

Theorem 1.57 (Fay's Trisecant Identity)**.** *For all C and for all p, q, r, s, the Kummer images*

$$\frac{p+q-r-s}{2}, \frac{p+r-s-q}{2}, \frac{p+q-r-s}{2}$$

in $\mathbb{C}^{2^g}$ are collinear.

The existence of this theorem is very surprising, which becomes evident when we go into higher dimensions.

Consider $\mathbb{CP}^{2^g-1}$, which has dimension $2^g - 1$, and a Kummer variety with dimension $\dim(\mathrm{Kum}(\mathrm{Jac}(C))) = g$. Given three points on the Kummer Variety, the Welter's conjecture states that they are collinear.

Furthermore, recall that we can pick p, q, r, s on the curve, allowing us to make them coincide. This results in a differential equation satisfied by the theta function in the limiting case. We will not discuss this approach in further detail here.

Theorem 1.58 (Gunning, 1981)**.** *If $(A, \Theta) \in A_g$ is such that Kum(A) has a 1-dim family of trisecant lines, then $A =$ Jac(C) for some $C \in \mathcal{M}_g$.*

This result provides a solution to the Schottky problem: it tells us when a principally polarized abelian variety is the Jacobian of a curve. However, there is an important caveat: we require a 1-dimensional family of trisecant lines to start with. If we are given both the abelian variety and the curve, then we know that A is the Jacobian of the curve. This is a very useful starting point, but it comes with certain limitations.

Theorem 1.59 (Welters, 1986)**.** *A germ of a 1-dim family of trisecants suffices.*

Welters also conjectured the following:

Conjecture 1.1 (Welters)**.** *If Kum(A) has one trisecant, $A =$ Jac(C).*

At first glance, this may appear to be a very different statement, as we are not given a curve or a germ of a curve as a starting point. However, this conjecture was later proven by Krichever:

Theorem 1.60 (Krichever, 2015)**.** *The Welters conjecture holds.*

We present this as one theorem, but it can be understood as three distinct cases of trisecants:

1. A fully discrete trisecant
2. A trisecant tangent at one point
3. A **flex line**, ie. tangency contact of multiplicity 3.

The proofs for all three are different. We will see the proof of the flex line case only. We will use Gunning's result freely without proof because it requires a whole set of techniques we haven't developed, even though the proof is not hard.

1.10 The Kadomtsev-Petviashvilli Equation

1.10.1 Baker-Akhiezer Functions

We begin with a point $p \in C$. Let k^{-1} be a local coordinate on C around p, with $k(p) = \infty$. This is a lot of data. In comparison, the moduli space of

curves is finite-dimensional, and adding a point adds one complex dimension to the moduli space. However, here we are making an infinite-dimensional choice, since we are selecting a local coordinate up to all orders in the Taylor series expansion. More precisely, the space of such data is infinite-dimensional and admits a forgetful map to the moduli space of curves, with infinite-dimensional fibers.

Definition 1.61. *For all C, p, k^{-1}, for all general $D = p_1 + ... + p_g$,, the **Baker-Akhiezer function** $\psi(x, y)$ is a function defined for all $x \in \mathbb{C}, z \in C$ satisfying:*

- *ψ is a meromorphic function on $C - p$ with simple poles at D, holomorphic on $C - \{p \cup D\}$.*
- *ψ has an essential singularity at p, such that $\psi(x, p)e^{-k(z)\cdot x}$ is holomorphic around p and equal to 1 at p.*

Proposition 1.62. *The Baker-Akhiezer function exists and is unique.*

Proof. Uniqueness: if ψ, ψ' are two such functions, then $\frac{\psi}{\psi'}$

- has no essential singularity at p, and is in fact equal to 1 at $p \implies \frac{\psi}{\psi'} \equiv 1$.
- is holomorphic on $C - p$.

Existence: The explicit formula for ψ is given by

$$\psi(z) := \exp\left(\int_p^z x \cdot \omega\right) \cdot \frac{\theta(z - D + ux)}{\theta(z - D)}$$

where

- $\cdot\omega$ is a differential on C, with a douple pole of the form dk at p and holomorphic elsewhere, with all A-periods zero, and
- u is a vector of B-periods of ω.

□

1.10.2 Generalized Baker-Akhiezer function

We can generalize the Baker-Akhiezer function as follows:

Definition 1.63. *Let p_0 be a point on a curve C, and let k^{-1} be a local coordinate on C around p_0. Given parameters $t_1, t_2, ... \in \mathbb{C}$ and a divisor $D = p_1 + ... + p_g \in Div^g(C)$, there exists a unique function called the **generalized Baker-Akhiezer function** $\psi(t_1, t_2, \ldots, C, p_0, k)(p)$, which is a meromorphic function on $C - p_0$ with the poles at $p_1, \ldots, p_g$, holomorphic on $C - \{p_0, \ldots, p_g\}$ and satiesfies the following condition:*

$$\lim_{p > p_0} \psi(t_1, t_2, \ldots, C, p_0, k)(p) \cdot \exp\left(\sum_{i=1}^{\infty} k^i t_i\right) = 1.$$

Similar to earlier, we can prove it's existence with an explicit formula:

$$\psi(t_1, t_2, \ldots, C, p_0, k)(p) := \exp\left(\int^p \sum t_i \cdot \omega_{i+1}\right) \frac{\theta(p + \sum U_i t_i + Z)}{\theta(p + Z)}$$

where ω_1 is the meromorphic differential on C with an ith order pole at p_0 and all A-periods zero, $U_i = \left(\int_{B_1} \omega_i, ..., \int_{B_g} \omega_i\right) \in \mathbb{C}^g$, and Z is $-D$ as a point on the Jacobian of C. We want $\int_{k\to\infty} \omega_i \sim k^i$, so we also require that $\omega_i \approx d\left(k^i\right)$ is holomorphic as $k \to \infty$ at p_0.

We won't prove uniquess, but it can be done by showing that $\frac{\theta(p+\sum U_i t_i + Z)}{\theta(p+Z)}$ does not rely on A or B-periods.

Remark 1.64. *Two minor remarks:*

1. *Notice that we did not write a lower bound: we cannot integrate from p_0 or else it diverges. We have a choice of lower bound, and since we want the limit in the definition to be 1, technically we should write*

   $$\psi(t_1, t_2, \ldots, C, p_0, k)(p) := \exp\left(\int^p \sum t_i \cdot \omega_{i+1}\right) \frac{\theta(p + \sum U_i t_i + Z)\theta(p_0 + Z)}{\theta(p + Z)\theta(p_0 + \sum U_i t_i + Z)}$$

   to normalize it, or replace 1 in the definition by $\frac{\theta(p+\sum U_i t_i + Z)}{\theta(p+Z)}$. But then this causes more difficulties to arise.

2. *The statement made here is formal. Recall earlier that we solved differential equations using formal series in k^{-1}. Here, ψ is global and meromorphic on the Riemann surface, not merely formal. This difference is crucial, and the advantage of the new formulation is that it allows for geometric solutions.*

It is also a good question of why exp converges, but we will ignore it because we will stop after the first three terms.

The main point is that $\frac{d\psi}{dt_i}$ are functions of the same type for different values of ψ (not entirely of the same sort because we haven't stated precisely what the exponentials and essential singularity at p looks like). There is a theorem which states that the space of such functions is finite-dimensional, and therefore uniqueness follows. If multiple constructions yield the same essential singularity, they must correspond to the same function. This is a source of differential equations.

1.10.3 Differential Operators

Theorem 1.65. *For all meromorphic functions $E : C \to \mathbb{CP}^1$ with an nth order pole at p_0 that are holomorphic on $C - p_0$, there exists a unique nth order differential operator $L = \sum_{i=0}^n u_i(x) \frac{d^i}{dx^i}$ such that*

$$L\psi(t_1, C, p_0, k)(p) = E(p)\psi(t_1, t_2, \ldots, C, p_0, k)(p).$$

Proof. We present the proof idea only: use the study of formal solutions to the differential equation $L\psi = E\psi$ to ensure that the essential singularity of both sides at p_0 is the same, and then use uniqueness of the Baker-Akhiezer function to complete the argment.

□

Theorem 1.66. *Let $t_1 = x, t_2 = y, t_3 = t, t_4 = t_5 = ... = 0$ and consider $\psi(x, y, t, C, p_0)(p)$. There exists unique differential operators*

$$L_1 = \sum_{i=0}^{2} u_i(x, y, t)\frac{\partial^i}{\partial x^i}$$

and

$$L_2 = \sum_{i=0}^{3} v_i(x, y, t)\frac{\partial^i}{\partial x^i}$$

such that

$$L_1\psi = \frac{\partial\psi}{\partial y}, \qquad L_2\psi = \frac{\partial\psi}{\partial t}.$$

Remark 1.67. *This is a shadow of a more general result: For $\exp(kx+Q(k)y+R(k)t)$ where $Q, R \in \mathbb{C}[k]$, there exists unique differential operators L_1, L_2 where $\deg L_1 = \deg Q$ and $\deg L_2 = \deg R$.*

Theorem 1.68. *The commutator of L_1 and L_2 satisfies the following relation:*

$$[L_1 - \frac{\partial}{\partial y}, L_2 - \frac{\partial}{\partial t}] = 0.$$

Proof. Since ψ is a kernel of the commutator, it provides an ∞-dim kernel of a differrential operator in x only, implying the commutator is 0.

□

1.10.4 The Kadomtsev-Petviashvilli Equation and Beyond

Example 1.69. *Consider*

$$\psi(x, y, t) = \exp(kx + k^2y + k^3t)(1 + \xi_1(x, y, t)k^{-1} + \xi_2(x, y, t)k^{-2} + ...)$$

near p_0. What are the corresponding differential operators?

We have

$$L_1 = \frac{\partial}{\partial x^2} - u, \qquad L_2 = \frac{\partial^3}{\partial x^3} - \frac{3}{2}u\frac{\partial}{\partial x} - w$$

where

$$u = 2\frac{\partial\xi_1}{\partial x}, \qquad w = 3\frac{\partial\xi_2}{\partial x} + 3\frac{\partial\xi_1}{\partial x^2} - \frac{3}{2}u\xi_1.$$

Now, we can compute ξ_1, ξ_2 to show that

$$u = -2\frac{\partial^2}{\partial x^2} \ln \theta(U_1 x + U_2 y + U_3 t + Z)|_{p_0}.$$

Then,

$$[L_1 - \frac{\partial}{\partial y}, \qquad L_2 - \frac{\partial}{\partial t}] = 0$$

gives a differential equation on u, ie. on Θ. Computing this gives

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{3}{2}u \cdot u_x - \frac{1}{4}y_{xxx}\right)$$

which is called a ***Kadomtsev-Petviashvilli (KP) equation****.*

Remark 1.70. *Note that we set $t_4 = t_5 = ... = 0$ for simplicity. The generalization with infinitely nonzero variables gives an integrable hierarchy of an infinite sequence of equations.*

Theorem 1.71 (Krichever)**.** *There exists a differential operators in one variable L_1, L_2 with $\gcd(ord\ L_1, ord\ L_2) = 1$ such that*

$$[L_1, L_2] = 0.$$

Furthermore, there exists a curve $C = \{Q(\alpha, \beta) = 0\} \subset \mathbb{C}^2$ satisfied by L_1, L_2, p_0 (point at ∞), and k, such that $\psi(x, c, p_0, k)(p)$ is a common eigenfunction of L_1 and L_2.

Proof. We present the proof idea only. We learned earlier that there is a polynomial equation $Q(L_1, L_2) = 0$ but this only holds for compact curves. So we need to compactify, and we need to decide whether we want $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$. It turns out we need to add only one point at ∞ and it's a singular point at ∞ which has n branches coming together. So $k = z^{-\frac{1}{n}}$, and we need to check that this is a local coordinate. Then we just need to check that the number of branches is correct and the corresponding ψ is a solution of L_1 and L_2.

□

1.11 The Flex Line Case

1.11.1 The Building Blocks

Recall the Welters' trisecant conjecture. We can rewrite it as:

$$\mathrm{Kum}(p + q - r - s) \wedge \mathrm{Kum}(p + r - q - s) \wedge \mathrm{Kum}(p + s - r - q) = 0.$$

where $p, q, r, s \in C$.

To get it into the flex case, take the limit $s \to p$. This gives:

$$\mathrm{Kum}(q - r) \wedge \mathrm{Kum}(r - q) \wedge \mathrm{Kum}(2p - r - q) = 0.$$

This is somewhat trivial since $\mathrm{Kum}(q-r) = \mathrm{Kum}(r-q)$, so we need to take the next order term as $s \to p$. Let $s = p - U\epsilon$, and we get

$$\mathrm{Kum}(q - r + U\epsilon) \wedge \mathrm{Kum}(r - q + U\epsilon) \wedge \mathrm{Kum}(2p - r - q) = 0$$

Taking the limit $\lim_{\epsilon \to 0}$ yields:

$$\mathrm{Kum}(q - r) \wedge \partial_U \mathrm{Kum}(q - r) \wedge \mathrm{Kum}(2p - r - q) = 0.$$

Next, take $C \hookrightarrow \mathrm{Jac}(C), p \mapsto 0$ and consider the limit $r \to p$. Let U and V be the first and second order derivative of the Abel-Jacobi map $C \to \mathrm{Jac}C$ at p, respectively. To interpret the derivatives in the map $C \hookrightarrow \mathrm{Jac}(C)$, we can take the explicit normalized basis $\omega_1, \ldots, \omega_g \in H^{1,0}(C, \mathbb{C}), z \mapsto \left(\int_p^z \omega_i\right)$, so that $U = (\omega_1(p), \ldots, \omega_g(p))$.

This gives

$$\mathrm{Kum}(q-p-U\epsilon-V\epsilon^2) \wedge \partial_U \mathrm{Kum}(p+U\epsilon+V\epsilon^2-q) \wedge \mathrm{Kum}(p-q-U\epsilon-V\epsilon^2) = 0/$$

Exercise 1.72. *Show that the lowest order nontrivial term is*

$$Kum(q) \wedge \partial_U Kum(q) \wedge \left(\partial_U^2 + \partial_V\right) Kum(q) = 0$$

for all $q \in C$.

This is an equation of a 1-parameter family of flex lines. Take $q \to p$, and expand the equation from the exercise to all orders.

Exercise 1.73. *Show that lowest order term is*

$$\left(\partial_U^4 - \partial_U \partial_V + \frac{3}{4}\partial_V^2 + c\right) Kum(0) = 0$$

for some constant c, where W is the third order derivative of the Abel-Jacobi map $C \to JacC$ at p.

Remark 1.74. *If we have a hyperelliptic curve and start with a Weierstrass curve, this becomes simpler.*

Exercise 1.75. *Use Riemann's bilinear relation to deduce from this the lowest order term equation for $\Theta(z)$.*

1.11.2 The Major Theorems

Theorem 1.76 (Gunning)**.** *If $A \in \mathcal{A}_g, C \subset A$, and $z_1, z_2, z_3 \in A$ satisfy*

$$Kum(p + z_1) \wedge Kum(p + z_2) \wedge Kum(p + z_3) = 0$$

for all $p \in C$, then $A = Jac(C)$.

This is weaker than a solution to the Schottky problem because the curve is given to us as we have a 1-parameter family of trisecants which are all obtained by translating each other.

Theorem 1.77 (Welters)**.** *If $A \in \mathcal{A}_g, C \subset A, z \in A$ is fixed and $Kum(p+z)$ is a flex point for all $p \in C$. Then $A = Jac(C)$.*

We can strengthen the condition from needing a geometric curve to having an ∞-order germ of a family of curve.

Theorem 1.78 (KP Hierarchy)**.** *An ∞-order formal germ of a formal family of flexes suffices.*

Theorem 1.79 (Ambarello-De Concini)**.** *For all g, there exists $N = N(g)$ such that an Nth order formal germ of a family of flexes characterizes Jacobians.*

Theorem 1.80 (Conjectured by Novikov, Proved by Shiota)**.** *For all g, the existence of a 4th order germ of a family of flexes chracterizes Jacobians ($\leftrightarrow$ KP equation).*

Morally, we want to begin with a fourth-order germ and keep increasing the order of the germ. At each step, there will be some obstruction. However, since the jet bundles are extensions of one another, we expect that if we can solve the first order, we can solve the second order, and so on.

We present one final theorem:

Theorem 1.81 (Krichever)**.** *The existence of one flex line characterizes Jacobians.*

1.11.3 Convergence of the Two Stories

We have seen two views:

1. The geometry of flex lines, which gives the equation (stated with slightly different notation compared to before)
$$\left(\partial_V - \partial_U^2 - 2p\partial_U + (E - p^2)\right) \cdot \mathrm{Kum}\left(\frac{r}{2}\right) = 0 \qquad \text{(equation } K\text{)}$$
2. The story of commuting differential operators, which gives the following equation (arising from $L_1\psi = \frac{\partial \psi}{\partial t}$):
$$\left(\partial_x^2 - u(x)\right)\psi = \partial_t \psi \qquad \text{(equation } L\text{)}$$

for
$$\psi = \frac{\theta(r + Ux + Vt + Z)}{\theta(Ux + Vt + Z)} \cdot e^{px+Ey},$$
where $u = -2\partial_x^2 \ln \theta(Ux + Vt + Z)$ for all $Z \in A$.

These two stories ultimately end up at the same point.

Theorem 1.82 (Krichever)**.** *If $(A,\Theta) \in \mathcal{A}_g$ satisfies equation K (equivalently, equation L), then $A = Jac(C)$ for some curve $C \in \mathcal{M}_g$.*

Lemma 1.83. *Equation L or equation K implies*

$$\left((\Theta_V)^2 - (\Theta_{UU})^2\right) + 2\left(\Theta_{UU}\Theta_{UUU} - \Theta_V\Theta_{UV}\right)\Theta_U + (\Theta_{VV} - \Theta_{UUUU})(\Theta_U)^2 = 0$$

along the theta divisor.

Remark 1.84. *The converse is unknown. The natural attempt would be to take a divisor of a function, take a partial derivative, and restrict it to the locus, yielding a section of the same bundle. However, we are not provided with a method to extend this section.*

1.12 Proof

The proof strategy is as follows: we have a partial differential equation for the theta divisor. If we had another differential equation that commutes with this one, then we could construct a spectral curve. . Once we have a spectral curve, we hope that the abelian variety will be the Jacobian of that spectral curve. The goal is to construct commuting differential operators, as the rest of the argument involves technical computations.

1.12.1 Finding Formal Solutions

Consider ψ as a solution of the equation L, where $u = -2\partial_x^2 \ln \tau(x,t)$, of the form

$$\psi(x,t,k) = e^{kx+k^2t}\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t)k^{-s}\right).$$

Assume that the zeroes x_i of $\tau(x,t)$ are simple as a function of t. Thus, we aim to show that $\tau(x,t) = \prod(x - x_i t)$ for $x_i \neq x_j$. Expand near $q = x_1$, we get

$$\psi = \frac{z}{(x-q)^2} + v + \omega(w-q) + ...$$
$$\psi = \frac{\alpha}{x-q} + \beta + \gamma(x-q) + ...$$

Substitute this into equation L, perform the necessary eliminations, and compute. Finally, differentiate with respect to t, using $\ddot{q} = z\omega$, $\dot{f} = \frac{\partial f}{\partial t}$, and $f' = \frac{\partial f}{\partial x}$.

Lemma 1.85. *Assuming $\tau(x,t)$ has simple zeroes at $x = q$, there exists a solution ψ of equation L that has simple poles at $x = x_i$ and is holomorphic everywhere else.*

Proof. We have the form $\xi_s = \frac{r_s}{x-q} + r_{s_0} + r_{s_1}\cdot(x-q) + ...$ Equation $L \implies z\xi'_{s+1} = \dot{\xi}_s + u\cdot\xi_s - \xi''_s$. For ξ_{s+1} to exist in this form, the residue of the previous equation at $x = q$ must vanish.

There is a miracle step: if this is true for ξ_s (no zero residue at previous step), then the residue at the next step must also vanish. This is not difficult, but it is tedious, so we omit the detailed computation.

□

1.12.2 Constructing U-Periodic Solutions Ψ

We've constructed Ψ as a formal function (which has no current meaning), but we want to extend it to a less formal solution so that it is invariant under translations by U. Let A be an abelian variety, and let A_U be the Zariski closure $\{U_z\}_{z\in\mathbb{C}} \subset A$. It turns out that $1 \leq \dim A_U \leq g$. We want "ψ to be a solution on A_U." If this holds, we might be done: if $\dim A_U = g$, we have the whole abelian variety and have constructed a global function, allowing the remaining machinery to quickly work. When $\dim A_U = 1$, we get an elliptic Calogero-Moser system. The real difficulty lies when $1 < \dim A_U < g$.

Let's define the bad locus $\sum := \{z : \theta(z) = \theta_U(z) = 0\}$.This represents a problematic set because our solutions will fail somewhere in the previous step. Since this involves two equations on abelian varieties, their codimension is either 1 or 2. If it is 1, they must share an irreducible component. However, on an indecomposable abelian variety, the divisor is irreducible, so the only way the codimension could be 1 is if the partial derivative vanishes identically on the entire divisor. This is impossible, as the partial divisor is not a section of a line bundle globally. Therefore, $\operatorname{codim}_A \sum = 2$ for indedcomposable A indecomposable (and fails for decomposable ones). We can choose a translate of A_U so this translate is not contained in $\sum$.

Lemma 1.86. *If equation D holds for the zeroes of $\tau(x,t)$, then equation L has a solution with*

$$u(x,t) = u(Ux+Z,t), \psi = e^{kx+k^2t}\phi(U_x+z,t,k),$$

where $\phi(z,t,k) = e^{bt}\,(1+\sum \xi_s(z,t)k^{-s})$, $\xi_s(z,t) = \frac{\tau_s(z,t)}{\tau(z,t)}$ for τ_s holomorphic are quasiperiodic with respect to to A_U.

This means that $\phi(z+\lambda,t,k) = \exp(...)\phi(z,t,k)$, where λ is a period in the lattice of A_U. This solution $\lambda \in$ the period lattice of A_U. This solution is unique up to some normalization, which can be explicitly written, but we will omit it for simplicity.

1.12.3 Constructing The Differential Operator

Lemma 1.87. *There exists a unique pseudo-differential operator*

$$\mathcal{L} = \partial_x + \sum_{s=1}^{\infty} \omega_s(Ux+Vt+Z)\partial_x^{-s}$$

where $z \in A$ such that

$$\mathcal{L}(U_x + Vt + z, \partial_x)\psi = k\psi$$

for ψ as in the previous lemma.

Note that ω_s is a meromorphic function of $z \in A$ with poles only along the theta divisor.

Remark 1.88.

1. *The unique existence of $\mathcal{L}$ is by computation for any $z \in \mathbb{C}^g \setminus \sum$, so $\omega_s \cdot \theta$ will be holomorphic on $\mathbb{C}^g \setminus \sum$. By Hartogs' theorem, this extends to all of $\mathbb{C}^g$.*
2. *$\mathcal{L}$ is independent of the normalization.*

1.12.4 Constructing Many Differential Operators

Attempting to construct a differential operator from a pseudo-differential operator by discarding the pseudo-differential parts does not work. Instead, define $L^m := (\mathcal{L}^m)_+ =$ the $(\partial_x^m, \ldots, \partial_x, \text{const})$ terms of $\mathcal{L}^m$. We know that $[\partial_t - \partial_x^2 + u, \mathcal{L}^m] = 0$. We can now compute

$$\left[\partial_t - \partial_x^2 + u, L_m\right] = 2\partial_x \left(\text{res}_{\partial_x} \mathcal{L}^m\right)$$

where $\text{res}_{\partial_x}\mathcal{L}^m$ denotes the ∂_x^{-1}th term. Let $F_m := \text{res}_{\partial_x}\mathcal{L}^m$.

Lemma 1.89. *For all m, F_m has at most a second order pole along Θ.*

The point of this is for all $m, F_m \in H^0(A, 2\Theta)$. Since $\dim H^0(A, 2\Theta) = 2^g$, for all but finitely many m, F_m is a linear combination of some fixed finite collection $F_{i_1} ..., F_{i_{\geq g}}$. Consequently, $F_m = a_{m_1}F_{i_1} + ... + a_{m_{i_{2^g}}}F_{i_{2^g}}$. Therefore

$$[\partial_t - \partial_x^2 + u, L_m - a_{m_1}F_{i_1} - ... - a_{m_{i_{2^g}}}F_{i_{2^g}}] = 0.$$

This implies that $L\psi = \frac{\partial}{\partial t}\psi$ gives the commutation relation $[L_m, L]$, and thus the operators satisfy a polynomial equation. This leads to the construction that for all curves and values of a, there exists a Bethe Ansatz such that an eigenfunction of L and L_m generates a curve C, which in turn leads to the abelian variety $A = \text{Jac}(C)$.

Finally, we may declare victory.

2 Pavel Etingof: The Hitchin System and its Quantization

Abstract

Let G be a simple complex Lie group. We will review the classical Hitchin integrable system on the cotangent bundle to the moduli space $\mathrm{Bun}_G(X)$ of principal G-bundles on a smooth complex projective curve X (possibly with punctures), as well as its quantization by Beilinson and Drinfeld using the loop group LG. We will explain how this system unifies many important integrable systems, such as Toda, Calogero-Moser, and Gaudin systems. Then I'll discuss opers (for the dual group $G^\vee$), which parametrize the (algebraic) spectrum of the quantum Hitchin system. Finally, we will discuss the analytic problem of defining and computing the spectrum of the quantum Hitchin system on the Hilbert space $L^2(\mathrm{Bun}_G(X))$, and will show that (modulo some conjectures, known in genus 0 and 1) this spectrum is discrete and parameterized by opers with real monodromy. Moreover, we will see that the quantum Hitchin system commutes with certain mutually commuting compact integral operators $H_{x,V}$ called Hecke operators (depending on a point $x \in X$ and a representation V of $G^\vee$), whose eigenvalues on the quantum Hitchin eigenfunction ψ_L corresponding to a real oper L are real analytic solutions $\beta(x, \bar{x})$ of certain differential equations $\partial\beta = 0$, $\bar{\partial}\beta = 0$ associated to L and V. This constitutes the analytic Langlands correspondence, developed in my papers with E. Frenkel and Kazhdan following previous work by Braverman-Kazhdan, Kontsevich, Langlands, Nekrasov, Teschner, and others. We will review the analytic Langlands correspondence and explain how it is connected with arithmetic and geometric Langlands correspondence.

Contents

2.1 Principal G-bundles

2.1.1 Motivation

This course is about Hitchin systems, both classical and quantum, as well as the Langlands correspondence. These topics involve analysis, geometry, and arithmetic, primarily in the context of function fields. All of this work centers around a geometric object known as $\mathrm{Bun}_G(X)$, which is the moduli space (or technically a stack) of principal G-bundles over a curve X.

$$\left\{\begin{array}{c}\text{Hitchin integrable systems: classical and quantum}\\ \text{Langlands correspondence for function fields}\end{array}\right\}$$
$$\updownarrow$$
$$\left\{\begin{array}{c}\text{analysis, geometry, and arithmetic}\\ \text{on } \mathrm{Bun}_G(X)\end{array}\right\}$$

Here, X is an irreducible smooth projective curve over some field, and G is a reductive algebraic group. To this data, we assign a very complex geometric object, the moduli stack of principal bundles. This stack is extremely rich in structure. When we study its geometry, analysis, or arithmetic, we uncover many profound results, including those related to integrable systems. One major topic is the **Hitchin integrable system**, a large class of finite-dimensional integrable systems associated with this data. This system includes many (perhaps most) of the interesting finite-dimensional integrable systems known. It has also played a significant role in the work of Edward Witten and others.

Rather than spending too much time on motivation, it's more effective to start with basic definitions and build up from there. Let's review some basic algebraic geometry related to this moduli stack, which is indeed complicated. Many details will be simplified or moved to the exercises.

Consider X, an algebraic variety over a field k. Often, X will be a one-dimensional smooth projective irreducible curve over k. We can keep $\mathbb{C}$ in mind as the base field, but it's also important to consider non-algebraically closed fields, such as finite fields or p-adic fields. G will be a reductive algebraic group, assumed to be connected and split. In this case, G is defined over any field you choose, essentially over the integers, and is attached to a root datum. The basic examples to consider include the general linear group GL_n, the special linear group SL_n, the projective general linear group PGL_n, symplectic groups Sp_n, and orthogonal groups O_n. One particularly important special case is GL_1, the multiplicative group. We can also consider tori, which are products of several copies of the multiplicative group: $\mathrm{GL}_1 \times ... \times \mathrm{GL}_1$.

2.1.2 Definition and Clutching Functions

To define a principal bundle, we do not need to assume that the algebraic group is reductive, connected, or split. This will be the starting point for our

exploration.

First, let G be any affine algebraic group over k.

Definition 2.1. *A **principal G-bundle** (or G-torsor) on a variety X consists of a variety P equipped with a morphism $\pi : P \to X$ and a right action of a group G on P that preserves π. Locally, in the étale topology on X, the bundle is isomorphic to the right action of G on the product $G \times X \to X$, where G acts on the first component by right multiplication.*

This means there exists an étale open cover U_i of X such that, on each U_i, we have the following isomorphism:

$$\begin{array}{ccc} P|_{U_i} & \cong & G \times U \\ & \searrow^{\pi} \quad \swarrow & \\ & U_i & \end{array}$$

where the structure is compatible with the G-action in a G-invariant manner.

The term "étale" requires further explanation. In algebraic geometry, open subsets of X typically refer to open subsets in the Zariski topology. However, étale charts are somewhat more general:

Definition 2.2. *An **étale chart** $U \to X$ consists of:*

1. *A Zariski open subset $\overline{U} \subset X$.*
2. *A finite unramified cover $U \to \overline{U}$.*

Thus, an étale chart is a map from a finite unramified cover of an open subset of X, which generalizes the concept of a standard open set. For most purposes, we can treat étale charts as if they were simply open subsets of X, as they are sufficiently well-behaved for our constructions. For example, the intersection of two étale charts U_i and U_j is given by the fiber product $P|U_i = P \cdot \times U_i$. This is not overly complicated; it simply means that the U_i's are finite unramified covers of open sets in X. However, we can treat these U_i as though they were open in the Zariski topology for convenience.

Principal G-bundles are classified by **clutching functions** $g_{ij} : U_i \cap U_j \to G$, where $U_i \cap U_j \neq$, satisfying the following conditions:

$$g_{ij} \circ g_{ji} = \mathrm{id}$$
$$g_{ij} \circ g_{jk} \circ gki = \mathrm{id}^{\times}$$

on the triple intersection $U_i \cap U_j \cap U_k$, modulo the transformation

$$h_i : U_i \to G, \quad g_{ij} \mapsto h_i \circ g_{ij} \circ h_j^{-1}.$$

Here, g_{ij} defines a 1-cocycle in the sheaf $Z^1(X, \mathcal{O}X, G)$, where $\mathcal{O}X, G$ is the sheaf of G-valued regular functions on X. The cocycle condition ensures that we are gluing trivial G-bundles on each U_i in a consistent way. Changing the functions g_{ij} on the left by an element h_i and on the right by h_j does not affect the isomorphism class of the bundle. Therefore, two cocycles that are cohomologous correspond to isomorphic bundles, implying that G-bundles are classified by the group cohomology $H^1_{\text{ét}}(X, \mathcal{O}_{X,G})$.

2.1.3 Bundles

Next, we consider the notion of an associated bundle. Given a group homomorphism $\varphi : G \to H$, an associated H-bundle can be constructed from any G-bundle by applying φ to the transition functions. Specifically, for $G = \mathrm{GL}_n$, a principal G-bundle is equivalent to a vector bundle of rank n, since the transition maps from $G \times U_i$ to $G \times U_j$ can be viewed as maps between vector spaces. This leads to a categorical equivalence between the category of principal GL_n-bundles and the category of vector bundles of rank n.

Additionally, given an algebraic representation $\rho : G \to \mathrm{GL}(V)$, every G-bundle E on X gives rise to an associated vector bundle E_ρ of rank n, where the action of G on E is translated into an action on the vector bundle.

In topology and geometry, the definitions of principal bundles are similar but involve smooth functions instead of regular functions. In the topological setting, X is a topological space and G is a topological group, while in differential geometry, X is a manifold and G is a Lie group. In complex geometry, X is a complex manifold and G is a complex Lie group, with holomorphic functions replacing regular ones. In all these contexts, principal bundles can be described by gluing data that respects the relevant structure of functions (continuous, smooth, or holomorphic). While the underlying geometric spaces may vary, the fundamental concepts of principal bundles and their classifications remain the same.

In complex geometry, a similar story applies to holomorphic functions. In topology, let X be a topological space and G a topological group, or alternatively, let X be a manifold and G a Lie group, in which case we consider smooth functions. In analytic geometry, X is a complex manifold and G is a complex Lie group, and we work with holomorphic functions. In all of these settings, we can use ordinary open subsets and obtain a principal bundle by gluing. If we are given a usual open cover, we can literally glue: if we have a point in chart U_i and a point in chart U_j which correspond to the same point on X, then we glue fibers at those points by using the corresponding map. If we have étale covers, it is slightly more complicated, and we use **faithfully flat descent** which is a similar procedure with coverings of open sets, but we will skip this because it is not relevant for our purposes.

Finally, for any point $x \in X$, the **fiber** of the principal bundle P at x is the fiber $\pi^{-1}(x)$, denoted P_x. This is a principal homogeneous space for G, meaning that

G acts on P_x transitively and simply. Note that this discussion assumes that X is defined over an algebraically closed field. If we do not have an algebraically closed field, then we should consider points not just over the original field (which may be none), but over some field extensions.

2.1.4 Serre's GAGA

One important result known as the GAGA theorem bridges the gap between analytic and algebraic structures.

Theorem 2.3 (Serre's GAGA)**.** *If X is a smooth complex projective variety, then the category of analytic G-bundles on X is equivalent to the category of algebraic G-bundles on X, where the equivalence in the direction from algebraic to analytic is given by analytification.*

This equivalence implies a one-to-one correspondence where every (smooth) algebraic variety over $\mathbb{C}$ can be viewed as a complex manifold. Similarly, any G-bundle in the algebraic context corresponds to an analytic G-bundle. This generalization extends the classical result from complex analysis that meromorphic functions on the projective line are rational functions.

2.1.5 Étale Charts

Why is étale cohomology necessary? In classical topology, we generally do not need to consider the étale topology because open neighborhoods of points are contractible and have trivial cohomology. This allows us to use these neighborhoods to compute global cohomology. However, the situation is different in algebraic geometry, particularly in the Zariski topology. For example, in the case of curves or lines, Zariski open sets are typically complements of finite sets of points. On the complex line, for instance, one might remove finitely many points, but there is no way to make these sets contractible in the same sense. This introduces complications when trying to compute cohomology.

To illustrate this, consider the principal G-bundle $P = \mathbb{C}^\times \to C^\times = X$, where the map is given by $z \mapsto z^2$. This is a μ_2-bundle, where $\mu_2 = \mathbb{Z}/2$ is the cyclic group of order 2. This bundle is not Zariski locally trivial on any non-empty Zariski open set in $\mathbb{C}^\times$. The reason for this is that the monodromy of the bundle, when traversing the circle in the punctured complex plane, results in multiplication by -1. Removing finitely many points does not eliminate this monodromy, and thus the bundle cannot be trivialized by Zariski localizations. Thus, in order to describe such bundles, we require that every bundle has an atlas of charts such that on each chart, the bundle becomes trivial. This is precisely why we need to take finite coverings to trivialize this monodromy.

For connected groups, the situation is more manageable. However, even in the case of connected groups, non-trivial subtleties can emerge, especially when dealing with more complicated varieties such as surfaces.

For connected, reductive groups, there is a well-known result, which we state here without proof:

Theorem 2.4 (Borel, Springer)**.** *If X is a smooth curve and G a connected, reductive group, any principal G-bundle on X is Zariski locally trivial.*

Remark 2.5. *For certain connected reductive groups, such as GL_n and SL_n, the result always holds. For GL_n, Hilbert's Theorem 90 guarantees this for any variety X. However, this is not true for all groups; for example, in the case of PGL_n and orthogonal groups, the situation is more subtle.*

In the case of curves, however, the story is different, and we can focus on ordinary Zariski covers without needing to invoke the étale topology. Specifically, any G-bundle on a smooth curve will trivialize after removing a finite number of points. This fact is a consequence of the structure of the Picard group for curves, which we explore further below.

For semisimple groups, we have a more refined result. The following theorem, which was later generalized by Drinfeld and Simpson to families of curves, is non-trivial:

Theorem 2.6 (Harder)**.** *If G is semisimple, then a G-bundle on a smooth affine curve is trivial. In particular, such a bundle can always be trivialized by removing a single point.*

This result does not hold for non-semisimple groups, such as GL_1. To understand this, consider the set of isomorphism classes of G-bundles on X when $G = \mathrm{GL}_1$. These isomorphism classes correspond to the set of line bundles on X, which is the Picard group $\mathrm{Pic}(X)$. For curves, the Picard group has several important properties:

- Pic is a group under the tensor product operation
- This group fits in an exact sequence

  $$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

  where $\mathrm{Pic}_0(X)$ is the connected component of the identity in $\mathrm{Pic}(X)$, and $\mathrm{Pic}_0(X) = \mathrm{Jac}(X)$ is the Jacobian of X, which is a complex torus of dimension g, where g is the genus of X.
- The Picard group without a point, denoted $\mathrm{Pic}(X \setminus x)$, is related to the original Picard group by:

  $$\mathrm{Pic}(X \backslash x) = \mathrm{Pic}(X)/\langle \mathcal{O}(x) \rangle.$$

  where $\mathcal{O}(x)$ is the line bundle corresponding to a divisor at x (a point with a pole of order at most 1). Furthermore, this exact sequence splits, so we have: $\mathrm{Pic}(X \backslash \{x\}) = \mathrm{Jac}(X)$. Therefore, if $g(X) > 0$, the bundle generally cannot be trivialized by removing a point. More precisely, for any finite set of points removed from X, there may exist a bundle that does not become trivial when those points are removed.

2.2 Moduli of G-Bundles on Smooth Projective Curves

2.2.1 Stacks

For any variety X and algebraic group G, we denote by $\mathrm{Bun}_G(X)(k)$ the set of isomorphism classes of principal G-bundles on X. Although this is just a set, there is much more to explore. We can extend this not only for a base field k, but also for any field extension of k or more generally for ring extensions. In algebraic geometry, we often extend scalars and consider bundles defined over a field extension of k, meaning that the transition functions g_{ij} will have coefficients in this extension. If A is a commutative k-algebra, we define $\mathrm{Bun}_G(X)(A)$ as the set of equivalence classes of principal G-bundles over X with coefficients in A. This defines a functor of points $\mathrm{Bun}_G(X)$, which sends $A \mapsto \mathrm{Bun}_G(X)(A)$. This functor is a central object in algebraic geometry, especially when defining affine schemes. More generally, schemes are defined as functors that satisfy certain properties, one of which is called **representability** by a scheme. However, $\mathrm{Bun}_G(X)$ is not representable by a scheme because bundles can have automorphisms. Instead, $\mathrm{Bun}_G(X)$ is an algebraic stack. The main distinction between schemes and stacks is that schemes correspond to sets of points with no additional structure, while stacks include automorphism groups for each point.

To illustrate this, consider G to be a finite group. We can then examine principal G-bundles over a point, denoted by $*$. The stack $BG := */G$, called the **classifying stack**, is a model for the moduli space of principal G-bundles. The functor of points for this stack is given by the category of principal G-bundles over an affine scheme, such as Spec$A$. This corresponds to a morphism from Spec$A$ into $*/G$, which defines a principal G-bundle over Spec$A$. We have framed the definition of bundles on varieties for simplicity, but the more general and accurate setting is to define bundles on schemes. The essential concept remains unchanged, with the term "variety" replaced by the more general term "scheme".

This stack $*/G$ consists of a one point with automorphism group G. If we consider a specific point in $\mathrm{Bun}_G(X)$ corresponding to a specific bundle, this point is not just an ordinary point, but an object like $*/G$. More generally, we can consider stacks which are global quotients. Algebraic stacks are globalizations of the notion of a quotient of a variety by group action. If Y is a variety and H is an algebraic group acting on Y, we can associate the stack Y/H with this action. The functor of points for this stack is $\mathrm{Map}(S, Y/H)$, which corresponds to a principal H-bundle $P \to S$ with a commutative diagram

$$\begin{array}{ccc} P & \to & Y \\ \downarrow & & \downarrow \\ S & \to & Y/H \end{array}$$

where $P \to Y$ commutes with the H-action.

The purpose of using stacks is to handle situations where the group H does not act freely on Y. If the group action is free, the quotient is straightforward.

However, if there are stabilizers, the quotient space may exhibit undesirable topological properties. The theory of stacks allows us to work with the quotient in a way that preserves the group action while avoiding the problematic aspects of the quotient space itself. In this framework, we consider Y with the H-action and express everything in terms of Y and H, rather than directly in terms of the quotient.

Next, we will define $\mathrm{Bun}_G(X)$. This is not a global quotient, nor is it of finite type when Y is a variety or scheme of finite type. However, it can be represented as a nested union of open subsets $\mathrm{Bun}_G(X)_j$ of the form Y/H, where Y is a smooth variety and H is a group (e.g., GL_n). Locally, one can work with such quotients in neighborhoods of points. Although there is no natural representation, various representations exist. For example, if H is a subgroup of another group H', the quotient Y/H can be "multiplied" by H', yielding an action of H', and we have the isomorphism $Y/H \cong Y \times_H H'/H'$. This reveals that there is no particularly nice or canonical representation, which makes the study of these stacks subtle. Specifically, for every field K (such as the complex numbers), $\mathrm{Bun}_G(X)(K)$ is a set, but it is not a well-behaved object beyond this.

$\mathrm{Bun}_G(X)(K)$ forms a topological space, as X/H yields a topological space. However, the topology is very non-separated, where points need not be closed. For example, in the case of the projective line $X = \mathbb{P}^1$, the closure of a point may include all points. This contrasts with the situation for schemes, which are also topological spaces but are more separated (although not Hausdorff, they have closed points over fields).

2.2.2 Examples

- SL_n-bundles are vector bundles of rank n with trivial determinant.
- PGL_n-bundles are vector bundles of rank n, modulo tensoring with line bundles.
- For GL_1-bundles with $k = \mathbb{C}$, we have:

$$\mathrm{Bun}_G(X) = \mathrm{Pic}(X) \cong \mathrm{Pic}_0(X) \times \mathbb{Z}$$

  This is a stack because for any line bundle L, the automorphism group $\mathrm{Aut}(L) = \mathbb{C}^\times$. While individual points typically have no automorphisms, in this case, every point shares the same automorphism group. This suggests that the stack is, in many ways, not much different from a scheme. To "rigidify" the situation, we can consider bundles along with a chosen point in one of the fibers. By fixing this point, the automorphism group $\mathbb{C}^\times$ is effectively trivialized, and the result is a scheme. More specifically, it is an infinite-type scheme due to the presence of infinitely many connected components, but it remains a well-behaved scheme where each component is an abelian variety.

- For SL_2 bundles, even on relatively simple curves like $\mathbb{P}^1$, the situation becomes significantly more complicated due to the potential for automorphism groups of arbitrary dimension. As the bundle degenerates, the size of its automorphism group increases correspondingly. To understand this concretely, consider line bundles on $\mathbb{P}^1$. We have $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$ because $\mathrm{Jac}(\mathbb{P}^1) = 0$, implying that all line bundles are of the form $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ for some degree $n \in \mathbb{Z}$.

  The line bundle $\mathcal{O}(1)$ can be described using two charts on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$: $U_\infty = \mathbb{P}^1 - \{0\}$ and $U_0 = \mathbb{A}^1 \cup \{\infty\}$. On $\mathbb{A}^1$, every vector bundle is trivial, a result that follows from a theorem in commutative algebra (due to Quillen, in response to a question by Serre), which states that vector bundles over a polynomial ring in one variable are equivalent to finite projective modules, and all such modules are free.

  Now, observe that the intersection $U_0 \cap U_\infty = \mathbb{A}^1 - \{0\} = \mathbb{G}_m$, the multiplicative group of nonzero scalars. To glue the two charts, we need a regular function $g : U_0 \cap U_\infty \to \mathbb{G}_m$, called the clutching map, which describes the transition from the ∞-chart to the 0-chart. This function must be non-vanishing, and it takes the form $g(z) = c \cdot z^n$ for some constant $c \neq 0$. Since this constant can be absorbed by rescaling, we may choose $c = 1$, giving the clutching function $g(z) = z^n$. The corresponding line bundle is therefore $\mathcal{O}(n)$.

In particular, if $P \to X$ is a G-bundle, the sections are maps that split the projection, meaning their composition gives the identity. We can discuss sections for principal bundles, vector bundles, and line bundles. For the line bundle $\mathcal{O}(n)$, the sections are pairs (f_0, f_∞) such that $f_0 = z^n f_\infty$, where f_0 is a polynomial in z and f_∞ is a polynomial in z^{-1}. From this relation, we see that f_0 is determined by f_∞, and since f_∞ is a polynomial of degree at most n, the space of sections has dimension $n + 1$.

Now, suppose we have a GL(2) or SL(2) bundle of the form $\mathcal{O}(n) + \mathcal{O}(-n)$, so that the transition matrix is

$$g = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

The automorphisms of E consists of matrices of the form

$$\mathrm{Aut}(\mathcal{E}) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid \begin{array}{l} a_{11}, a_{22} \in \mathbb{C}, \\ a_{21} = 0, \\ a_{12} \in \Gamma(\mathcal{O}(2n)) \end{array} \right\}$$

where the second line follows from $f_0 = z^n f_\infty$ which implies that

$$\dim \Gamma(\mathcal{O}(n)) = \begin{cases} n+1, n \geq 0 \\ 0, n < 0 \end{cases}.$$

The third condition follows from the fact that

$$\mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(-n)) = \mathrm{Hom}(O, \mathcal{O}(-2n)) = \Gamma(\mathcal{O}(-2n)) = 0,$$

for $n > 0$, and $\Gamma(\mathcal{O}(2n))$ is a space of dimension $2n + 1$. Therefore, the automorphism group $\mathrm{Aut}(E)$ consists of matrices of the form:

$$\mathrm{Aut}(E) = \{\begin{pmatrix} c & \alpha \\ 0 & c^{-1} \end{pmatrix}\}, \quad \dim \mathrm{Aut}(E) = 2n + 2$$

where α has dimension $2n+1$. We can now conclude that dim $\mathrm{Aut}(E) = 2n+2$. So this explains that the stack is complicated for SL_2 bundles.

When the genus of X is greater than 1, this stack becomes even more complicated, which is why Langlands theory and Hitchin systems theory are so rich.

2.2.3 Classification of Rank 2 Vector Bundles on $\mathbb{P}^1$

For those more familiar with analytic techniques, we can also approach this using complex numbers through the lens of geometry of holomorphic bundles.

Theorem 2.7. *Every rank 2 vector bundle E on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(m) \oplus \mathcal{O}(n)$ for some unique $m \leq n$.*

Proof. **Uniqueness**: We can recover the value of n from E by using the fact that $n = \max(i \mid \mathrm{Hom}(\mathcal{O}(i), E) \neq 0)$, because for $i > M, N$, there will be no non-zero maps into the direct sum.

Existence: Let E be a fixed rank 2 vector bundle on $\mathbb{P}^1$. It has a meromorphic section because it trivializes on some open set, either U_∞ or U_0, where there exists a meromorphic section with a pole at the missing point. This gives a map $\varphi \neq 0$, $\varphi : \mathcal{O}(m) \to E$, where φ is non-vanishing.

A subsection is a map from $\mathcal{O}(m) \to E$, but for the subsection to be meromorphic, it must have poles. To "kill" these poles, we take a negative m to account for them. This is a reasonable assumption because if we had a map where the entire fiber vanishes at some point, we would avoid this by rescaling m appropriately.

Thus, we have a short exact sequence:

$$0 \to \mathcal{O}(m_0) \to E' \to \mathcal{O}(m_j) \to 0.$$

The classification of extensions is given by:

$$\mathrm{Ext}^1(\mathcal{O}(n), \mathcal{O}(m)) = H^1(\mathbb{P}^1, \mathcal{O}(m-n)) \overset{\text{Serre dual}}{=} H^0(\mathbb{P}^1, \mathcal{O}(n-m) \otimes K),$$

where $K = \mathcal{O}(-2)$. If $m \geq n - 1$, then $n - m - 2 < 0$, implying that $\mathrm{Ext}^1 = 0$, and thus $E = \mathcal{O}(m) \oplus \mathcal{O}(n)$. Therefore, we have nothing further to prove in this case.

Assume $m < n-1$. Let r be the maximal integer such that $\mathrm{Hom}(\mathcal{O}(r), E) \neq 0$. This integer exists because it corresponds to a position in the long exact sequence.

We can realize this bundle by:

$$g = \begin{pmatrix} z^m & f(z) \\ 0 & z^n \end{pmatrix},$$

where $f(z)$ is a Laurent polynomial. We can change g by conjugating with matrices:

$$h_1 = \begin{pmatrix} 1 & \varphi(z) \\ 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & \varphi(z^{-1}) \\ 0 & 1 \end{pmatrix}.$$

It is left as an exercise to show that we can reduce f to a form where it consists of monomials z^s for $m < s < n$.

A map $\mathcal{O}(r) \to E$ is a subsection of $\mathcal{O}(-r) \otimes E$. The transition map is:

$$\begin{pmatrix} z^{m-r} & z^{-r} f(z) \\ 0 & z^{n-r} \end{pmatrix}.$$

A subsection consists of a pair:

$$\left(\begin{pmatrix} x_0(z) \\ y_0(z) \end{pmatrix}, \begin{pmatrix} x_\infty(z) \\ y_\infty(z) \end{pmatrix} \right),$$

with $x_0(z) = z^{m-r} x_\infty(z^{-1}) + z^{-r} f(z) y_\infty(z^{-1})$ and $y_0(z) = z^{n-r} y_\infty(z^{-1})$. These expressions have no monomials below degree r, so we can ignore x_0 and y_0 and only focus on x_∞ and y_∞. Thus, we can bound $\deg(y_\infty) \leq n - r$ and show that $f(z) y_\infty(z)$ has no terms of degree between $m+1$ and r.

With $n - r + 1$ unknowns (the coefficients of the polynomial $f(z) y_\infty(z)$) and $r - m$ linear homogeneous equations, nonzero solutions exist if $r - m < n - r + 1$, or equivalently, if $r \leq \frac{m+n}{2}$. Hence, we have:

$$0 \to \mathcal{O}(r_{\max}) \to E \to \mathcal{O}(r') \to 0,$$

where $r' = m + n - r_{\max} \leq \frac{m+n}{2}$. This sequence splits, and we conclude:

$$0 \to \mathcal{O}(r_{\max}) \to E \to \mathcal{O}(r') \to 0,$$

which implies that $E \cong \mathcal{O}(r_{\max}) \oplus \mathcal{O}(r')$. Thus, we are done.

□

Corollary 2.8 (Grothendieck)**.** *Rank n vector bundles on $\mathbb{P}^1$ are uniquely of the form $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$, where $m_1 \leq \cdots \leq m_n$.*

Proof. Proof is via induction: Suppose we have the sequence

$$0 \to \mathcal{O}(m_0) \to E \to \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n) \to 0.$$

This implies there exists a sub-bundle $E'_j \subset E$ such that:

$$0 \to \mathcal{O}(m_0) \to E' \to \mathcal{O}(m_j) \to 0.$$

Assume m_0 is the maximal value in E, so $m_0 \geq \frac{m_0+m_j}{2} \implies m_0 \geq m_j$. This implies that the extension splits.

□

Thus, when we consider the stack $\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)$, we observe that bundles become more and more degenerate as the index increases: $P_0 \cong \mathcal{O}(0)^2, P_1 \cong \mathcal{O}(1) \oplus \mathcal{O}(-1), P_2 \cong \mathcal{O}(2) \oplus \mathcal{O}(-2), \ldots$. The closure of each point in the stack is the union of successive bundles, and this suggests that the closure of the point P_0 contains all other bundles.

Next, we discuss the tangent bundle of Bun_G. When the genus of X is greater than or equal to 2, Bun_G, although a stack, has a large open set of stable bundles, which forms a smooth variety. This enables us to consider its cotangent bundle and perform symplectic geometry on it. In particular, this bundle carries an integrable system known as the Hitchin system, which we will construct, discuss its integrability, and explore its quantizations.

2.2.4 Principal G-bundles on $\mathbb{P}^1$

Let's generalize Grothendieck's theorem to an arbitrary connected reductive group, using a reformulation of the original theorem that makes it applicable to any connected reductive group.

Recall that rank-n vector bundles are the same as GL_n-bundles. Since GL_n is the group of invertible $n \times n$ matrices, it contains a maximal torus T, which is the set of diagonal matrices, or equivalently, $T = (\mathbb{C}^\times)^n$.

Theorem 2.9 (Grothendieck's Theorem, reformulated)**.** *Every GL_n-bundle on $\mathbb{P}^1$ is associated to a T-bundle.*

In other words, the structure group of the bundle reduces to the torus. Moreover, the same G-bundle can be realized as a collection of T-bundles, since the ordering of $m_1, \ldots, m_n$ is significant for T-bundles but not for G-bundles. Furthermore, if E_1 and E_2 are T-bundles on $\mathbb{P}^1$, then $E_1 \times_T GL_n \cong E_2 \times_T GL_n$ if and only if there exists a permutation $\sigma \in S_n$ such that $E_1 \cong \sigma(E_2)$. Now, if G is a connected reductive group, $T \subset G$ is a maximal torus, and $N(T) \subset G$ is the normalizer of T, the Weyl group $W = N(T)/T$ acts on T.

Theorem 2.10. *Any G-bundle on $\mathbb{P}^1$ is associated to a T-bundle E, and*

$$E_1 \times_T G \cong E_2 \times_T G$$

if and only if $E_1 \cong w(E_2)$ for some element $w \in W$.

Although we won't provide a proof, it can be shown by reducing to the case of vector bundles and considering the representations of G.

Now, suppose T is a torus. How can we classify T-bundles on $\mathbb{P}^1$? We know that $T \cong (\mathbb{C}^\times)^n$, meaning T can be viewed as an n-tuple of integers in a non-canonical way. More canonically, the **cocharacter lattice** is given by $X_\times(T) \stackrel{\text{def}}{=} \text{Hom}\,(\mathbb{C}^\times, T)$. It is clear that this is canonical when we recall that a T-bundle on $\mathbb{P}^1$ is defined by a transition map $g(z) : U_\infty \to U_0$ of the form

$$g : \mathbb{C}^\times \to T \simeq \left(\mathbb{C}^\times\right)^n$$

given by

$$z \to \begin{pmatrix} c_1 z^{m_1} \\ c_2 z^{m_2} \\ \vdots \\ c_n z^{m_n} \end{pmatrix}.$$

Thus, we have the identification

$$\{T\text{-bundles on } \mathbb{P}^1\} \cong X_\times(T).$$

The theorem states that G-bundles on $\mathbb{P}^1$ are classified by $X_\times(T)/W$, where W is the Weyl group.

Remark 2.11. *The cocharacter lattice $X_\times(T)$ is equivalent to the weight lattice $\Lambda^\vee$ of the Langlands dual group $G^\vee$. A connected reductive group, by definition, has a root datum consisting of a root system, a dual root system, a weight lattice, and a dual weight lattice. The root system defines an involution, allowing the exchange of roots with coroots and weights with coweights. This involution also swaps the group G with its Langlands dual group $G^\vee$, for instance, exchanging SLn with $PGLn$ and $Sp2n$ with $SO2n+1$. Langlands duality establishes a deep connection between objects associated with G and those related to $G^\vee$. A simple example of this duality is given by the identification*

$$X_\times(T)/W = \Lambda^\vee/W = \Lambda^\vee_+$$

where $\Lambda^\vee_+$ denotes the dominant integral weights, which correspond to the irreducible representations of $G^\vee$. Thus, we have a bijection between principal G-bundles on $\mathbb{P}^1$ and irreducible representations of $G^\vee$.

2.3 Double Quotient Realization of $\text{Bun}_G(X)$ and Number Theory

2.3.1 The Construction

Consider a smooth irreducible projective curve X, a split connected reductive group G over k, and attach to this data the moduli stack $\text{Bun}_G(X)$ of principal G-bundles on X. This is a very complicated object, but fortunately, most of the

complications are not relevant for our purposes. Earlier, we defined $\mathrm{Bun}_G(X)$ through its functor of points - let's now develop a more explicit way to define it.

For simplicity, assume that G is semisimple and k is an algebraically closed field, so that $E \in \mathrm{Bun}_G(X)$ trivializes once you remove any point (by Harder's theorem). We fix a point $x \in X$, and cover X by two charts: $X \setminus x$ and a (formal) disk D_x around x. With bundles and these two charts, there is no 1-cocycle condition, and we can simply fix one transition map. The intersection is the punctured disk $D_x^\times = (X \setminus x) \cap D_x$. To be more precise, let $R = \mathcal{O}(X \setminus x)$ be the ring of regular functions on the affine curve $X \setminus x$. If t is a formal coordinate at x, we can define

$$\mathcal{O} := \mathcal{O}(D_x) \cong k[[t]],$$
$$K := \mathcal{O}(D_x^\times) \cong k((t))$$

Note that $\mathcal{O}(X \setminus x) \subset K$ is similar to a ring of integers, so we have a non-Archimedean valuation ≥ 0. We can embed $R \hookrightarrow K$ by taking the Laurent expansion. To perform this Laurent expansion, we need to choose a coordinate, but it is easy to show that, canonically, the inclusion does not depend on the choice of coordinates. Bundles E are defined by the transition map $g(t) : D_x \to X \setminus x$, where $g \in G(K)$ is defined by $g \mapsto h_1 g h_2^{-1}$, with $h_1 \in G(R)$ and $h_2 \in G(\mathcal{O})$.

Thus, we have proved the following:

Proposition 2.12.

$$Bun_G(X) \cong G(R) \backslash G(K) / G(\mathcal{O}).$$

Consider the affine Grassmannian $\mathrm{Gr}_G = G(K)/G(\mathcal{O})$. The G-bundles correspond to orbits of $G(R)$ on Gr_G.

We can generalize this by removing finitely many points from X. Let $S \subset X$ be a finite subset with $S \neq \emptyset$. We have two charts: $U_1 = X \setminus S$ and $U_2 = \bigcup_{x \in S} D_x$, so

$$U_1 \cap U_2 = \bigcup_{x \in S} D_x^\times.$$

Thus, we can write

$$\mathrm{Bun}_G(X) = G(X \setminus S) \backslash \prod_{x \in S} G(D_x^\times) / \prod_{x \in S} G(D_x).$$

This description is valid for semisimple groups, which have the property that we can trivialize any bundle by removing just one point. However, for $G = \mathbb{C}^\times$, the simplest example of a reductive group, this is not the case, and there is no set

S that we can remove to trivialize all possible bundles. One might think that removing all points results in an empty set, but in algebraic geometry, we allow functions that have poles at all those points. Thus, we can consider arbitrary meromorphic or rational functions, and we should replace the ring R with the field of rational functions on the curve.

The further generalization is to take the colimit with respect to S, which gives

$$\mathrm{Bun}_G(X)(k) = G(k(X))\backslash \prod_{x\in X}{}' G(D_x^\times)/\prod_{x\in X} G(D_x).$$

The prime on the product indicates the restricted product, meaning that only finitely many coordinates are not in $G(D_x)$ (i.e., have a pole).

If k is not algebraically closed, we proceed similarly but with $S \subset X(\overline{k})$ that are Galois-invariant. In this case, we obtain

$$\mathrm{Bun}_G(X)(k) = G(k(X))\backslash \prod_{x\in X(\overline{k})/\mathrm{Gal}(\overline{k}/k)} G(D_x^\times)/\prod_{x\in X} G(D_x).$$

Example 2.13. *If k is finite, then $F := k(X)$ is a global field, and the first product is a product over all valuations of $k(x)$. We get*

$$Bun_G(X)(k) = G(\mathbb{F})\backslash G\left(\prod_{v\in Val(\mathbb{F})}{}' F_v\right)/G\left(\prod_{v\in Val(\mathbb{F})}{}' \mathcal{O}_v\right).$$

2.3.2 Number Theory

This is called the **arithmetic quotient** and the name comes from number theory. Here, we have a global field of characteristic p because k is a finite field, but there are also global fields of characteristic 0 which are the number fields: the finite extensions of $\mathbb{Q}$.

Definition 2.14. *If F is a global field, the **ring of adèles** is*

$$\mathbb{A} := \mathbb{A}_p := \prod_{v\in Val(F)}{}' F_v$$

Over global characteristic p, all valuations have this form. However, for number fields, there are two kinds of valuations: Archimedean (embeddings of the field into $\mathbb{R}$ and $\mathbb{C}$, with the usual absolute value) and non-Archimedean (such as p-adic valuations). The prime in the product in this case makes sense for non-Archimedean valuations.

Let $\mathcal{O}_{\mathbb{A}} = \prod_{v\in \mathrm{Val}_{\text{non-Archimedean}}(F)} \mathcal{O}_v$. Then

$$M := G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_{\mathbb{A}})$$

as a generalization. Therefore, if $F = k(X)$, we get $\mathcal{M} = \mathrm{Bun}_G(X)(k)$.

Example 2.15. *Take $F = \mathbb{Q}$ with $Val(F) = p, \infty$, where p corresponds to $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and ∞ corresponds to the usual absolute value valuation. We have*

$$\mathbb{A} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p, \quad \mathcal{O}_{\mathbb{A}} = \prod_p,$$

and

$$\mathcal{M} = G(\mathbb{Q})\backslash G(\mathbb{R}) \times \prod_p{}' G(\mathbb{Q}_p)/G(\mathbb{Z}_\text{ı}).$$

In number theory, it is important that this is equal to $G(\mathbb{Z})\backslash G(\mathbb{R})$.

- *If $G = Sp_{2n}$, we get*

  $$Sp(2n, \mathbb{Z})\backslash Sp(2n, \mathbb{R})/U(n) = \mathcal{A}_n$$

  .

- *If $G = SL_2$, we get*

  $$SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/U(1),$$

  but $SL_2(\mathbb{R})/U(1)$ is the upper half-plane (so we use modular forms), meaning this equals $\mathbb{A}_1$.

Langlands wanted to understand modular forms and their higher-rank generalizations. To do this, we often need to use algebraic geometry tools.

It is important to note that the arithmetic quotient is well-defined for any reductive group, and we no longer need the semisimple condition because we have removed all of the points. If we move to the field of functions, any G-bundle will be trivialized, since, as we saw earlier, they are all Zariski locally trivial.

If $G = \mathrm{GL}_1$, we get

$$\mathrm{Jac}(X) = \mathbb{C}(X)^\times \backslash \mathbb{A}^\times / \mathcal{O}_A^\times,$$

where $\mathbb{A}^\times / \mathcal{O}_{\mathbb{A}}^\times$ is the group of divisors on $X \cdot \mathbb{C}^\times$ and $\mathbb{C}(X)^\times$.

2.4 Hitchin Systems

2.4.1 Stable Bundles

For the purposes of this course, the Hitchin system is an integrable system. Integrable systems live on symplectic manifolds, so we need to introduce one.

The most common symplectic manifolds are cotangent bundles of ordinary manifolds. We want to define $T^*\mathrm{Bun}_G(X)$. Note that $\mathrm{Bun}_G(X)$ is neither a manifold nor a scheme - it is a smooth stack, so locally it is the quotient of a smooth variety by an algebraic group. For such objects, we can define the cotangent bundle and cotangent spaces as complexes, but we won't go into this here.

Assume that the genus $g \geq 2$ and G is simple and adjoint (i.e., it has no center). Consider an open set. $\mathrm{Bun}_G(X)$ is a stack, and the main feature of a stack compared to a scheme is that points have automorphisms. However, **generic bundles** have trivial automorphism groups and form a smooth algebraic variety $\mathrm{Bun}_G^0(X)$. The first part implies that they correspond to a locus, which means in our presentation the group acts freely and stabilizers are trivial. The second part implies that, as long as we restrict to an open set, this stack (which is a functor of points) is representable by some variety. There are many ways to choose this open set—one nice way is to consider stable bundles:

Consider $G = \mathrm{PGL}_n$. Then G-bundles are rank n vector bundles modulo line bundles. For stable bundles, the "modulo line bundles" condition doesn't matter because stability is preserved when we tensor the bundle with a line bundle.

If E is a vector bundle on X, then we have two integers associated with it: the degree $d(E)$ (the first Chern class) and the rank $r(E)$ (the dimension of the fiber). We can now define the slope:

Definition 2.16. *The **slope** of E is defined as*

$$\mu(E) = \frac{d(E)}{r(E)}.$$

The purpose of defining the slope is to show what it means for E to be stable.

Definition 2.17. *E is **stable** if for all sub-bundles, $O \neq E' \subsetneq E$, $\mu(E')\mu(E)$.*

Why did we define these?

Theorem 2.18. *Stable bundles form a smooth variety, which is an open subset of $Bun_G(X)$.*

Exercise 2.19. *If L is a line bundle and E is a vector bundle, show that E is stable if and only if $E \otimes L$ is stable.*

So, generic bundles are stable and have no nontrivial automorphisms. We define $\mathrm{Bun}_G^0(X)$ as the moduli space of stable bundles, which is a smooth variety. We can now consider $\mu_G^0 = T^*\mathrm{Bun}_G^0(X)$. This is where the Hitchin system will initially live. However, there is a partial compactification, which is also a smooth variety, that creates the Hitchin moduli space, and the Hitchin system naturally lives on this compactification.

2.4.2 Higgs Field

What is the dimension of $\mathrm{Bun}_G^0(X)$? To answer this, we need to ask: what is $T_E\mathrm{Bun}_G^0(X)$? This is the deformation space of E.

Exercise 2.20. *Deformations of E are classified by $H^1(X, ad(E))$, where $ad(E)$ is the adjoint bundle, the vector bundle associated with E corresponding to the adjoint representation.*

Example 2.21. *Consider $G = GL_n$ as a vector bundle. The deformations of E are similar to the deformations of projective modules over the sheaf of functions on X of rank n. Deformations of modules over any ring are classified by $Ext^1(E,E)$ because deformation means defining a module over the ring of dual numbers $k[[h]]/h^2$, and our module is free in this view. So, in our case, the deformations are classified by*

$$\begin{aligned} Ext^1(E,E) &= Ext^1(\mathcal{O}, E^* \otimes E) \\ &= H^1(X, ad(E)) \end{aligned}$$

.

Consider $T^*_E \mathrm{Bun}^0_G(X) = H^1(X, \mathrm{ad}E)$. By Serre duality, this is equivalent to $H^0(X, (\mathrm{ad}E)^* \otimes K_X)$. But there is a trace form on the Lie algebra $\mathfrak{g}$ given by the Killing form, so ad has an invariant inner product which implies $(\mathrm{ad}E)^* = \mathrm{ad}E$, leaving us with

$$H^0(X, (\mathrm{ad}E) \otimes K_X) = H^0(X, K_X \otimes \mathrm{ad}E).$$

What is the dimension of $H^0(X, K_X \otimes \mathrm{ad}E)$? Recall the Euler characteristic

$$\chi(X, K_X \otimes \mathrm{ad}E) = \dim H^0(X, K_X \otimes \mathrm{ad}E) - \dim H^1(X, K_X \otimes \mathrm{ad}E).$$

In geometry, this doesn't change under deformations. But $\dim H^1 = 0$ for generic bundles, which is intuitive because χ is invariant. If we take a non-generic bundle, $\dim H^1$ will be positive and $\dim H^0$ will be larger, which is what happens when we have singularities—the tangent space at a singularity is larger than the tangent space at a generic point. So, computing $\dim H^0(X, K_X \otimes \mathrm{ad}(E))$ is equivalent to computing the Euler characteristic.

If our group is adjoint we have to be slightly more careful because we have several connected components of Bun_G but if it is simply connect when can compute χ for the trivial bundle (because χ is deformation invariant), which is given by

$$\dim H^0(X, K_X) \dim \mathfrak{g} - \dim H^1(X, K_X) \dim \mathfrak{g}.$$

but $H^0(X, K_X) \dim \mathfrak{g} = g$ and $H^1(X, K_X) = H^0(X, 0)^* = \mathbb{C}$ by Serre duality, and we are left with

$$\dim H^0(X, K_X \otimes \mathrm{ad}E) = (g-1) \dim \mathfrak{g}$$

for semisimple groups.

If $G = \mathbb{C}^\times$, then $\mathrm{Bun}_G(X) = \mathrm{Jac}(X)$, which has dimension g. For G a general reductive group,

$$\dim \mathrm{Bun}_G(x) = (g-1) \dim \mathfrak{g} + \dim Z(\mathfrak{g}).$$

Example 2.22. *For $G = GL_n$,*

$$\dim Bun_G(x) = (g-1)n^2 + 1.$$

This has a clear and vivid geometric interpretation: it corresponds to the 1-forms on X with coefficients in $\mathrm{ad}(E)$. These objects are significant in physics, and they lead us to define the **space of Higgs fields**.

2.4.3 The Hitchin Integrable System

Let's (finally) define the Hitchin integrable system on $T^*\mathrm{Bun}_G^0(X)$.

Definition 2.23. *An **integrable system** is a collection of Poisson commuting functions that are functionally independent, where the number of functions equals half the dimension of it's symplectic manifold.*

We can think of it as follows: if we make these functions into a vector, we have map to a vector space of dimension $\dim \mathrm{Bun}_G^0(X)$, which is a Lagrangian fibration. Generically, a Lagrangian fibration corresponds to the functions being functionally independent and forming an involution.

Example 2.24. *For SL_n, $\dim Bun = (n^2-1)(g-1)$, and*

$$Bun_G^0(X) = \{(E, \phi)\}.$$

where E is a stable bundle and ϕ is a Higgs field.

Definition 2.25. *The **Hitchin map** is*

$$\begin{aligned} p : T^* Bun_G^0(X) &\to \bigoplus_{i=1}^{n-1} H^0(X, K_X^{\otimes i+1}) \\ (E, \phi) &\mapsto (Tr \wedge^2 \phi, Tr \wedge^3 \phi, ..., Tr \wedge^n \phi) \end{aligned}$$

*where $Hitch := \bigoplus_{i=1}^{n-1} H^0(X, K_X^{\otimes i+1})$ is called the **Hitchin base***

By Riemann-Roch, the $\dim H^0(X, K_X^{\otimes i+1}) = (2i+1)(g-1)$ so

$$\dim \oplus_{i=1}^{n-1} H^0(X, K_X^{\otimes i+1}) = (n^2-1)(g-1).$$

Since ϕ is a 1-form, $\mathrm{Tr}\phi^2$ is a quadratic differential, $\mathrm{Tr}\phi^3$ is a cubic differential, and so on, making the map well-defined.

Theorem 2.26 (Hitchin)**.** *The Hitchin map is an integrable system.*

Hitchin's theorem implies that coordinate functions on $\mathcal{B}$ are pulled back by p. Let $d = \dim \mathcal{B} = (n^2-1)(g-1)$. The basis $H_1, ..., H_d$ Poisson commute on $T^*\mathrm{Bun}_G^0(x)$: $\{H_i, H_j\} = 0$ and are functionally independent. Therefore, $p(E, \phi) = \sum_{j=1}^{d} H_j(E, \phi) b_j$.

Let's generalize to an arbitrary semisimple G. Before we do this, recall Chevalley's Theorem: $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \ldots, P_r]$ where $r = \operatorname{rank}(G)$, and the P_i are homogeneous with degree $\deg P_i = d_i$.

Example 2.27.

$$\mathbb{C}[\mathfrak{sl}_n]^{sl_n} = \mathbb{R}[TrA^2, TrA^3, ..., TrA^n].$$

Let $P \in \mathbb{C}[\mathfrak{g}]^G$ be a homogeneous element of degree m, and let (E, ϕ) be a Higgs pair, where E is a G-bundle and $\phi \in \Omega^1(X, \operatorname{ad}(E))$. To this data, we associate $P(\phi)$.

The fibers of $\operatorname{ad}E$ are copies of $\mathfrak{g}$ and we have clutching maps which are conjugations by some element of this group, so the fiber of this bundle cannot be canonically identified with $\mathfrak{g}$. Fortunately, we can identify it canonically with $\mathfrak{g}$ up to conjugation, so if we have a conjugation invariant function on $\mathfrak{g}$, we can canonically compute it for ϕ. Since ϕ is a 1-form, we obtain an m differential and we can conclude that $\phi \in H^0(X, K_X^{\otimes m})$.

We now define the Hitchin base $\mathcal{B} = \bigoplus_{i=1}^r H^0(X, K_X^{\otimes d_i})$, where d_i are the degrees associated with the group G. Recall that $2d_i - 1$ corresponds to the degrees of the generators of the cohomology ring of G. Therefore, the dimension of $\mathcal{B}$ is given by

$$\dim \mathcal{B} = \sum_i (2d_i - 1)(g-1) = (g-1)\dim G = \dim \operatorname{Bun}_G^0(X).$$

We may now define

$$\begin{aligned} p : T^*\operatorname{Bun}_G^\circ(X) &\to B \\ (E, \phi) &\mapsto (P_1(\phi), \ldots, P_r(\phi)) \end{aligned}$$

Theorem 2.28 (Hitchin)**.** *This is an integrable system.*

Hitchin proved it for classical groups, and later others completed the proof for exceptional groups. We will explain the proof in the case SL_n or GL_n. The proof consists of two parts: these functions define an involution, and they are functionally independent. For the second part, functional independence is equivalent to showing that the map is a dominant map.

2.4.4 Marsden-Weinstein Symplectic Reduction

For $G = \mathrm{SL}_n$.

Part 1: Poisson Commuting

Let's start by explaining Hamiltonian reduction in the simple case of cotangent bundles. Let Y be a manifold (or variety), and H a Lie algebra (or group) acting on Y on the right. Then H acts by Hamiltonian transformations on T^*Y because the symplectic form on the cotangent bundle is canonical and

therefore preserved by the action of any group that acts on Y. Moreover, this action is Hamiltonian, meaning the action is defined by some Hamiltonian or equivalently, there exists a moment map

$$\mu : T^*Y \to \mathfrak{h}^*$$

dual to the action map

$$a : \mathfrak{h} \to \mathrm{Vect}(Y) = \Gamma(Y, TY),$$

such that

$$\mu(x,p)(b) = (p, a(b)x), \quad \forall (x,p) \in T^*Y,\, b \in \mathfrak{h}.$$

Theorem 2.29 (Marsden, Weinstein)**.** *Suppose H acts freely on Y. Then $\mu^{-1}(0)/H \cong T^*(Y/H)$ has a natural symplectic structure.*

We can use this to construct integrable systems: Suppose $\dim Y/H = n$, and $F_1, \ldots, F_n$ are Poisson commuting functions on T^*Y. $\{F_i, F_j\} = 0$ and F_i are H-invariant, so they descend to functions $\overline{F_i}$ on $T^*(Y/H)$. Then $\{\overline{F_i}, \overline{F_j}\} = 0$, where $F : T^*Y \to \mathbb{C}$ are H-invariant and $\overline{F}$ is the descent of $F|_{\mu^{-1}(0)}$ to $\mu^{-1}(0)/H \cong T^*(Y/H)$. It is easy to check that these functions are an involution downstairs on $T^*(Y/H)$, and there are exactly the right number of them to form an integrable system. If they happen to be functionally independent, then $\overline{F_1}, \ldots, \overline{F_n}$ form an integrable system on $T^*(Y/H)$. Note that there are too few functions to form an integrable system on T^*Y, so we must descend to $T^*(Y/H)$.

This is a powerful method to construct integrable systems, and we will use them in our case, except we will start with something of infinite dimension. This method is good because it can be hard to check that certain functions are an involution, especially if we don't have an effective way of writing them down explicitly. However, often it is the case that upstairs on Y, $F_1, \ldots, F_n$ Poisson commute for some trivial reason. For example, if Y is a vector space, on T^*Y we have coordinate and momentum variables, and often $F_1, \ldots, F_n$ only depend on momentum so they Poisson commute. When we go downstairs to the quotient, things become less trivial. Luckily, we don't have to check they Poisson commute because it follows from the construction.

Let's now apply this to the Hitchin system. Recall that $\mathrm{Bun}_{G^0}(X) = G(X \setminus x) \backslash G(K)/G(O)$, $K = \mathbb{C}(D_x^*) \cong \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[D_x] \cong \mathbb{C}[[t]]$, and set $G^0(K) \subset G(K)$ as the preimage of $\mathrm{Bun}_G^0(X)$. On $G^0(K)$, this acts on $G(X \backslash x) \times G(0)$ almost freely, with stabilizer $Z(G)$. So $T^*\mathrm{Bun}_{G^0}(X)$ is the Hamiltonian reduction of the loop group $Y = T^*G^0(K)$ by $H = G(X \backslash x) \times G(0)$.

To construct an integrable system downstairs, we need to find some commuting elements upstairs. Y is a Lie group, so we have

$$T^*G^0(K) = G^0(K) \times \mathfrak{g}((t))^*.$$

But for $a \in \mathfrak{g}((t)), b \in \mathfrak{g}((t))\,dt$, we can define the pairing

$$(a,b) = \mathrm{Res}_{t=0}\langle a(t), b(t)\rangle,$$

so we have $\mathfrak{g}((t))^* = \mathfrak{g}((t))\,dt$ and

$$T^*G^0(K) = G^0(K) \times \mathfrak{g}((t))\,dt.$$

We have considered (g,ϕ) where ϕ is the Higgs field x on $D^\times$, so we can take $P_i(\phi) \in \mathbb{C}((t))(\,dt)^{\,di}$. Define

$$H_{i,n} := \mathrm{Res}(t^n P_i(\phi)), \quad \forall\, 1 \le i \le r \text{ and } n \in \mathbb{Z}.$$

Observe that $H_{i,n}$ depends only on momenta on $T^*G^0(K)$, which implies

$$\{H_{i,n}, H_{j,m}\} = 0.$$

Momenta on $T^*(G)$ commute like Lie algebra elements but over P_i, they are G-invariant so they Poisson commute. In other words, the dual space of a Lie algebra is a Poisson manifold (with Poisson structure coming from the Lie bracket), but invariant functions lie in the center of the Poisson algebra, so they still Poisson commute. The group is nonabelian and the momenta don't commute, but the invariant functions of momenta commute with momenta and thus with each other:

$$\begin{array}{ccc} T^*G^0(K) & \xrightarrow[\tilde{p}]{(H_1,\ldots,H_r)} & \bigoplus_i \mathbb{C}((t))(dt)^{d_i} \\ \uparrow & & \uparrow \\ \mu^{-1}(0) & & \\ \downarrow & & \\ T^*\mathrm{Bun}_G^0(X) & \xrightarrow[p]{} & \bigoplus_i H^0(X, K_X^{\otimes d_i}) \end{array}$$

commutes where the right map is the Taylor series expansion of differentials. Additionally, this map is injective by analytic continuation.

Part 2: Independence

Suppose $b = (b_2, \ldots, b_n) \in B = \bigoplus_{i=1}^{n-1} H^0(X, K_X^{\otimes i+1})$ where $b_j \in H^0(X, K_X^{\otimes j})$. For convenience, we will redefine

$$p(E,\phi) = (\mathrm{Tr} \wedge^2 \phi, -\mathrm{Tr} \wedge^3 \phi, ..., (-1)^n \mathrm{Tr} \wedge^n \phi).$$

Consider the polynomial $\lambda^n + b_2\lambda^{n-2} + ... + b_n = (\lambda - \lambda_1)...(\lambda - \lambda_n)$. Note that λ_i are all 1-forms on X, but we have an Galois group action so they are not single valued. When we go around the Hitchin base, the λ_i will be be permuted. Let $x \in X$. This factorization allows us to write $\{\lambda_1(x), \ldots, \lambda_n(x)\} \subseteq T_x^*X$.

This gives a subset $C_b \subset T^*X$ consisting of $\lambda_i(x)$ for various x. In fact, C_b is an algebraic curve inside the surface T^*X defined by the equation $\lambda^n + b_2(x)\lambda^{n-2} + ... + b_n(x) = 0$. Furthermore, we have a projection $\pi : C_b \to X$ of degree n.

Definition 2.30. *The curve C_b is called the **spectral curve** of b.*

Suppose we have $(E, \phi) \in T^*\mathrm{Bun}_G^0(X)$ and attach $p(E, \phi) = (b_2, \ldots, b_n)$, where b_i are coefficients of the characteristic polynomial of ϕ. Thus,

$$\lambda^n + b_2\lambda^{n-2} + ... + b_n = \det(\lambda - \phi),$$

where $\lambda_i(x)$ are the eigenvalues of $\phi(x)$ for $x \in X$. The name "spectral" comes from the fact that $C_{p(E,\phi)}$ is traced out by the spectrum of $\phi(x)$ when x varies along X. Because $\phi \in \Omega^1(X, \mathrm{End}E)$ is a 1-form, the eigenvalues are also 1-forms (not functions) and live in T^*X.

Why is this curve useful when studying the Hitchin integrable system? Because $C(E, \phi)$ depends only on the b of $p(E, \phi)$. Now, it is natural to ask: can we recover (E, ϕ) from C and something else?

Theorem 2.31 (Hitchin)**.** *C is smooth and irreducible for generic b.*

The proof is not hard but we will not present it.

If we have ϕ whose spectral curve is C, then we have an **eigenline bundle** L_ϕ on C: the fiber of L_ϕ at $\lambda \in C$ (which projects to $x \in X$) is the eigenline of $\phi(x)$ with eigenvalue λ. In the generic case where all of the eigenvalues are distinct, we can attach to a point $\lambda \in C$ a certain line which is the line of ϕ with eigenvalue λ, giving us a line bundle on the spectral curve. We have to be slightly more careful when the eigenvalues collide, but it is similar.

Proposition 2.32.

$$E \cong \pi_* L_\phi.$$

Algebraically, if we have a module over functions on $\mathbb{C}$, it is a module over functions on X by π. Geometrically, when the eigenvalues are distinct, $E_x = \bigoplus_{\lambda \in \pi^{-1}(x)} (L_\phi)_\lambda$. Suppose L_ϕ has degree d, which follows from the fact that the moduli space is connected and reducible, and the degree is a topological invariant. Thus,

$$L_\phi \in \mathrm{Pic}^d(C_b) \cong \mathrm{Jac}(C_b).$$

Moreover, if we know L_ϕ, we can recover (E, ϕ) from L_ϕ. This implies that $p^{-1}(b) \subset T^*\mathrm{Bun}_G^0(X)$ gets identified with a subset of $\mathrm{Jac}(C_b)$. This is exactly what we want because this shows that there is a linear flow some torus on the integrable system.

It remains to show for generic b,

$$\dim p^{-1}(b) = (n^2 - 1)(g - 1)$$

and not larger because then there would be undesired functional dependence. To estimate the dimension of the fiber, we know that the fiber is contained in the Jacobian of a curve, so we can just compute the genus.

Consider the group $G = \mathrm{GL}_n$ and a curve C defined by $\lambda^n + b_1\lambda^{n-1} + ... + b_n$.

Theorem 2.33. *The genus of C is*

$$n^2(g-1) + 1 = Bun^0_{GL_n}(x).$$

Proof. Compute the genus when $b_1, \ldots, b_{n-1} = 0$ and C is given by the equation $x^n + b_n(x) = 0$. We map $C \to X, n \to 1$ if $b_n(x) \neq 0$. Note that $b_n \in H^0(X, K_X^{\otimes n})$ has degree $(2n-2)n$, so generically, b_n has this many zeros. Thus, we compute the Euler characteristic of C as follows:

$$\begin{aligned}\chi(C) &= ((2-2g) - (2g-2)n)n + (2g-2)n \\ &= n^2(2g-2).\end{aligned}$$

This gives the genus of C:

$$g(C) = \frac{2 + n^2(2g-2)}{2} = 1 + n^2(g-1).$$

Therefore, we conclude that $p^{-1}(b)$ is dense in $\mathrm{Jac}(C)$, and the dimension of $p^{-1}(b)$ is bounded by

$$\dim p^{-1}b \leq 1 + n^2(g-1) = \dim \mathrm{Bun}^0_G(x),$$

completing the proof.

□

2.5 Bundles with Parabolic Structure

2.5.1 Principal Bundles

We continue our discussion of Hitchin integrable systems from earlier, prior to the proof. In this section, we will compute a few examples. Unfortunately, computing for curves of genus ≥ 2 is quite complex, as the moduli space Bun_G is icomplicated and difficult to express in explicit coordinates. While Hitchin managed this for GL_n, the computations are still quite challenging. Therefore, we will instead consider a generalization of Hitchin systems, which includes punctured curves. This generalization allows for a non-trivial structure when the genus is 0 or 1, providing a more desireable setting for explicit calculations.

Suppose G is a connected reductive group, $H \subset G$, and $\mathcal{E}$ is a G-bundle on X.

Definition 2.34. *An H-**structure** at x is an H-orbit in $\mathcal{E}_x$.*

There are exactly G/H choices for the H-structure and H only matters up to conjugation because right-multiplication by $g \in G$ transforms an H-orbit into a gHg^{-1}-orbit. Fix distinct $t_1, \ldots, t_N \in X$, and let $P_1, \ldots, P_n \subset G$ be parabolic subgroups (containing Borel subgroups, ie. the quotient G/P is a projective variety). For GL_n, a simple example arises when $n = n_1 + \cdots + n_r$ is a composition, corresponding to the upper triangular block matrices with block sizes $n_1, \ldots, n_r$. The smallest parabolic occurs when $n_i = 1$ for all i.

Definition 2.35. $\mathrm{Bun}_G(X, t_1, \ldots, t_N, P_1, \ldots, P_N)$ *is the moduli stack of G-bundles on X with a P_i-structure at t_i for $i = 1, \ldots, N$.*

For $G = \mathrm{GL}_n$, G-bundles correspond to vector bundles of rank n. Let E be the rank n vector bundle and $\mathcal{E}$ be the corresponding GL_n-bundle. Canonically, the fiber is $\mathcal{E}_x = \{\text{bases in } E_x\}$, which is a classic example of a torsor over GL_n because we can perform change of basis via a matrix. In particular, if P is a parabolic subgroup, i.e., the stabilizer of a partial flag $0 \subset V_1 \subset \cdots \subset V_r = V$, where the dimensions of the quotients V_i/V_{i-1} are given by n_i, then a P-structure is a set of bases compatible with this flag. Specifically, there is a subset of the bases that forms a basis for each of the spaces V_i. Thus, a P-structure is equivalent to fixing a flag.

Example 2.36. *For $G = GL_2$,*

$$P_i = B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.$$

Thus, a B-structure on E at $x \in X$ is simply a line $\ell \in E_x$. We then have a fibration $Bun_G(x, t_1, \ldots, t_N, P_1, \ldots, P_N) \to Bun_G(X)$ with fiber $G/P_1 \times G/P_2 \times \ldots \times G/P_n$.

We care about these structures because they allow us to consider the cases $g = 0$ and $g = 1$. Adding this data means that the automorphism groups of the bundles must preserve it, which makes the automorphism group smaller, thus creating more objects with a trivial automorphism group. If $N \geq 3$, then a generic G-bundle on $\mathbb{P}^1$ with parabolic structures has a trivial automorphism group. Let E be the trivial bundle. Then, $\mathrm{Aut}(E) = \mathrm{PGL2}$, since a holomorphic function on a compact Riemann surface must be constant. However, we have parabolic structures $\ell_1, \ldots, \ell_N$, where $\ell_i \in \mathbb{P}E_{t_i} = \mathbb{P}^1$. Thus, the set $\mathrm{Bun}_G^{\mathrm{triv}}(x, t_1, \ldots, t_N)$, the bundles with parabolic structures and trivial automorphism group, is given by $(\mathbb{P}^1)^N/\mathrm{PGL}_2$. This contains an even smaller open set $(y_1, \ldots, y_n)$. It is well known that PGL_2 acts trivially on triples of distinct points on $\mathbb{P}^1$, and there is a theorem stating that there is a unique element in PGL_2 that maps these points to 0, 1, and ∞. Therefore, we want

$$(\mathbb{P}^1)^{N-3} \subset (\mathbb{P}^1)^N/\mathrm{PGL}_2$$

which is a variety.

From now on, we will live on $(\mathbb{P}^1)^N/\mathrm{PGL}_2$, which is still a stack but simpler than the general version. Now, we can do simpler calculations using coordinates

$(Y_1..., Y_N) \in (\mathbb{P}^1)^N$ with $y_i \in \mathbb{P}^1 = \mathbb{C} \cup \infty$. In general,

$$\mathrm{Bun}_G^{\mathrm{triv}}(X, t_1, \ldots, t_N, P_1, \ldots, P_N) \cong \left(\prod_i G/P_i \right) /G_{\mathrm{diagonal}}.$$

2.5.2 Classical Hitchin Systems

Recall that we constructed the Hitchin system by realizing Bun_G as a double quotient and descending a certain collection of Hamiltonians upstairs on the quotient bundle of the loop group down to the double quotient. We can do the same with parabolic structure. Recall that

$$\mathrm{Bun}_G(X) = G(X \setminus \{t_1, \ldots, t_N\}) \backslash \prod_i G(D_{t_i}^\times) / \prod_i G(D_{t_i}).$$

Now, we take the same Hamiltonians $H_{i,j,n} := \mathrm{Res} P_{t_i}(\phi) z_j^n$ on $T^*G((H_{t_1}^\times)) \times ... \times T^*(D_{t_n}^\times)$, and do the reduction to $G(X \backslash \{t_i\}) \times \prod \tilde{P}_i$

If we want to fix parabolic structures, we should modify $\prod_i G(D_{t_i})$. We have $\mathrm{ev} : G(D) \to G$, where $\mathrm{ev}(g(z)) = g(0)$. Then we can define $\tilde{P}_i = \mathrm{ev}^{-1}(P_i)$, which gives an integrable system on $T^*\mathrm{Bun}_G(t_1, \ldots, t_N, P_1, \ldots, P_N)$. Points of this space are (E, ϕ) where E is a parabolic bundle and ϕ is a Higgs field with singularity. More precisely, we have the condition that

$$\phi \in \Omega^1(X \backslash \{t_i\}, \mathrm{ad}E)$$

has at most first order poles at t_i and the residue strictly preserves (lies in the unipotent of $\mathrm{Stab} F_i$) the flag F_i at T_i. In the general case GL_n, this means flags are a filtration on our space, and strictly preserves means that it maps i-th piece of the filtration to $i-1$, so it acts by 0 in the associated gradient.

Exercise 2.37. *Check that we have the right condition.*

2.5.3 The Garnier System

For PGL_2, ϕ has simple poles at t_i, and $\mathrm{Res}_{t_i} \phi$ is nilpotent and act by 0 on $\ell_i \subset E_t$. Now let's compute the Hitchin system for PGL_2 in genus 0. Assume $t_1, \ldots, t_N \in \mathbb{A}^1 \subset \mathbb{P}^1$ and the parabolic structure at t_i is $y_i \in \mathbb{A}$. The Higgs field ϕ is a 1-form with simple poles at t_1 valued in $\mathfrak{sl}_2$:

$$\phi = \sum_{i=1}^N \frac{A_i}{z - z_i} \, dz$$

with $A_i \in \mathfrak{sl}_2$ with

- Regular at ∞ because there must not be a puncture there. Our form is regular at ∞ if it has 2nd order decay. Our condition is equivalent to $\sum_{i=1}^N A_i = 0$.

- A_i are nilpotent: $A_i \begin{pmatrix} y_i \\ 1 \end{pmatrix} = 0$. Given $A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_i \\ 1 \end{pmatrix} = 0$ which implies $ay + b = 0, cy + d = 0$ giving us

$$A = \begin{pmatrix} cy & -cy^2 \\ c & -cy \end{pmatrix} \implies A_i = p_i \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix}$$

  where p_i momenta coordinates with symplectic form $\sum dy_i \wedge d\, p_i$.

We still have a PGL_2 symmetry, so we will some system in terms of y and then take the reduction with respect to PGL_2. Let's compute the Hitchin Hamiltonians:

$$\begin{aligned} H_{n,2} &= \frac{1}{2}\mathrm{Tr}\phi^2 (\,dz)^2 \\ &= \frac{1}{2}\mathrm{Tr}\left(\sum_{i,j} \frac{A_i}{z - t_i} \cdot \frac{A_j}{z - t_j} \right) (\,dz)^2 \\ &= \sum_{i<j} \frac{\mathrm{Tr} A_i A_j}{(z - t_i)(z - t_j)} (\,dz)^2 \\ &= \sum_{i \neq j} \frac{\mathrm{Tr} A_i A_j}{(t_i - t_j)(z - t_i)} (\,dz)^2 \\ &= \sum_{i \neq j} \frac{p_i p_j \mathrm{Tr} \begin{pmatrix} y_i & -y_i^2 \\ 1 & -y_i \end{pmatrix} \mathrm{Tr} \begin{pmatrix} y_j & -y_j^2 \\ 1 & -y_j \end{pmatrix}}{(t_i - t_j)(z - t_i)} (\,dz)^2 \\ &= \sum_{i \neq j} \frac{p_i p_j (y_i - y_j)^2}{(t_i - t_j)(z - t_i)} (\,dz)^2 \end{aligned}$$

where the third line follows from $i = j$ dropping out as $A_i^2 = 0$, the fourth line follows from the identity

$$\frac{1}{(z - a)(z - b)} = \frac{1}{a - b} \left(\frac{1}{z - a} - \frac{1}{z - b} \right).$$

We can take residues (since scaling doesn't matter to us):

$$G_i - \mathrm{Res}_{t_i} H_2 = \sum_{i \neq j} \frac{p_i p_j (y_i - y_j)^2}{t_j - t_i}$$

which gives us the **Garnier system**.

2.6 The Twisted Hitchin/Garnier System

We can create even more generalized integrable systems by "twisting" Hitchin systems for bundles with parabolic structure. To do this, let's explain Hamiltonian reduction along an orbit. Let M be a symplectic manifold with H acting on

M. We can generalize a moment map $\mu : M \to \mathfrak{h}^*$ to $\mu^{-1}(\mathcal{O})/H$ with $\mathcal{O} \subset \mathfrak{h}^*$, an H-orbit which we call a **coadjoint orbit**. We can check that $\mu^{-1}(\mathcal{O})/H$ has a canonical symplectic structure and use the same construction: if F_i are involutions on M, then F_i H-invariant $\mapsto$ descent to $\overline{F_i}$ on $\mu^{-1}(\mathcal{O})/H$ also are involutions. The proof is the same.

Consider

$$\prod_i G(D_{t_i}^\times) \supset G(X \backslash \{t_1, \dots, t_N\}) \times \prod_i G(D_{t_i}).$$

with ker denoting the evaluation map $\prod_i G(D_{t_i}) \to G^N$, evaluated at $t_1, \dots, t_N$. We can first reduce by ker and then take the residual action of G^N, after which we descend the Hitchin Hamiltonians to $\mu^{-1}(\mathcal{O})/G^N$ for $\mathcal{O} \subset (\mathfrak{g}^*)^N$.

The points of $\mu^{-1}(0)/G^N = (E, \phi)$. For $G = \mathrm{PGL}_2$, fix the orbits in $\mathfrak{sl}_2^* \cong \mathfrak{sl}_2$ at t_i: the twisted Hitchin system corresponds to the coadjoint orbit of a nilpotent element

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix},$$

while the original Hitchin system with parabolic structure corresponds to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that these two are in the same conjugacy class with $\lambda_i \to 0$.

Thus, the condition on ϕ is that ϕ has simple poles at t_i:

$$\mathrm{Res}_{t_i} \phi|_{\ell_i} = \lambda_i \cdot \mathrm{Id}.$$

Letting $\phi = \sum_i \frac{A_i}{z - t_i}\, dz$ gives

$$A_i \begin{pmatrix} y_i \\ 1 \end{pmatrix} = \lambda_i \begin{pmatrix} y_i \\ 1 \end{pmatrix}$$

which gives

$$A_i = \begin{pmatrix} -\lambda_i + p_i y_i & 2\lambda_i - p_i y_i^2 \\ p_i & \lambda_i - p_i y_i \end{pmatrix}.$$

Thus, we have

$$\mathrm{Tr}(A_i A_j) = -(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j$$

and

$$G_i(\lambda_1, \dots \lambda_N) = \sum_{j \neq i} \frac{-(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j}{t_i - t_j}.$$

These equations define the **twisted Garnier system**.

2.6.1 Elliptic Calogero-Moser System

For genus 1, let X be an elliptic curve with $0 \in X$. Consider generic bundles of degree 0 and rank n: these are direct sums of line bundles. However, in this case, the line bundles have parameters and are elements of the Jacobian. Since the Jacobian is isomorphic to the curve itself, we have

$$L_a = \mathcal{O}(a) \otimes \mathcal{O}(0)^{-1}$$

with meromorphic sections of the form $\frac{\theta(z-a)}{\theta(z)}$. Atiyah showed that generic rank n bundles are of the form

$$E = L_{a_1} \oplus ... \oplus L_{a_n}.$$

Now, consider $G = \mathrm{PGL}_n$, place a puncture at 0, and perform a twisted reduction for $\mathcal{O} \subset \mathfrak{g}^*$ with

$$\mathcal{O} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -n+1 \end{pmatrix}.$$

This is the smallest semisimple orbit, consisting of matrices with n eigenvalues equal to 1 and one eigenvalue equal to $-n+1$, which forces $\mathrm{Tr} = 0$.

When $c \to 0$, the orbit degenerates to

$$\left(\begin{array}{c|c} 0 & * \\ \hline 0 & 1 \end{array}\right),$$

which is a nilpotent matrix of rank 1. This corresponds to the maximal parabolic subgroup

$$P = \left(\begin{array}{c|c} (n-1) \times (n-1) & * \\ \hline 0 & 1 \times 1 \end{array}\right)$$

so $G/P = \mathbb{P}^1$.

What is the automorphism group of this bundle? The automorphism group is given by $\mathrm{Aut}(E) = (\mathbb{C}^\times)^{n-1}$, which acts on $\mathbb{P}^{n-1}$. Therefore, we need to consider the free orbits of the vector

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Next, let $\phi = (\phi_{ij})$, where ϕ_{ij} is a section of $L_{q_i} \otimes L_{q_j}^{-1} = L_{q_i - q_j}$, which has a first-order pole at $z = 0$, with the residue preserving the vector $\begin{pmatrix} 1 & 1 & \vdots & 1 \end{pmatrix}$ acting with eigenvalue c. This bundle has only one such section, up to scaling, so

$$\phi_{ij} = a_{ij} \frac{\theta(z - q_i + q_j)}{\theta(z)\theta(q_i - q_j)}$$

for $i \neq j$, and $\phi_{ii} = p_i$ represents the momenta.

The condition for the A_{ij} must satisfy

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

with $a_{ii} = 0$, and A has only two eigenvalues. Therefore, all $a_{ij} = C$ for some constant C. Substituting this into the expression for the trace, we get

$$\mathrm{Tr}\phi^2 = \sum p_i^2 + C \sum_{j \neq i} \frac{\theta(z - q_i + q_j)(z - q_j + q_i)}{\theta(z)^2 \theta(q_i - q_j)^2}.$$

An interesting identity is

$$\sum_{j \neq i} \frac{\theta(z - q_i + q_j)(z - q_j + q_i)}{\theta(z)^2 \theta(q_i - q_j)^2} = \wp(z) - \wp(q_i - q_j)$$

where $\wp$ is the Weierstrass elliptic function. This leads us to the **elliptic Calogero-Moser system**. Moreover, taking $\mathrm{Tr} \wedge^i \phi$ yields higher Calogero-Moser Hamiltonians.

2.7 Quantizations

2.7.1 Quantum Integrable System

What is a quantum integrable system, and what does it mean to quantize a classical integrable system? Classical integrable systems are defined on symplectic manifolds. Before addressing this, we should first understand what it means to quantize a symplectic manifold.

This discussion can be framed in the context of smooth real manifolds, complex analytic manifolds, or algebraic varieties, even over arbitrary fields of positive characteristic. For simplicity, we will gloss over some technical details as our aim is to provide motivation. Suppose M is a symplectic manifold. Then, $\mathcal{O}(M)$ is a Poisson algebra. In the case of non-affine algebraic varieties or non-Stein complex manifolds, we may need to consider sheaves. Given two functions f and g, we can assign the Poisson bracket $f, g \mapsto \{f, g\}$ which is

a Lie algebra structure and a derivation with respect to each argument. In classical mechanics, M is a phase space of some Hamiltonian system and $\mathcal{O}(M)$ consists of the classical observables. The quantization of M involves replacing these observables with operators, which still form an algebra but no longer commute.

Definition 2.38. *The **quantization** of M is an algebra over $k[[\hbar]]$ or $k[\hbar]$, noncommutative, such that $A/\hbar A \cong \mathcal{O}(M)$, and*

$$\lim_{h\to 0} \frac{f * g - g * f}{\hbar} = \{f, g\}.$$

We have intentionally glossed over some details here, and a full course could be dedicated to this topic. However, for the sake of motivation, we will not delve deeper into the technicalities.

Example 2.39. *Let $M = T^*Y$. Then the natural quantization is the algebra of differential operators on Y, denoted $\mathcal{D}(Y)$. More precisely, $\mathcal{D}_\hbar(Y)$ is generated locally by the coordinates x_i and the derivatives $\hbar\partial_i$. Recall that $\mathcal{O}(T^*Y)$ is generated locally by the coordinates x_i, p_i, satisfying Heisenberg's uncertainty relations*

$$[\hat{p}_i, x_i] = \hbar.$$

Classically, an integrable system is defined as follows:

Definition 2.40. *An **integrable system** is a collection of functions $H_1, ...H_n$ that Poisson commute ($\{H_i, H_j\} = 0$) and are functionally independent (or algebraically independent).*

This defines a map $p : T^*Y \to \mathbb{A}^n$ with the pullback $p^* : \mathcal{O}(\mathbb{A}^n) = \mathbb{C}[X_1, \ldots, X_n] \to \mathcal{O}(T^*Y)$, where $X_i \mapsto H_i$, defining a Poisson-commutative subalgebra.

Theorem 2.41. *Any function that Poisson commutes with $H_1, \ldots, H_n$ is algebraically dependent on them.*

This suggests that, locally, the system is maximal (up to finite algebraic extension).

If Y is a smooth algebraic variety, then classical integrable systems on T^*Y correspond to Poisson-commutative subalgebras of $\mathcal{O}(T^*Y)$, with the trace degree equal to the dimension of Y.

Suppose A is a noncommutative algebra quantizing $\mathcal{O}(T^*Y)$. We then have an inclusion $\mathbb{C}[X_1, \ldots, X_n] \hookrightarrow A = \mathcal{D}(Y)$, where the commutative subalgebra is given by $X_i \mapsto H_i$. There is a similar result in the quantum setting:

Theorem 2.42 (Makar-Limanov)**.** *If $[H, H_i] = 0$, then H is algebraically dependent on $H_1, \ldots, H_n$.*

To construct a quantum integrable system, we seek a maximal (up to algebraic extension) commutative subalgebra within $\mathcal{D}(Y)$. This quantizes a classical

system if, as $\hbar \to 0$, the quantum system converges to its classical counterpart, with the swapping $\hat{p_i} \leftrightarrow p_i$.

The naive quantization approach is to replace p_i with $\hat{\partial}_i$. However, this is problematic due to the ordering issue: partial derivatives do not commute with coordinates. For instance, should we map $x_i p_i$ to $t_i x_i \partial_i$ or $\hbar \partial_i x_i = \hbar x_i \partial_i + \hbar$? In many cases, this procedure fails, though it can work in certain situations, such as for the Garnier system:

2.7.2 Quantizing the Garnier System

Let

$$G_i = \sum_{j \neq i} \frac{-(y_i - y_j)^2 p_i p_j + 2(\lambda_i p_j - \lambda_j p_i)(y_i - y_j) + 2\lambda_i \lambda_j}{t_i - t_j}.$$

The quantized system is

$$\begin{aligned} \frac{1}{\hbar^2} \hat{G}_i &= \sum_{j \neq i} \frac{-(x_i - x_j)^2 \partial_i \partial_j + 2(x_i - x_j)\left(\frac{\lambda_i}{\hbar} p_j - \frac{\lambda_j}{\hbar} p_i\right) + 2\frac{\lambda_i}{\hbar}\frac{\lambda_j}{\hbar}}{t_i - t_j} \\ &= \sum_{j \neq i} \frac{-(x_i - x_j)^2 \partial_i \partial_j + 2(x_i - x_j)\left(\Lambda_i \partial_j - \Lambda_j \partial_i\right) + \frac{1}{2}\Lambda_i \Lambda_j}{t_i - t_j}. \end{aligned}$$

by setting $\Lambda_i = \frac{2\lambda_i}{\hbar}$, which gives the **Gaudin system for $\mathbf{PGL}_2$**.

The representation theoretic way of writing this: the action of $\mathcal{U}(\mathfrak{sl}_2)$ by differential operators on the line via

$$\begin{aligned} f &\mapsto -\partial_x \\ h &\mapsto 2x\partial_x + \Lambda \\ e &\mapsto x^2 \partial_x + \Lambda x. \end{aligned}$$

This action on the line extends naturally to an action on the projective line, which then maps to twisted differential operators acting on sections of some Λ-power of the bundle $\mathcal{O}(1)$, where Λ is an integer.

The **Casimir tensor** $\Omega \in (\mathfrak{sl}_2 \times \mathfrak{sl}_2)^{\mathfrak{sl}_2}$ is given by

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h.$$

where e, f, and h are the standard generators of $\mathfrak{sl}_2$. The numerators in the formula for $\hat{G}_i$ are simply Ω_{ij} for Λ_i. Therefore, we can express the operators as

$$\hat{G}_i = \sum_{j \neq i} \frac{\Omega_{ij}}{t_i - t_j} \in \mathcal{U}(\mathfrak{g})^{\otimes n}$$

Now, we can verify that $[\Omega_{12}, \Omega_{13}+\Omega_{23}]$ holds, which is true for any (semi)simple Lie algebra $\mathfrak{g}$ because $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is invariant. This invariance implies that $[\hat{G}_i, \hat{G}_j] = 0$, so the elements $\hat{G}_i$ commute. These commuting elements in $\mathcal{U}(\mathfrak{sl}2)^{\otimes n}$ can be promoted to operators by having them act on representations. For formal Ω_{ij} satisfying the condition $[\Omega_{12}, \Omega_{13}+\Omega_{23}]$, arbitrary complex numbers t_i, t_j will commute, which gives the Gaudin Hamiltonians for $\mathfrak{sl}_2$.

If $V_1, \ldots, V_n$ are representations of $\mathfrak{g}$, then $\hat{G}_i \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ commute with $\mathfrak{g}$, meaning they act on the invariant subspace $(V_1 \otimes \cdots \otimes V_n)^{\mathfrak{g}}$. This gives rise to an interesting family of commuting operators.

For Lie algebras of higher rank, however, we no longer obtain a quantum integrable system, as higher-order operators must be considered. Thus, while this procedure works in some cases, it is not universally applicable. As an illustration, consider the following example:

2.7.3 Quantizing the Elliptic Calogero-Moser System

Consider the elliptic Calogero-Moser system with Hamiltonian

$$H_2 = \sum_i p_i - \sum_{j \neq i} \wp(q_i - q_j).$$

Quantizing gives

$$\frac{1}{\hbar^2}\hat{H}_2 = \sum_i \partial_i^2 - \frac{1}{\hbar^2}\sum_{j \neq i} \wp(q_i - q_j).$$

Theorem 2.43. *This defines a quantum integrable system, which is the centralizer of $\hat{H}_2$ in $\mathcal{D}$.*

However, this does not follow directly from the classical case. Consider the Hamiltonian

$$H_3 = \sum_i p_i^3 + \sum_i p_i f_i(q) + g(q).$$

There is still the ordering problem with the term $\sum_i p_i f_i(q)$: does $p_i f_i(q)$ correspond to $\partial_i f_i(q)$, $f_i(q)\partial_i$, or something else?

In general, there is no universal method for quantizing an integrable system. We must return to the definition of the classical system and explore whether the way we obtain quantizations can be adjusted.

2.7.4 The Quantum Hitchin Integrable System

Recall how we constructed classical Hitchin on $\mathrm{Bun}_G^0(X)$:

Step 1: Represent $\mathrm{Bun}_G(X)$ as a double quotient, for example, $G(X\backslash x)\backslash G(K)/G(\mathcal{O})$.

Step 2: Construct some commuting Hamiltonians on $T^*G(K)$, invariant under the left and right actions of $G(K)$.

Step 3: Perform Hamiltonian reduction by $G(X\backslash x)\times G(\mathcal{O})$ to obtain $T^*\mathrm{Bun}_G(X)$, which gives the Hitchin integrable system.

To retrace our steps, we now need to discuss quantum Hamiltonian reduction along an orbit. Classically, suppose H acts on a symplectic manifold M with a moment map $\mu : M \to \mathfrak{h}^* \supset 0$. The Hamiltonian reduction is given by $\mu^{-1}(0)/H$, which results in a symplectic manifold if the action is "nice."

Quantum mechanically, we need to understand the quantum analogs of these constructions. We have H acting on A, a noncommutative algebra.

Returning to the classical case, we have the following data: a moment map $\mu : S(\mathfrak{h}) = \mathcal{O}(\mathfrak{h}^*) \to \mathcal{O}(M)$, which is a Poisson homomorphism between Poisson algebras. Furthermore, $\mu^{-1}(0)/H \subset M/H$ is defined by the equation $\mu(m) = 0$, so $\mathcal{O}(M/H) = \mathcal{O}(M)^H \subset \mathcal{O}(M)$ is a Poisson subalgebra, and $\mathcal{O}(\mu^{-1}(0)/H)$ is the quotient of $\mathcal{O}(M/H) = \mathcal{O}(M)^H$ by the ideal defined by the equation $\mu(m) = 0$.

Now, we can extend this to the quantum case:

Definition 2.44. *The **quantum Hamiltonian reduction** of A by H is the algebra $A^H/(A\mu(h))^H$.*

$A\mu(H)$ is only a left ideal in A, but $(A\mu(h))^H$ is a two-sided ideal in A^H. Furthermore, if H is reductive (which is not the case for the Hitchin system), this is the same as $(A/\mathrm{Aut}(\hbar))^H$. Notice that $A/\mathrm{Aut}(\hbar)$ is an A-module (not an algebra), but taking H-invariants gives it an algebra structure.

If we replace 0 with an orbit $\mathcal{O} \subset \mathfrak{h}^*$, the equation $\mu^{-1}(\mathcal{O})/H$ becomes $\mu(m) \in \mathcal{O}$. In the quantum mechanical setting, we need to find an ideal in $\mathcal{U}(\mathfrak{h})$ that quantizes $\mathcal{O}$, so that $\mathcal{U}(\mathfrak{h})/I$ is a quantization of $\mathcal{O}$.

The quantum Hamiltonian reduction is then given by $A^H/(A\mu(I))^H$. In the case where $\mathcal{O} = 0$, I is the augmentation ideal, $\ker(\mathcal{U}(g) \to \mathbb{C})$, which is the same as $A\mu(I) = A\mu(\mathfrak{h})$.

In the classical case, the condition for the moment map is

$$\forall z \in \mathfrak{h}, \qquad z \circ a = \{\mu(z), a\}$$

The quantized moment map satisfies

$$z \circ a = [\mu(z), a]$$

where μ is H-invariant.

We can now attempt to implement this for the Hitchin system. For simplicity, let's use the version without punctures. We take the loop group $A : \mathcal{D}(G(\mathbb{C}((tt))))$. The quantum Hitchin system should be obtained from certain two-sided differential operators on $G(K)$. This is an infinite-dimensional group, and considering differential operators on it involves complex differential geometry. However, Beilinson and Drinfeld have figured out how to handle this. For

now, we will ignore these details and treat $L = G(K)$ as if it were a finite-dimensional Lie group. This leads to the following question: What are the two-sided invariant differential operators on L?

The left invariants are given by $\mathcal{U}(\ell)$ with $\ell \in L$. The two-sided invariants are given by $\mathcal{U}(\ell)^L = Z(\mathcal{U}(\ell))$, the center of the enveloping algebra. We have $\mathrm{Lie}(G((t))) = \mathfrak{g}((t))$, with G simple. It turns out that $Z(\mathcal{U}(\mathfrak{g}((t)))) = \mathbb{C}$, which is disappointing. This Lie algebra is infinite-dimensional, so we should ask whether we need some completions. The bad news is that this does not help us, and the center remains trivial. The problem arises because the Hitchin Hamiltonian system upstairs was defined by very simple formulas, and we cannot lift them into the quantum world.

Classically, $H_2 = \frac{1}{2}\mathrm{Tr}\phi^2$, with $\phi = \phi(z), dz \in \mathfrak{g}((z))$. Writing $\phi = \sum \phi_n z^{-n}$ gives

$$H_2 = \frac{1}{2}\sum z^n \left(\mathrm{Tr}\sum_m \phi_m \phi_{n-m}\right)(\,dz).$$

The trace here means summing over some orthonormal basis a_i of $\mathfrak{g}$, with $\phi = \sum \phi^i a_i$, which givess a_i of $\mathfrak{g}$ with $\phi = \sum \phi^i a_i$, which gives

$$H_2 = \sum_{n,i} z^n \left(\sum_m \phi^i_m \phi^i_{n-m}\right).$$

Here, z is just a parameter on the Higgs field defined on the punctured formal disc. We want z to be invariant under the action of the Lie algebra, so we require that $[\phi^j_p, H_2] = 0$. Expanding the commutator gives an infinite sum, and we need ordering to make this expression meaningful on the highest-weight representations. However, this commutation relation ultimately fails.

In fact, the issue arises because Beilinson and Drinfeld showed that every globally defined differential operator on $\mathrm{Bun}_G(X)$ is a scalar. To address this, we need to recall some physics: for $M = T^*Y$, we want classical observables to quantize operators on $L^2(Y)$. However, to define this, we need a measure on Y. We do this by taking $L^2\left(Y, \Omega^{\frac{1}{2}}\right)$, where Ω denotes bundles of densities, with

$$\|f(y)|dy|^{\frac{1}{2}}\|^2 = \int |f(y)|^2 |dy|.$$

Thus, the most natural quantization is $\mathcal{D}(Y, K^{\frac{1}{2}})$.

Let Y be a smooth variety and L a line bundle on Y. We can define differential operators acting on subsets of L, denoted $\mathcal{D}(Y, L)$. Note that this is not equivalent, but Morita equivalent, to the usual differential operators on Y.

Next, we can define $\mathcal{D}(Y, L^{\otimes n})$, generated by functions and vector fields, with relations involving n as a parameter. Locally, we pick a connection with curvature ω, and the relations on L between functions and vector fields are the same

as usual, but we now have

$$[\nabla_V, \nabla_U] = \nabla[V, U] + n\omega(v, u)$$

so we can take any $n \in \mathbb{C}$.

Now, we want to replace $\mathcal{D}(\mathrm{Bun}_G(x))$ by $\mathcal{D}(\mathrm{Bun}_G(X), K^{\frac{1}{2}})$. To do this, we need to understand how to obtain $K^{\frac{1}{2}}$ as a line bundle on $G((t))$. In ordinary Lie groups, we don't have interesting line bundles because $H^2 = 0$. However, H^2 of a loop group, by transgression, is the same as H^3 of the underlying group G, which is 1-dimensional if G is simple. Therefore, we do have an interesting line bundle on the loop group. This line bundle is related to the central extension, and Kac-Moody groups naturally arise. There exists a Kac-Moody group with the sequence

$$\hat{G} \mapsto \mathbb{C}^\times \to \hat{G} \to G((t)) \to 1,$$

which is just the central extension and defines a principal $\mathbb{C}^\times$ bundle on $G((t))$, denoted $\mathcal{E}$.

Thus, Lie$\hat{G}$ is the affine Kac-Moody algebra $\mathfrak{g}((t)) \oplus \mathbb{C}K$, with the commutator relation

$$[a(t), b(t)] = [a, b](t) + \mathrm{Res}_{t=0}(a(t)\, db(t))K.$$

Theorem 2.45. *K_{Bun_G} is obtained by reduction of $\mathcal{E}^{-2h^\vee}$, where $h^\vee$ is the dual Coxeter number of G.*

For example, E_8 has dual Coxeter number 30, so the reduction is given by $\mathcal{E}^{-60}$.

If we want to consider differential operators on $K^{\frac{1}{2}}$, we should look at the two-sided invariant elements in $\mathcal{D}(G((t)), \mathcal{E}^{-h^\vee})$, which is

$$Z(\hat{\mathcal{U}}(\hat{\mathfrak{g}})/K = -h^\vee).$$

Theorem 2.46 (Feigin and Frenkel, 1991)**.** *$P_i(\theta)$ lift to this center and can be quantized.*

In conformal field theory, this implies the existence of a **quantum anomaly**: if $K = 0$, we do not have a center due to the ordering problem. However, if we apply normal ordering, we obtain an anomaly that vanishes. The value of K is referred to as the **level**, and $h^\vee$ is the **critical level**. The representation theory of Kac-Moody algebras is fully differential at this level due to the existence of this center. It turns out that this is the only value of K for which the center is non-trivial.

Theorem 2.47 (Beilinson-Drinfeld)**.** *The two-sided invariant differential operators on $G((t))$ acting on $\mathcal{E}^{-h}$ descend to differential operators on $Bun_G(x)$. This map is surjective onto $\mathcal{D}(Bun_G(X), K^{\frac{1}{2}})$, and the algebra is a polynomial algebra of $(g-1)\dim G$ generators that quantizes the Hitchin system.*

Both the Gaudin and elliptic Calogero-Moser systems are examples of this theorem. This resolves the ordering problem - if we compute the action of the Feigin-Frenkel central elements (which are difficult to compute but can be done for GL_n, for example) and evaluate their image, we obtain the correct ordering that gives an integrable system.

2.7.5 The Sugawara Construction

The behavior of the quadratic operators gives the Sugawara construction, which plays a crucial role in representation theory and conformal field theory.

Classically, we have $\frac{1}{2}\mathrm{Tr}\phi^2$. In the quantum mechanical case, $T_n = \frac{1}{2}\sum_{m,i} :\phi_m^i \phi_{m-n}^i:$, where :: denotes normal ordering with the larger index placed first. This works as an operator in the highest weight representations.

Definition 2.48. *The **Sugawara construction** is given by the following commutation relations:*

$$[\phi_p, T_n] = p(K+h^\vee)\phi_{n+p},$$
$$[T_n, T_m] = (K+h^\vee)(n-m)T_{m+n} + \frac{n^3-n}{12}K(K+h^\vee)\dim\mathfrak{g}\cdot\delta_{n-m}.$$

Theorem 2.49 (Sugawara)**.** *When $L_n = \frac{T_n}{K+h^\vee}$, with $K \neq h^\vee$, this gives a Virasoro algebra with central charge*

$$c = \frac{K\dim\mathfrak{g}}{K+h^\vee}.$$

When $L = -h^\vee$, we obtain the central case.

There are higher-order operators involving cubics, etc., which arise from higher central elements, but we won't cover those here. Ultimately, Beilinson and Drinfeld used the Feigin-Frenkel theorem as input to construct this quantum Hitchin system.

This is a good place to stop.

3 Nikita Nekrasov: Integrable Many-Body Systems and Gauge Theories

Abstract

Elliptic Calogero-Moser and Toda systems, Gaudin and other spin chains are algebraic integrable systems which have intimate connections to gauge theories in two, three, and four dimensions. I will explain two such connections: first, classical, through Hamiltonian reduction and second, quantum, through dualities of supersymmetric gauge theories.

Contents

3.1 Introduction

3.1.1 Overview

This course explores the relationship between integrable systems (which we refer to as many-body systems, since quantum and classical many-body systems are prime examples) and their connections to classical and quantum gauge theories.

From a physics perspective, gauge theories are fundamental because they describe most of the known interactions of elementary particles. They also have significant mathematical importance, as many concepts in differential topology have emerged over the past 40 years from studying solutions to partial differential equations motivated by gauge theory. By examining the moduli spaces of these solutions, alongside cohomology theory, intersection numbers, and enumerative geometry, we have made considerable progress in understanding the invariants of four-manifolds, knot invariants, and categorification. These mathematical theories have developed alongside advances in physics.

The connection between integrable systems and gauge theory will be presented in two main forms, with an emphasis on duality - a concept that has seen significant development in recent months. In the context of physics, duality is a central theme, while in mathematics, it is a well-established notion that has taken many forms, from the Fourier transform and the Legendre transform to Langlands duality. A simplified version of Langlands duality will be discussed here.

We can identify two types of relationships between these fields:

- "Gauge theory is equivalent to an integrable system." This occurs in rare cases where we can directly analyze the Lagrangian of the gauge theory, solve the constraints in the appropriate variables, and observe that the resulting dynamics correspond to those of a many-body system. This is sometimes useful, and we will explore more examples later.
- Indirect relation: "Correlation functions of quantum gauge theory obey classical (or quantum) equations of some integrable system."

The key difference between these two relationships lies in the connection between the quantization parameters. In the first case, the Planck constant of the gauge theory is equal to that of the integrable system: $\hbar_G = \hbar_I$. This establishes an equivalence between the two systems, both classically and quantum mechanically. In the second case, the quantization in one theory is not directly related to the quantization in the other, meaning the Planck constants differ: $\hbar_G \neq \hbar_I$. This difference is intriguing because it suggests that we can study a quantum system in one theory by solving the classical equations of motion of a simpler integrable system, and potentially vice versa.

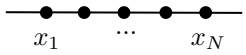
In some instances, the integrable systems for both direct and indirect relations may be the same, while the gauge theories differ. This points to a duality between the gauge theories: simple questions in one gauge theory correspond to

more complex questions in another, and vice versa. This duality is a key reason why studying these connections is so valuable.

3.2 Examples of Integrable Systems

3.2.1 The Calogero-Moser-Sutherland System

The Calogero-Moser-Sutherland many-body system describes particles moving in one dimension.



Consider 1D indistinguishable particles. These are non-relativistic, and their Hamiltonian is given by

$$H_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2}\nu^2 \sum_{i<j} \frac{1}{(x_i - x_j)^2}$$

where the phase space is endowed with the symplectic form

$$\omega = \sum_{i=1}^{N} dp_i \wedge dx_i$$

and the phase space is

$$P = T^*(\mathbb{R}^n \backslash \Delta)/S(n).$$

with Δ representing the diagonal in $\mathbb{R}^N$. Since the particles are indistinguishable, the configuration space is the set of positions modulo permutations. To correctly account for this symmetry, we consider the quotient by the symmetric group acting on both the coordinates and momenta.

If $\nu^2 > 0$, there is repulsion between the particles. This repulsion ensures that in finite-energy configurations, the particles cannot collide, which is why we exclude the diagonal from the phase space.

The system is Hamiltonian, and the equations of motion follow from the Hamiltonian:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}$$

Remarkably, these equations can be packaged in one equation, of Lax form:

$$\dot{L} = [L, A]$$

where L is an $N \times N$ Hermitian matrix defined by

$$L = \|p_i \delta_{ij} + \sqrt{-1}\nu \frac{1}{x_i - x_j}(1 - \delta_{ij})\|.$$

The term $\frac{1}{x_i - x_j}(1-\delta_{ij})$ is antisymmetric, but we multiplied it by $\sqrt{-1}$ to ensure that L is Hermitian.

The crucial claim here is that there exists a matrix, depending on $p(x)$, such that the time derivative of this matrix operator satisfies the commutation relation. This implies that the eigenvalues of L are constants of motion. As the particles repel each other, in the limit as $t \to \pm\infty$, their positions become well-separated: $x_{\sigma(1)} \ll x_{\sigma(2)} \ll \cdots \ll x_{\sigma(N)}$. In this limit, the off-diagonal terms in L become negligible, and the eigenvalues of L correspond to the asymptotic momenta $p_1^{\pm}, p_2^{\pm}, \ldots, p_N^{\pm}$. While the individual eigenvalues at $t \to \pm\infty$ need not be the same, they are permuted versions of the same set. These eigenvalues are not typically useful functions of p and x directly, as they are the solutions to an algebraic equation. Instead, we form polynomial functions of the momenta and rational functions of the coordinates by taking powers of the operator L and computing the trace:

$$H_k = \frac{1}{k}\mathrm{Tr}L^k$$

which are the integrals of motion. This is the simplest example of a direct relationship between gauge theory and an integrable system.

3.2.2 Symplectic Quotients

A more complex evolution of this system arises from a projection of a simpler dynamics within a larger phase space. The larger phase space hidden in this problem is

$$T^*(\mathrm{U}(N) \times \mathbb{R}^N),$$

which is a symplectic vector space of dimension $2(N^2+N)$. We can parameterize elements of this phase space as

$$\{(\sqrt{-1}P, \sqrt{-1}X; w, v)\}$$

where $X = X^T$ and $P = P^T$ are $N \times N$ matrices, $v \in \mathbb{R}^N$, and $w \in (\mathbb{R}^N)^*$. On this larger space, we study two aspects:

1. First, we introduce very simple dynamics. In this context, P is conjugate to X and w is conjugate to v, where we use "conjugate" in the sense of a symplectic form $w = \mathrm{Tr}, dP \wedge dX + dw \wedge dv$. We define the Hamiltonian $H_k = \frac{1}{2}\mathrm{Tr}P^k$, which generates the following evolutions:

$$X(t) = X(0) + \sum_{k=1}^{N} t_k P^{k-1}(0),$$

$$P(t) = P(0),$$

$$v(t) = v(0),$$

$$w(t) = w(0).$$

2. In principle, k can be arbitrary, but the evolution will be generated by the H_k when $k > N$. For $k > N$, the evolution can be reduced to one already covered. This describes a free motion in the large space.

3. Next, we assume that our problem has a symmetry and perform the symplectic quotient with respect to this symmetry. The symmetry is generated by $G = \mathrm{U}(N)$, and it acts as follows: for $g \in G$, $(P, X, z) \mapsto (g^{-1}Pg, g^{-1}Xg, g^{-1}z)$, where $z = v + \sqrt{-1}w$. The only effect of the symmetry is to transform z under the action of $\mathrm{U}(N)$. This is a symmetry that preserves the symplectic form, and it also preserves the Hamiltonians. Therefore, if we take a simple motion in the large phase space P_L with G-orbits, we can project it onto the smaller phase space $P_S = P_L/G$. On the small phase space, a trivial motion from the larger phase space may now appear more complicated, as we are identifying points according to the group action.

   The simplest example of this construction, when $N = 2$, is not difficult to discuss: consider a free motion of a particle in $\mathbb{R}^3$. $\mathbb{R}^3$ is the Lie algebra of $\mathrm{SU}(2)$. Take the free motion $\vec{r} = \vec{r_0} + \vec{v}t$ and project it onto the quotient space $\mathbb{R}^3/\mathrm{SO}(3)$. The invariant here is the distance from the origin. Therefore, if we look at the time dependence of $r(t) = |\vec{r_0} + \vec{v}t|$, we observe that the particle (represented by the radial coordinate of the free particle in $\mathbb{R}^3$) experiences a force that repels it from the origin. This force is the centripetal force.

3.2.3 The Momentum Map

In the standard setting of spaces with symmetry, it is common to take a quotient by identifying points within an equivalence class, effectively contracting the entire orbit of the symmetry group to a single point. However, in classical mechanics, where we work with symplectic manifolds, taking the quotient of a symplectic manifold by the action of a group does not, in general, yield another symplectic manifold. This issue is addressed through a construction known as the momentum map, which we will now describe.

The momentum map, denoted by $\mu : P_L \to \mathrm{Lie}(G)^*$, can be thought of as the collection of Hamiltonians that generate the symmetry. Rather than discussing the general procedure, we will compute it explicitly for our example.

For a free particle moving in 3D space, the phase space is $T^*\mathbb{R}^3 = (\vec{p}, \vec{r})$, a 6-dimensional space. The symmetry group in this case is $\mathrm{SO}(3)$, and the associated generators of the symmetry are the components of the angular momentum, $\vec{L} = \vec{p} \times \vec{r}$. The angular momentum is a classic example of a momentum map: a map from the 6D phase space to the 3D space, which is dual to the Lie algebra of $\mathrm{SO}(3)$.

To compute the infinitesimal version of this symmetry, consider a perturbation of the symmetry group given by $g = 1 + \xi + \ldots$, where $\xi \in \mathrm{Lie}, \mathrm{U}(N)$. This

leads to a vector field of the form:

$$V_\xi(P, X, z) = ([P, \xi], [X, \xi], -\xi \cdot z),$$

Since the vector field preserves the symplectic form, there has to be a Hamiltonian that generates this vector field. Since each component of the vector field is linear, each component of the Hamiltonian must be linear. We have

$$H_{V_\xi} = \mathrm{Tr}\xi([P, X] + \sqrt{-1}(z \otimes z^T)),$$

which preserves the symplectic form. By the properties of symplectic geometry, we know that there must be a Hamiltonian that generates this vector field. Since each component of the vector field is linear in ξ, the Hamiltonian that generates this vector field must also be linear. Therefore, we obtain:

$$\mu(P, X, z) = [P, X] + z \otimes z^T.$$

It is important to note that the momentum map is not unique because the relationship between the vector field and the Hamiltonian is not one-to-one. Specifically, if H differs by a constant, the resulting vector field will be the same. However, if we are given the vector field and wish to reconstruct the Hamiltonian, we can do so up to a constant. Consequently, we construct $\mu(\xi)$ up to an additive constant, which is linear in ξ. There is also a cohomological issue that arises when adjusting these constants in such a way that the G-equivariance of the momentum map holds. In particular, we require the following condition:

$$\{\mu(\xi_1), \mu(\xi_2)\} = \mu([\xi_1, \xi_2]).$$

This condition imposes some restrictions on the choice of constants, but there remains some flexibility, especially if the group G is not simple. If G contains a center, for instance, we may find that the momentum map takes the form $\mu(P, X, z) = [P, X] + \sqrt{-1}(z \otimes z^T - \nu \cdot 1_N)$, where ν is a constant associated with the center of G.

We are almost ready to project. Since $\mu^{-1}(0)$ is G-invariant, we can take the quotient $\mu^{-1}(0)/G$.

Proposition 3.1. *$\mu^{-1}(0)/G$ is a symplectic manifold.*

Now, let's attempt to solve $\mu = 0$, which gives us the condition $[P, X] = \sqrt{-1}(\nu \cdot 1_N - z \otimes z^T)$, modulo G-symmetry. Since P and X are Hermitian matrices and G acts by conjugation, we can choose a representative from the G-orbit of (P, X, z), where we can set $X = \mathrm{diag}(x_1, \ldots, x_N)$. Fixing this representative, there is still residual symmetry remaining. Specifically, we are left with transformations that preserve the diagonal form of X, reducing G to $\mathrm{U}(1)^N \ltimes S(N)$.

In the basis when X is diagonal, we have

$$[P, X]_{ij} = P_{ij}(x_i - x_j) = \sqrt{-1}(\nu\delta_{ij} - z_i\overline{z}_j).$$

If $i = j$, then everything becomes 0, so $\nu = |z_i|^2$ for all $i = 1, \ldots, N$. Now, we use $g = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N})$ to make $z_i = \sqrt{\nu}$ for all $i = 1, ...N$, reducing G to $\mathrm{S}(N)$.

At this point, we can solve for P. The solution is given by:

$$P = p_i S_{ij} + \sqrt{-1}\nu \frac{\delta_{ij} - 1}{x_i - x_j}$$

The conclusion is that if we parametrize the orbits of the group action by the diagonal of X, P takes the form above, and the motion becomes nontrivial. The eigenvalues of X evolve according to the equation:

$$x_i(0)\delta_{ij} + t_i(p_i\delta_{ij} + \sqrt{-1}\nu \frac{\delta_{ij} - 1}{x_i - x_j}).$$

In particular, the transformations of X and P under the group action are:

$$X \mapsto g^{-1}(X + tP)g$$
$$P \mapsto g^{-1}Pg$$

where $g = g(t)$.

This is a gauge symmetry - a symmetry transformation that depends on time. To represent this using a Lagrangian, we start with the standard Lagrangian that corresponds to the symplectic structure and the Hamiltonian:

$$L = \mathrm{Tr}(P\dot{X} + z^T\dot{z}) - \frac{1}{2}\mathrm{Tr}P^2 + \nu(A_t)$$

where A_t is a Lagrange multiplier, a 1-form valued in the Lie algebra $\mathrm{Lie}(G)$, introduced to reduce the number of degrees of freedom. We can quantize this action and write it as:

$$S = \mathrm{Tr}\left(P\,dX + z^T\,dz\right) - \frac{1}{2}\,dt\,\mathrm{Tr}(P^2) + \mu(P, X, z)\,(A_t\,dt)$$

where S is a functional defined on $\mathrm{Maps}(\mathbb{R}, P_L) \times \Omega^1(\mathbb{R} \otimes \mathfrak{g})/\mathrm{Maps}(\mathbb{R}, G)$. If we perform the transformation

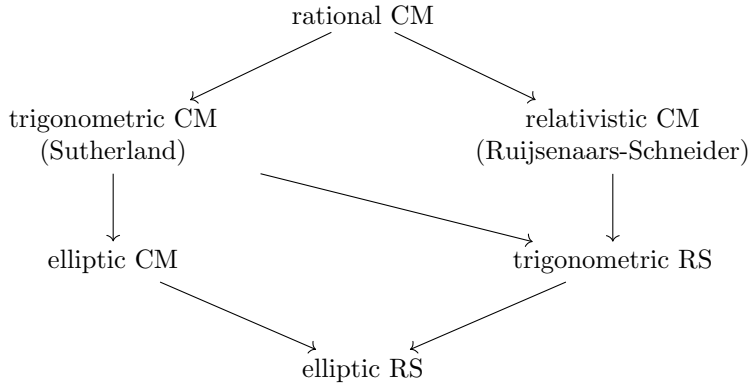
$$(P(t), X(t), z(t)) \mapsto (g^{-1}(t)P(t)g(t), g^{-1}(t)X(t)g(t), g^{-1}(t)z(t)),$$

the action is not invariant under this transformation. However, by adjusting $A_t \mapsto g^{-1}A_t g + g^{-1}\partial_t g$, we achieve invariance, providing us with our first glimpse of gauge invariance.

3.2.4 Generalizations

This many-body system is a realization of the simplest 1D gauge theory: it is a gauge theory where the spacetime is just time (with no spatial dimensions).

However, the variables have matrix structure, suggesting that both P and X represent components of a higher-dimensional gauge field. Specifically, we interpret $A_x = X$ and $A_y = P$, which implies that this rational Calogero-Moser system may have a generalization or deformation in which these extra dimensions will become apparent. In fact, there are many such generalizations:



Each model corresponds to a different gauge theory living in different space time dimensions:

- The trigonometric CM corresponds to the 2D Yang-Mills theory.
- The elliptic CM corresponds to the hybrid hopological holomorphic 2D Yang-Mills theory.
- The trigonometric RS corresponds to the 3D Chern-Simons theory.
- The elliptic RS corresponds to the 4D hybrid topological holomorphic Chern-Simons theory.

Example 3.2 (The trigonometric CM system)**.** *For the trigonometric CM system, the systems now live on the circle so we map* $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ *and the Hamiltonian is*

$$H_2 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{8} \nu^2 \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j))}.$$

We can think of the system where the symmetric group (which was permuting particles) gets promoted to an affine Weyl group, so in addition to permutations, we also have shifts (with period 2π*).*

A well-known construction by Kazhdan, Kostant, and Sternberg explains how this system arises through a projection method. Specifically, the method involves replacing the Lie algebra of $U(N)$ *in the large phase space with the corresponding group, and repeating the entire construction. However, this approach does not account for the emergence of additional dimensions. To do this, we need to promote* $U(N)$ *to the affine group* $\widehat{U(N)} = \{(E(x), c) | x \in S^1, E(x) \in U(N), c \in$

$\mathbb{R}\}$ *with the Lie bracket operation*

$$[(E_1(x), c_1), (E_2(x), c_2)] = \left([E_1(x), E_2(x)], \int_{S^1} TrE_1\, dE_2\right).$$

One should verify that this bracket satisfies the Jacobi identity.

Next, consider the dual space

$$T^*\widehat{U(N)} = \{(E(x), c; k\partial_x + A_x) | k, c \in \mathbb{R}, E, A_x \text{ are } \mathfrak{g}\text{-valued on } S^1\}.$$

Here, we combine $k\partial_x$ and A_x into the first differential operator because we have a natural pairing

$$kc + \int_{S_1} TrEA_x\, dx.$$

When acting on the adjoint representation, the coadjoint action is computed so that this pairing is preserved, and the coadjoint action acts on the differential operator. At the level of the group, the LG-action on (E, A_x, k, c) is given by

$$(g^{-1}(x)E(x)g(x), g^{-1}(x)A_xg(x) + kg^{-1}(x)\partial_x g(x), k, c + ...).$$

Here, we can think of E as analogous to P, and A_x as analogous to X.

Now, let's fix $k = 1$. In this case, the commutator $\mu = [P, X]$ becomes $\mu = D_xE = \partial_xE + [A_x, E]$. In physics, E is called the electric field, A_x is called the vector potential, and the new commutator is a form of Gauss's law (though it may appear slightly different from the usual statement). The term $\nu \cdot 1_N - z \cdot z^T$ arises from the n-dimensional representation of $U(N)$ which we used exactly once: when fixing the gauge, we did not fully utilize all of the symmetry, leaving behind some residual symmetry associated with diagonal transformations. We introduce this vector in $\mathbb{C}^N$ to absorb this remaining symmetry. The analog of $X = diag(x_1, \ldots, x_N)$ is now $A_x = diag(x_1, \ldots, x_N)$, where each x_i are constant.

This is a result of Floquet-Lyapunov theory: consider an anti-Hermitian $N \times N$ matrix-valued function $A_x(x)$ defined on S^2. If we define the gauge transformation $A_x(x) \mapsto g^{-1}(x)A_x(x)g(x) + g^{-1}(x)\partial_x g(x)$, where $g : S^1 \to \mathrm{U}(N)$, what is the canonical normal form of A_x?

It turns out that we can always diagonalize A_x and make it constant. The strategy is to "kill" A_x by finding a gauge function g such that $A_x = \partial_x g g^{-1}$. This leads to the first-order ordinary differential equation $\partial_x g = A_x g$, which can be easily solved with an appropriate initial condition.

The issue arises when we wrap around the circle and solve for $g(x)$. Specifically, when solving $g(x) = P\exp\left(\int_0^x A_x, dx\right)$, the solution is not the same as the original g, and the holonomy becomes $P\exp \oint_0^{2\pi} A_x, dx = G$. Thus, there is no g such that $\partial_x g = A_x g$ globally, but we can always find a g such that $\partial_x g = A_x g - gX$, where X is a matrix with constant diagonal entries, effectively undoing the holonomy.

Once we find the gauge, there are still transformations that preserve the fact that A_x is diagonal, but which change the eigenvalues. For example, consider $g(x) = \mathrm{diag}(e^{in_1 x}, \ldots, e^{in_N x})$, where $n_i \in \mathbb{Z}$. Under such a transformation, the eigenvalues x_i transform as $x_i \mapsto x_i + n_i$. Additionally, we can introduce transformations of the form $g(x) = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N})$, which do not affect the eigenvalues. To absorb the remaining symmetry, we need an analog of the vector z. This explains why, at the end, the expression $\nu \cdot 1_N - z \cdot z^T$ becomes $\delta(x)(\nu \cdot 1_N - z \otimes z^T)$, and the Hamiltonian takes the form $H_2 = \frac{1}{2}\int dx, \mathrm{Tr}(E^2)$.

In the examples discussed, we observed how gauge theory is directly equivalent to a quantum or classical mechanical system of interacting particles. In the next section, we will explore how this framework extends from 2-dimensional Yang-Mills theory on smooth manifolds to Yang-Mills theory on forms, where many integrable systems arise.

3.3 Complexification

3.3.1 Classical Mechanics

Complexification is a process that sends $\mathbb{R} \to \mathbb{C}, \mathbb{Z}/2 \to \mathrm{U}(1), \mathbb{C}^\times$. What happens in classical mechanics? Earlier, we discussed paths in the phase space. We would like to complexify the space of paths in a phase space P. One way to do this is to complexify the phase space $P_\mathbb{C}$, but keep the paths real.

Definition 3.3. *A **complex phase space** is a complex manifold with a holomorphic symplectic structure $\omega_\mathbb{C} \in \Omega^{2,0}(P_\mathbb{C})$ such that $\dim_\mathbb{C} P_\mathbb{C} = 2n$ and $\omega_\mathbb{C}^n \neq 0$.*

Even if a problem involves dynamics in a real phase space, one might want to do statistical mechanics where we integrate over the phase space with some probability measure (induced by the Liouville form and the Hamiltonian). However, we are interested in integral quantities like averages. To compute an integral, we can use Stokes' theorem and deform the integration contour, which often results in integrating over contours in the complexification of the original space.

If we take complex paths in P (compared to real paths in $P_\mathbb{C}$), this leads to string theory (where we replace worldlines with worldsheets), which gives a different kind of complexification. These two approaches are related: when one approaches quantization as a study of periods, we end up studying not just paths but maps of high-dimensional objects. But this is a story for later.

This simple process allows us to freely talk about things that are otherwise difficult to discuss in full generality. Let's see an example:

3.3.2 Example: Many-body Elliptic Calogero-Moser System

Consider the Hamiltonian

$$H_2 = \sum \frac{1}{2} p_i^2 + \nu^2 \sum_{i<j} \wp(x_i - x_j; \tau)$$

where $\wp$ is the Weierstrass $\wp$-function on $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $\tau_2 := \operatorname{Im} \tau > 0$. In general, the function inside the summation is not real, so it would make the dynamics look awkward unless we allow $(p_i, x_i)_{i=1}^N$ to be complex.

Now, an interesting phenomenon occurs: when both position and momentum are real, and we study configurations of finite energy, it becomes clear that we cannot allow the points to collide because doing so would require infinite energy. However, once position and momentum become complex, it's possible to keep the energy finite by allowing particles to approach each other. This can be achieved by making the momenta very large and complex in such a way that the potential energy compensates for the kinetic energy.

This change makes the phase space much more interesting, as it is now $\widetilde{(T^*E)^N}/\mathrm{S}(N)$, where E is an elliptic curve. This is a smooth hyperkähler manifold. Inside the elliptic curve, there is a circle S^1. If we start with a configuration of points, all of which are real and lie on this circle, and then set some initial conditions for the momenta (also real), the complex Hamiltonian will force these particles to move. They will not remain confined to the circle. However, we can still restrict their motion by adjusting the momenta in such a way that the coordinates stay real.

When we quantize this system, we can look for solutions where the momenta and coordinates remain real, even as the particles move. In quantum mechanics, we would construct a differential operator from this function $p_i = \sqrt{-1}\hbar\frac{\partial}{\partial x_i}$ acting on wave functions, which are analytic and defined in a neighborhood around the circle. We can require that the wave functions be normalizable only when x is on the circle. Nevertheless, we don't have to restrict the parameters to be purely imaginary. The Hamiltonian will generally have complex eigenvalues, so the operator $\hat{H}_2$ will, in general, take complex values as a function of the parameters ν and τ.

The interesting and nontrivial part is that on this phase space, the symplectic form ω, despite looking like an exact form, is actually not exact. In particular, the coordinates p and x are not global coordinates, meaning they do not cover the entire phase space. As a result, $[\omega_{\mathbb{C}}]$ is nontrivial in cohomology, and there exists a 2-dimensional sphere inside the space such that the period of $\int_{S^2} \omega_{\mathbb{C}} \sim \nu$. The coupling constant (the coefficient in front of the potential term) measures the topological non-triviality of the symplectic form. This non-triviality arises due to the resolution of singularities.

Let's discuss the case where $n = 2$ to see how this works in more detail. For simplicity, consider the center of mass frame, where $p_1 + p_2 = 0$. This means we perform the symplectic quotient $P_{\mathbb{C}}//E$ acting by $(x_1, \ldots, x_N) \mapsto (x_1 + z, \ldots, x_N + z)$ for $z \in E$, and the moment map is $\mu = p_1 + \cdots + p_N$. The phase space is acted upon by the group, which in this case is the elliptic curve itself. The group acts by shifting all coordinates by the same amount. As an exercise, we can perform the reduction by fixing the total momentum to zero and identifying configurations of points that are related by an overall shift. In

this case, the reduced phase space is $(T^*E)/(\mathbb{Z}/2)$. The elliptic curve here is given by the coordinate $x_1 - x_2$, which is invariant under the transformations of $\mathbb{Z}_2$ that exchange x_1 and x_2. These transformations act on both the coordinates and the momenta, exchanging the momenta p_1 and p_2.

Now, there is a naive phase space, which is just the quotient of the cotangent bundle over the elliptic curve by the involution that changes the sign of the coordinates. This involution has fixed points, and we need to understand how many fixed points there are.

We have four fixed points. To visualize this, the elliptic curve has four points where the involution $(p, x) \to (-p, -x)$ fixes them. These points lie in the fiber of the cotangent bundle. The involution also acts by changing the sign in the fiber, so at these fixed points, the zero in the fiber times the fixed points gives the final set of fixed points. The set of points is

$$(p, x) = (0, 0); \left(0, \frac{1}{2}\right), \left(0, \frac{\tau}{2}\right), \left(0, \frac{1+\tau}{2}\right).$$

The number of fixed points for the involution is important in understanding the structure of the phase space. In fact, there are 16 fixed points for the involution on the 4-dimensional torus, which is crucial when studying the K3 manifold — we're focusing on a simpler version. The key feature here is that the local geometry near each of these fixed points is identical. This space generally has more complex structures than we can easily visualize, but it's crucial to note that each of these fixed points can be resolved by gluing in a 2-dimensional sphere at each of these points. Normally, we could make these spheres completely independent, and that would give us a four-dimensional parameter space, but in our case, due to the symmetries, we only have one deformation accessible to us.

Now, let's focus on the local geometry. We can model the local structure near these points as $\mathbb{C}^2/(\mathbb{Z}/2)$, where $\mathbb{C}^2$ is a complex symplectic manifold with symplectic form $dz_1 \wedge dz_2$. We want to consider the orbifold of $\mathbb{C}^2$ by the cyclic group of order 2, which acts by $(z_1, z_2) \mapsto (-z_1, -z_2)$. Algebraically, the quotient is $\mathrm{Spec}[X, Y, Z].\langle YZ - X^2\rangle$, where $X = z_1 z_2$, $Y = z_1^2$, $Z = z_2^2$. Any polynomial in z_1 and z_2 that remains invariant under the sign change can be expressed as a polynomial in these three variables. However, there's a relation between these functions: $X^2 = YZ$, which describes a cone in $\mathbb{C}^3$.

This cone structure is central to understanding the geometry. We can blow up this singularity to make it into the total space of an $O(-2)$ bundle over $\mathbb{P}^1$, where the singularity is replaced by a non-contractible 2-dimensional sphere. We can also do a complex deformation by adding a constant term: $\mathrm{Spec}[X, Y, Z].\langle YZ - X^2 - \zeta^2\rangle$.

This procedure replaces the space with a new one, making it a non-singular quadric $X^2 - YZ = \zeta^2 \neq 0$. Now, we can examine the two-form $\omega_{\mathbb{C}} = dz_1 \wedge$

$dz_2 = \frac{dY,dZ}{4X}$. At this point, we keep the form but deform the relation between X, Y, Z. It turns out this form has a non-trivial period, which we can see by parameterizing:

$$\begin{aligned} X &= \zeta x, \\ Y &= \zeta(y + \sqrt{-1}z), \\ Z &= -\zeta(y - \sqrt{-1}z). \end{aligned}$$

This turns the quadric into a complexification of the sphere $x^2 + y^2 + z^2 = 1$. In addition, we have

$$\omega_{\mathbb{C}} = \frac{\sqrt{-1}\zeta}{2} \frac{dy \wedge dz}{x}$$

Proposition 3.4.

$$\frac{dy \wedge dz}{x}|_{S^2} = vol_{S^2}$$

This is the standard volume form on the two-sphere, and it has a nontrivial period, which is π. Therefore, $\frac{\sqrt{-1}\zeta}{2} \frac{dy \wedge dz}{x}$ has a nontrivial period proportional to ζ.

3.3.3 Complexification In The Elliptic Case

Now, let's go back to the original example. We will do a similar analysis in the elliptic case. Suppose we have two variables p and x, where x is an elliptic variable, defined up to periods. We want to find the invariant functions on this space, allowing functions with poles in x, because the elliptic curve is compact. One such function, which is famously invariant under $x \to -x$, is the Weierstrass $\wp$-function:

$$\wp(x;\tau) = \frac{1}{x^2} + \sum_{(m,n)=(0,0)} \left(\frac{1}{(x + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right) + c$$

where we choose a constant so that the expansion of this function near 0 starts at the x^2 term.

In this notation, the Hamiltonian is $H = p^2 + \nu^2 \wp(x;\tau)$. This Hamiltonian has a symmetry that changes the sign of the variables, but this is not a symmetry of the symplectic form $\omega_{\mathbb{C}} = dp \wedge dx$. Let

$$X_1 = \wp(x;\tau) = \wp(-x\tau).$$

Since $\wp$ is an even function, its derivative is an odd function. If we multiply by p, we get another invariant function

$$Y = p\wp'(x;\tau)$$

and a third invariant function

$$X_2 = p^2.$$

Any invariant function of p and x, meromorphic with respect to x, can be expressed as a polynomial in these three functions. However, these functions are not independent. So, what is the relation between them?

Let's compute Y^2. We have:

$$Y^2 = X_2(4X_1^3 - g_2(\tau)X_1 - g_3(\tau)).$$

where g_2 and g_3 are constants.

Proposition 3.5. *This function satisfies the kDV equation.*

Proof. To verify this, we expand both sides of the equation in terms of X near zero. After subtracting the right-hand side from the left-hand side, the result should no longer have any singularities in X, implying that the difference is constant. By adjusting this constant, we can make the difference zero. □

Remark 3.6. *The constants g_2 and g_3 are expressed in terms of the fourth and sixth Eisenstein series.*

Thus, this equation replaces the quadric equation we had in the local picture earlier. Expanding this cubic polynomial in terms of its roots will lead to similar structures near each of the roots. There is also another important singularity to consider when $X_1 \to \infty$.

Moving on, the phase space now resembles a complex surface, with an affine part. There is a region of this space described by the equation we just discussed, but this equation does not cover the entire elliptic curve. The elliptic curve itself is compact, and our equation describes the complement of a point. This is an algebraic curve, so to make this a complete description, we need to either make the equation homogeneous or add a point to it. By adding some points, we can partially compactify the surface.

The final result of this analysis is that $P_{\mathbb{C}}$ is a two-dimensional complex manifold, and we want to represent it as a fibration over a one-dimensional complex base. The base is a complex manifold of dimension one, and the fibers above each point in the base are one-dimensional complex spaces. These fibers are elliptic curves $\mathcal{E}_u$, with their complex structures varying depending on the base point u.

Recall that the Hamiltonian is

$$H = X_2 + \nu^2 X_1.$$

We can rewrite this as

$$Y^2 = 4(H - \nu^2 X_1)(X_1 - e_1)(X_1 - e_2)(X_1 - \zeta)$$

where e_1, e_2, e_3 are the roots of the polynomial $(\wp')^2$. Similar to how g_2 and g_3 are Eisenstein series, e_1, e_2, e_3 are theta constants that sum to zero.

Now, rescale $H = \nu^2 u$ to get:

$$Y^2 = 4\nu^2(u - X)(X_1 - e_1)(X_1 - e_2)(X_1 - \zeta).$$

If we fix the value of the Hamiltonian, the fiber (the set of points in the phase space with the same energy) is a solution to this equation. The deformation consists of adding a point X_1 at infinity, and Y tends to infinity as X_1^2.

By considering the affine curve described by this equation, we get an elliptic curve missing two points. Adding those points back complexifies the fiber, and the complex structure of the elliptic curve depends on $[u : e_1 : e_2 : e_3]$. By compactifying the curve with a point at infinity, the base becomes $\mathbb{CP}^2$, and the fibers degenerate at points e_1, e_2, e_3.

We see that once we complexify the problem, the fibers degenerate in different ways depending on the base point. This regularity becomes apparent only after complexification, as real slices do not capture the irregular behavior of elliptic curves. Once complexified, everything becomes regular and beautiful.

3.4 Lax Representation

The goal of this section is to show that the structure we've discussed extends to all values of n, not just $n = 2$.

3.4.1 The Basics

To establish this, we need to explore the Lax representation for eCM-systems, as developed by Krichever in the 1980s. By formulating this representation, Krichever identified what later became known as the Hitchin systems. In the context of these lectures, where we are working within the framework of gauge theory, we will present this construction as an analog of the two-dimensional theory we examined earlier. However, we have since complexified the problem, so I will proceed with this complexified approach. In particular, this construction can be seen as a complexification of the 2D Yang-Mills framework.

Previously, we considered the infinite-dimensional symplectic manifold

$$P_\infty = \{(E, A) | E \in \text{Maps}(S^\perp, \mathfrak{g}), \{A \in \Omega^1(S^1) \otimes \mathfrak{g}\} \otimes \mathcal{O}//LG\}$$

where $\mathcal{O}$ is a finite-dimensional orbit. For our purposes, we take $\mathfrak{g} = \mathfrak{su}(N)$, $\mathcal{O} = \mathbb{CP}^{N-1}$, and $G = \text{SU}(N)$.

Let us now review the action of the loop group on this data. For $x \in S^2$, the action is given by

$$g(x) \cdot (E, A) = (g^{-1}Eg, g^{-1}Ag + g^{-1}dg)$$

and

$$g(x) \cdot (z_1 : ... : z_N) = ((g^{-1}(0)z)_1 : ... : (g^{-1}(0)z)_N)$$

where 0 is the base point of S^1. The moment map for this system is

$$\mu = D_X E + J\delta(x)$$

where

$$J : \mathcal{O} \to \mathfrak{g}^*$$
$$J(z_1 : ... : z_n) = \|\sqrt{-1}\nu(z_i\overline{z}_j - \delta_{ij})\|_{i,j=1}^N$$

assuming that $\sum_{i=1}^N |z_i|^2 = 1$. Note that J is traceless and anti-Hermitian, and it forms an $N \times N$ matrix.

Earlier, we showed that if we solve

$$H_2 = \frac{1}{2}\int_{S^1} dx \mathrm{Tr} E^2,$$

we obtain the trigonometric CM system. We now move to a complexified version of this by replacing S^1 with an elliptic curve.

3.4.2 Transition to the Complexified Version: Elliptic Curves and Gauge Fixing

Let

$$P_\infty^{\mathbb{C}} = \{(E, \overline{A})\} \times \mathcal{O}_{\mathbb{C}}$$

where E is a function on T^2 valued in $\mathfrak{sl}_N(\mathbb{Z})$, depending on z and $\overline{z}$, and $\overline{A}$ is a $(0,1)$-connection.

The gauge group action now depends on the choice of complex structure. Upon complexification, the degrees of freedom increase. $\mathcal{O}_{\mathbb{C}}$ is the complexification of $\mathbb{CP}^{N-1}$, and since z and $\overline{z}$ are now independent, we have

$$\mathcal{O}_{\mathbb{C}} = \{(z \in \mathbb{C}^N, w \in (\mathbb{C}^N))^* | w(z) = 1\}/\mathbb{C}^\times.$$

with the group action

$$(z, w) \mapsto (tz, t^{-1}w)$$

for $t \in \mathbb{C}^\times$. Furthermore, the moment map is given by

$$J(z, w) = |\sqrt{-1}\nu(z_i w_i - \delta_{ij})|_{i,j=1}^N$$

which is a traceless $N \times N$ matrix.

3.4.3 The Lax Matrix and Its Spectral Invariance

Next, we want to make a group of maps from T^2 (the 2-torus) to the complexification of my group, $\mathrm{SL}(n)$. The group $\mathrm{SL}(n)$ acts on this phase space as follows: mark a point and call it z, and the group G will act on the quadruple $(E, \bar{E}, Z, W)$ by pointwise conjugation:

$$g(z, \overline{z})(E, \overline{A}; (z, w)) = (g^{-1}Eg, g^{-1}\overline{A}g + g^{-1}\overline{\partial}g, (g^{-1}(0)z, wg(0)))$$

The action is by evaluation at the marked point, and it allows us to compute the momentum map, which will be used for the reduction process. The moment map is

$$\mu = \overline{\partial} E + [\overline{A}, E] + J\delta^2(z, \overline{z}).$$

We want to solve this equation and then divide by the group's action to perform the reduction. To solve this equation, the trick is to find the representative normal form for $\bar{A}$, meaning we need to to find the canonical form where we can make the connection on the trivial bundle look like a 0-1 connection by performing complex gauge transformations.

There exists a "complexified version" of gauge theory, where $\overline{A}$ can be written as $\text{diag}(\alpha_1, \ldots, \alpha_N)$, with $\alpha_i \in \mathbb{C}$ being constant with respect to z and $\overline{z}$. This gauge choice works for a generic $\overline{A}$, meaning that with probability 1, we can diagonalize the matrix. However, in some exceptional cases, something interesting happens: the diagonal elements may not correspond to connections on trivial U(1) bundles, but instead could have non-trivial first Chern classes. While this phenomenon is significant in condensed matter physics, we will not delve into it here.

Instead, we focus on the concept of diagonalization: a rank r vector bundle with trivial degree generically splits into a direct sum of line bundles, each with degree zero:

$$\mathcal{E} = \bigoplus_{i=1}^{N} L_i.$$

In practical terms, this means that, via a complex gauge transformation, we can diagonalize most of the matrix $\overline{A}$, turning it into a diagonal matrix with constant coefficients. However, as we saw earlier, there remain some gauge transformations of the form of diagonal matrices where the diagonal elements are periodic functions of z and $\bar{z}$:

$$g(z, \overline{z}) = \text{diag}\left(e^{2\pi i n_i} \frac{z - \overline{z}}{\tau - \overline{\tau}} - 2\pi i m_i \frac{z\overline{\tau} - \overline{z}\tau}{\tau - \overline{\tau}} \right).$$

These transformations affect the α values:

$$\alpha_i \mapsto \frac{2\pi i}{\tau - \overline{\tau}}(n_i + m_i \tau).$$

It's convenient to introduce new variables $x_i = \frac{\tau_2}{\pi}\alpha_i$, where $x_i \sim x_i + m_i + n_i\tau$.

Next, we need to solve the equation involving $\overline{A}$. We do this in analogy to the way we approached it for the circle. In the basis where $\overline{A}$ is diagonal, we have

$$\overline{\partial} E_{ij} + (\alpha_i - \alpha_j) E_{ij} + J_{ij}\delta^2(z, \overline{z}) = 0.$$

3.4.4 When $i = j$

When $i = j$, the equation simplifies to a condition on the diagonal matrix element of E: it must be holomorphic outside zero and possess only a first-order pole at the origin. Specifically, the equation $\overline{D} = \delta$ is solved by a meromorphic function with a first-order pole. This is a complex analog of the equation

$$\delta_x E_{ii} + J_{ii}\delta(x) = 0,$$

where a function on the circle with a jump discontinuity corresponds to a first-order pole in the complex case.

Drawing an analogy with the real case, this is similar to a function on a circle with a jump discontinuity, where the "jump" corresponds to the residue of the pole. To ensure smooth behavior, the residue must vanish, which forces E_{ii} to be constant ouside $x = 0$. Additionally, we have the condition:

$$E_{ii}(+0) - E_{ii}(-0) = J_{ii}.$$

This is analogous to the fact that a function on an elliptic curve can have at most a first-order pole at a single point. The residue of this pole must vanish, which implies that the function is constant everywhere. This serves as a complexification of the theorem, leading to the conclusion that $J_{ii} = 0$ and $E_{ii} = P_i$ is constant.

Then, we have

$$J_{ii} = \sqrt{-1}\nu(z_i w_i - 1) = 0$$

which implies $z_i w_i = 1$ for all i. This is analogous to the situation on the circle, where we have additional transformations. Specifically, when $g = \mathrm{diag}(t_1, \ldots, t_N)$, with $t_i \in \mathbb{C}^\times$ and $\prod_i t_i = 1$, these transformations allow us to map $z_i \mapsto 1$ and $w_i \mapsto 1$.

3.4.5 When $i \neq j$

When $i \neq j$, the situation becomes more interesting. Away from the origin, the solution is

$$E_{ij}(z\overline{z}) = e^{(z-\overline{z})(\alpha_i - \alpha_j)} L_{ij}(z),$$

for some function L that depends only on z and satisfies the following properties:

- L_{ij} has a first-order pole at zero, since $L_{ij} \sim \frac{J_{ij}}{z} = \frac{\sqrt{-1}\nu}{z}(1 - \delta_{ij})$.
- $L_{ij}(z + 1) = L_{ij}(z)$.
- $L_{ij}(z + \tau) = e^{(\overline{\tau}-\tau)(\alpha_i - \alpha_j)}$.

Proposition 3.7. *These three properties uniquely determine L as*

$$L_{ij}(z) = \sqrt{-1}\nu \frac{\theta(z + x_i - x_j)\theta'(0)}{\theta(z)\theta(x_i - x_j)},$$

which we call the ***Lax matrix****.*

Unlike the trigonometric case, where the electric field E had an exponential dependence but the prefactor did not depend on the position of the circle, here the function exhibits a nontrivial z-dependence. This leads to a remarkable fact:

The electric field depends on both z and $\overline{z}$, but the Casimir has no z-dependence: $\mathrm{Tr}E^2$ is a meromorphic function of z with a second-order pole at $z = 0$. The space of meromorphic functions with a second-order pole at a single point is two-dimensional. These functions can take the form

$$\nu^2 \wp(z;\tau) + H_2.$$

If we return to the phase space, which includes variables like E, $\overline{A}$, and others, and begin to study the classical evolution generated by the Hamiltonian—essentially an averaged version of the trace squared—we find that the second-order pole has important implications. The meromorphic nature of the Casimir allows us to define the principal value integral of the trace squared in a well-defined way, yielding a well-defined Hamiltonian:

$$H_2 = \int_{T^2} d^2z \, \mathrm{Tr}E^2.$$

This Hamiltonian results in a simple evolution in the high-dimensional phase space of variables. We can then project the evolution by the action of the gauge group, leading to a highly non-trivial evolution of the eCM particles.

At this point, we can identify the algebraic integrable structure of this phase space, which is now known to have the structure of a fibration over an N-dimensional base. The fibers of this fibration are abelian varieties, one of the possible complexifications of Liouville tori in real integrable systems.

The abelian varieties are defined as follows: we aim to extract as much as possible from the Lax operator. The spectral invariance of this operator provides conserved quantities, because in this infinite-dimensional space, any flow generated by a suitably regularized Casimir of the electric field will commute with the flow generated by E^2. This is because anything built from the electric field will Poisson commute with each other: these are commuting flows. Furthermore, by reducing by the gauge transformations—which do not affect the spectral invariance—we obtain conserved quantities and integrals of motion.

Since we have something that depends on a point on an elliptic curve, the spectral invariance varies from point to point. The object that is convenient to study in this context is the spectral curve, which resides on the elliptic curve:

$$\det(\lambda - L(z)) = 0.$$

Over a generic point on the elliptic curve, this equation has n solutions. However, when $z = 0$, L has a pole, and the residue at this pole is a matrix of a special form:

$$L(z) \sim \frac{\sqrt{-1}\nu(1 - z \otimes w)}{z}.$$

If we shift it, there's a way to resolve the structure near $z = 0$, but we will not go into the details.

This curve can be partially compactified, sitting in the partial complexification of E minus the point where $z = 0$. To resolve the structure near this point, we need to perform a blow-up. The key claim is that this results in a genus $(N - 1)$ curve that covers the elliptic curve E, and the Jacobian of this curve depends on a meromorphic polynomial $R(\lambda, z)$. As a polynomial, it has $n - 1$ non-trivial coefficients, each of which is a meromorphic function of z. There are only a limited number of functions that have a singularity at $z = 0$, so we can count the number of parameters, yielding exactly n parameters. Let's call the coefficients in this polynomial u. For any given value of u, we have a specific curve, and its Jacobian is the fiber of this fibration.

This is just one example of integrable systems that have a Lax representation with the spectral parameter. Here, we have seen a gauge theory description of such systems. There are other systems where, instead of the base curve being an elliptic curve with one point, it can be a genus-0 curve with many points. These systems are the complexification of systems you would get by marking several points on a circle.

Later on, we will begin describing a completely different story. At first, it may seem like a different topic, with origins in gauge theory. However, the end point will bring us right back to the same structure we've been discussing. We will recover these integrable systems, both classical and quantum, by applying probability theory to some finite (or infinite but filtered) sets. The statistics of the expectation values in this probability problem will be described by the system we've been working with.

3.5 Partitions

We now turn the page to introduce a new cast of characters, which will ultimately turn out to be the full cast of characters. As we proceed, we'll eventually reconnect the second half of the course with the first. To speed things up, we will make a jump and discuss supersymmetric gauge theory in 4, 5, and 6 dimensions. Without delving into the details, we can translate these theories into a class of statistical models. In essence, we will be studying complexified probability measures μ: measures on the set of multi-partitions. These measures will depend on several parameters, so it will take some time to introduce all the details.

3.5.1 Partitions via Young Diagrams

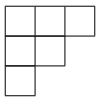
Let's begin by discussing partitions:

Definition 3.8. *A **partition***

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{\ell(\lambda)} > 0)$$

has size $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_{\ell(\lambda)}$ *and length* $\ell(\lambda)$.

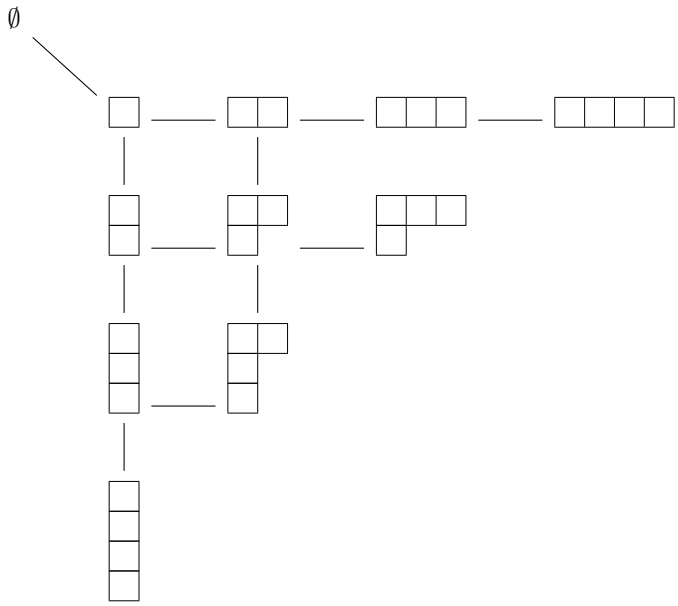
Geometrically, partitions can be visualized through a Young diagrams:



where the total number of boxes is $|\lambda|$, and the ith row contains λ_i boxes. We assign coordinates (i, j) to each box, where i counts vertically and j counts horizontally. Specifically, note that

$$(i, j) \in \lambda \iff 1 \leq j \leq \lambda_i \iff 1 \leq i \leq \lambda_j^T.$$

Definition 3.9. *The **Young graph** is an infinite graph whose vertices and partitions and two partitions are connected by an edge if there is a way to obtain one from another by adding or removing a box:*



For any tableau in the Young graph, let $\partial_+\lambda$ be the number of corners where we can add a box, and let $\partial_-\lambda$ be the number of corners where we can remove a box.

Proposition 3.10.

$$\#\partial_+\lambda - \#\partial_-\lambda = 1.$$

3.5.2 Partitions via Characters

Another way to describe the shape of a partition is through its character:

Definition 3.11. *The **character***

$$Ch\lambda = X_\lambda(q_1, q_2) = \sum_{(i,j)\in\lambda} q_1^{i-1} q_2^{j-1}$$

This character becomes more meaningful if we identify λ with a monomial ideal in $\mathbb{C}[z_1, z_2]$ of codimension $|\lambda|$.

In algebraic geometry, when you want to specify a position in space and time, you can't just say, "I'm here." You must also specify that if you move in one direction, you will be multiplied by a function that vanishes in that direction. If you make another move, you'll be multiplied by a function that vanishes in the next direction, and so on. This idea is central to the concept of a resolution, which we will now use to understand the boundaries of partitions.

Definition 3.12. *A **monomial ideal** is an ideal of the form*

$$Y = \langle z_1^{a_i} z_2^{b_i} \rangle_{i=1,\ldots,N}.$$

For example, in the infinite diagram

z_2^2	$z_1 z_2^2$	$z_1^2 z_2^2$
z_2	$z_1 z_2$	$z_1^2 z_2$
1	z_1	z_1^2

everything above and to the right of $z_1^a z_2^b$ forms a monomial ideal.

Proposition 3.13. *The elements of $\partial_+\lambda$ are the generators of the monomial ideal J_λ:*

$$J_\lambda = \langle z_1^{i-1} z_2^{\lambda_2} \rangle.$$

Note that given $(i, j) \in \lambda$,

$$\langle z_1^{i-1} z_2^{j-1} \rangle \in \mathbb{C}[z_1, z_2]/J_\lambda = K_\lambda,$$

where $\dim K_\lambda = |\lambda|$.

he number of elements in ∂_+ could be smaller than what's shown here because this is an exhaustive list, but it's not necessarily minimal. $\#\partial_+$ corresponds to the minimal set of generators.

What about ∂_-? The key point is that $\partial_+\lambda$ corresponds to the generators of J_λ, whereas $\partial_-\lambda$ corresponds to the relations in J_λ.

Proposition 3.14. *J_λ is invariant under $\mathbb{C}^\times \times \mathbb{C}^\times$ action via*

$$(z_1, z_2) \mapsto (t_1 z_1, t_2 z_2).$$

Thus, K_λ is a representation of the 2-dimensional torus. Moreover,

Proposition 3.15. *Chλ is a character of the $\mathbb{C}^\times \times \mathbb{C}^\times$ action on K_λ.*

Definition 3.16. *Let*

$$\begin{aligned} S_\lambda &= 1 - (1-q_1)(1-q_2)X_\lambda(q_1, q_2) \\ &= \sum_{\square \in \partial_+\lambda} e^{c(\square)} - q_1 q_2 \sum_{\blacksquare \in \partial_-\lambda} e^{c(\blacksquare)} \end{aligned}$$

where $e^{c(i,j)} - q_1^{i-1} q_2^{j-1}$.

Proposition 3.17. *Given the character of the partition, performing this algebraic manipulation extracts two characters, precisely those of $\partial_+\lambda$ and $\partial_-\lambda$.*

This essentially repeats what we discussed regarding the generators and relations: it's the calculation of the Koszul resolution, but at the level of characters. Note that

$$\begin{aligned} \frac{S_\lambda}{(1-q_1)(1-q_2)} &= \sum q_1^{i-1} q_2^{j-1} \\ &= \frac{1}{(1-q_1)(1-q_2)} X_\lambda(q_1, q_2) \\ &= \frac{\sum_{\square \in \partial_+\lambda} e^{c(\square)}}{(1-q_1)(1-q_2)} - \frac{q_1 q_2 \sum_{\blacksquare \in \partial_-\lambda} e^{c(\blacksquare)}}{(1-q_1)(1-q_2)} \end{aligned}$$

where in the first line we are summing over all (i, j) such that $z_1^{i-1} z_2^{j-1} \in J_\lambda$.

Let's proceed with a bit of homological algebra.

Proposition 3.18. *There is a tautological ($\mathbb{C}^\times \times \mathbb{C}^\times$-equivariant) complex associated with the partition λ:*

$$K_\lambda \xrightarrow{d_1} K_\lambda \otimes Q \oplus \mathbb{C} \xrightarrow{d_2} K_\lambda \otimes \Lambda^2 Q$$

where $Q = \mathbb{C}^2$, with $d_1(k) = ((z_1 k), (z_2 k); 0)$ and $d_2(k_1, k_2, u) = u + z_2 k_1 - z_1 k_2$. This complex satisfies the condition:

$$d_2 \circ d_1 = 0.$$

What is the cohomology here?

Proposition 3.19. *The cohomology of this complex is given by:*

$$\chi_{H^1}(q_1, q_2) = \sum_{\square \in \partial_+ \lambda} e^{c(\square)},$$

and

$$\chi_{H^0}(q_1, q_2) = q_1 q_2 \sum_{\blacksquare \in \partial_- \lambda} e^{c(\blacksquare)}.$$

Here, $H^0 = \text{Ker } d_1$, which corresponds to the elements of the quotient space that vanish under multiplication by z_1 and z_2. Specifically, multiplication by z_1 shifts us down, while multiplication by z_2 shifts us to the right. Therefore, Ker d_1 is spanned by the vectors corresponding to the $\blacksquare$ elements.

3.5.3 Multi-Partitions

Now that we are familiar with partitions, let us introduce multi-partitions. Since we want to study systems in multiple planes of reality, investigating multi-partitions is a natural next step.

Definition 3.20. *A **mutli-partition** is a collection $(\lambda^{(\alpha)})$ of partitions where A is a finite set.*

Next, we discuss the partition function, which will involve a sum over a multi-set of partitions. The corresponding measure depends on the shape of the partitions and certain parameters, and is given by:

$$Z = \sum_{(\lambda^{(\alpha)})} \prod_{\alpha \in A} q_\alpha^{|\lambda^{(\alpha)}|} \mu_{(\lambda^{(\alpha)})}(a_\alpha, \epsilon_1, \epsilon_2, \dots).$$

There are 3 theories to consider here for μ:

- Topological theory H: Uses rational functions and corresponds to 4-dimensional space.
- K-theory K: Uses trigonometric functions and corresponds to 5-dimensional space compactified over S^1.
- Elliptic theory Ell: Uses elliptic functions and corresponds to 6-dimensional space compactified over an elliptic curve E.

The origin of these measures comes from localization in elliptic cohomology of moduli spaces of solutions to partial differential equations on $\mathbb{R}^4$. By adding two dimensions corresponding to the elliptic curve, we form a 6-dimensional theory. In the K-theory case, there's a hidden S^1 that leads to a 5-dimensional theory.

Several tools are available for this purpose, but we will focus on a few. For example, there are measures associated with Hodge surfaces, which admit a cohomological treatment but cannot be expressed in K-theory.

3.5.4 Example: $\hat{A}_0$

Consider $A = \{1, \ldots, N\}$, where the elements are called **colors**. The parameters are $(a_1, \ldots, a_N) \in \mathbb{C}^N$, $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \mathbb{C}^4$ with $\sum_{a=1}^4 \epsilon_a = 0$, and $q = \exp(2\pi\sqrt{-1}\tau)$, where $\mathrm{Im}\ \tau > 0$. Given the random variables $(\lambda^{(1)}, \ldots, \lambda^{(N)})$, the measure is expressed as:

$$\mu = q^{\sum_{\alpha=1}^N |\lambda^{(\alpha)}|} \prod_{\alpha,\beta=1}^N \prod_{(i,j)\in\lambda^{(\beta)}} \frac{\theta(\epsilon_3 + \epsilon_1(i - \lambda_j^{(\beta)}) + \epsilon_2(1 + \lambda_i^{(\alpha)} - j) + a_\alpha - a_\beta)}{\theta(\epsilon_1(i - \lambda_j^{(\beta)}) + \epsilon_2(1 + \lambda_i^{(\alpha)} - j) + a_\alpha - a_\beta)}$$

$$\times \prod_{(i,j)\in\lambda^{(\alpha)}} \frac{\theta(\epsilon_3 + \epsilon_1(\lambda_j^{(\alpha)t} + 1 - i) + \epsilon_2(j - \lambda_i^{(\beta)}) + a_\alpha - a_\beta)}{\theta(\epsilon_1(\lambda_j^{(\alpha)t} + 1 - i) + \epsilon_2(j - \lambda_i^{(\beta)}) + a_\alpha - a_\beta)}.$$

Here, the θ function is defined as:

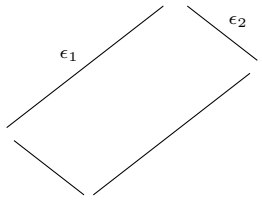
$$\theta(x) = \begin{cases} x & \text{(for linear functions)} \\ 1 - e^{-x} & \text{(for exponential functions)} \\ \prod_{n=1}^\infty (1 - p^{n-1}e^{-x})(1 - p^n e^x) & \text{(for elliptic functions)} \end{cases}$$

depending on the theory in use, where $p = \exp(2\pi\sqrt{-1}\sigma)$ and $\mathrm{Im}\ \sigma > 0$.

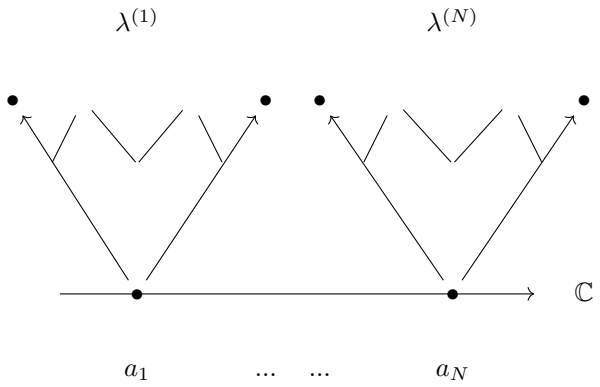
In the above product, if $\alpha = \beta$, $\mathrm{arm}(i, j) = \lambda_i^{(\alpha)}$ and $\mathrm{leg}(i, j) = \lambda_j^{(\alpha)t} - i$.

3.5.5 Visualizing with Young Diagrams

We can visualize this in terms of Young diagrams. When we rotate a Young diagram by 135°, it resembles a pile of bricks,correspond to complex points $A_1, \ldots, A_N$,



thrown in a gardening cart:



3.5.6 Growth Process on the Complex Plane

We have a growth process on the complex plane.

The growth starts with seeds at the points $A_1, A_2, \ldots, A_n$, and the growth steps are governed by ϵ_1 and ϵ_2. The Young diagram is simply the boundary of this growth process. The boundary of this process interacts with itself, and there is a "potential" between the corners of the diagram: the $\square$ and $\blacksquare$ we discussed earlier, akin to dipole-dipole interactions. This is a discrete process, meaning that the "charges" (the boxes) can only occupy specific positions. However, if we decipher this setup, we can think of it as a Coulomb gas (a system of dipoles interacting in a discrete way).

The simplest case is $N = 1$. The formula simplifies to

$$\sum_{\lambda} q^{|\lambda|} \frac{\theta(\epsilon_3 + \epsilon_1(-\text{leg}_\square) + \epsilon_2(\text{arm}_\square + 1))\theta(\epsilon_3 + \epsilon_1(\text{leg}_\square + 1) - \epsilon_2\text{arm}_\square)}{\theta(\epsilon_1(-\text{leg}_\square) + \epsilon_2(\text{arm}_\square + 1))\theta(\epsilon_1(\text{leg}_\square + 1) - \epsilon_2\text{arm}_\square)}$$

where $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$. Many physicists call ϵ_3 the "joint mass" and $\epsilon_4 = -\epsilon_3$. Although not immediately obvious, there is a symmetry that allows us to exchange $\epsilon_1 \leftrightarrow \epsilon_2$ and $\epsilon_3 \leftrightarrow \epsilon_4$.

Remarkably, this formula can be summed up. We won't bother with the elliptic case, but in the rational case when $\theta(x) = x$, this sums to

$$\left(\prod_{n=1}^{\infty}(1-q^n)\right)^{-\frac{(\epsilon_3+\epsilon_1)(\epsilon_3+\epsilon_2)}{\epsilon_1\epsilon_2}}.$$

There is a special case related to the representation theory of the symmetric group: $\epsilon_1 = -\epsilon_2$, $q \to 0$, $\epsilon_3 \to \infty$, and $\epsilon_3^2 q = \Lambda^2$ is fixed. Then the measure simplifies and becomes the Plancherl measure, giving us the famous hook formula:

Proposition 3.21.

$$\mu = \frac{(\dim \lambda)^2}{|\lambda|!} = \prod_{\square} \frac{1}{h_{\square}^2}$$

For the experts: we don't think we live in $\mathbb{R}^4$, rather $\mathbb{C}^2$. But, it's not just any $\mathbb{C}^2$; it's a $\mathbb{C}^2$ embedded inside $\mathbb{C}^4$. Since there is are four parameters $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$, there's a four-dimensional space hiding within this structure. Our $\mathbb{C}^2$ is just a two-dimensional slice inside this four-dimensional space, and a three-dimensional torus $(\mathbb{C}^\times)^3 \subset \mathrm{SL}(4)$ acts on it. So our space we're dealing with is abstracted in a certain way: it's $\mathbb{C}^2$ with a two-dimensional bundle on it, and the ϵ_3 and ϵ_4 parameters are the weights of this bundle.

Back to the $N = 1$ case, the formula becomes

$$\sum_{n=1}^{n} q^n \int_{\mathrm{Hilb}^{[n]}(\mathbb{C}^2)}^{\mathrm{Ell}} \sum_i (-1)^i \Lambda T^* \mathrm{Hilb}^{[n]}(\mathbb{C}^2).$$

Recalling information from Okounkov's lecture, we can recognize that this corresponds to considering the cotangent bundle and then looking at the zero section to find the zeros of a generic section. This relates to characteristic classes, but we want to make things a bit more interesting by introducing weight factors. Instead of just multiplying by $(-1)^n$, we use the following weight instead

$$\sum_{n=1}^{n} q^n \int_{\mathrm{Hilb}^{[n]}(\mathbb{C}^2)}^{\mathrm{Ell}} \sum_i (-q_3)^i \Lambda T^* \mathrm{Hilb}^{[n]}(\mathbb{C}^2)$$

where $q_3 = e^{\epsilon_3}$. We're allowed to break the symmetry between ϵ_3 and ϵ_4 because of the symplectic nature of the manifold we're dealing with.

For general n, it is not simple. For $n = 2$, Z describes conformal blocks of a 2-dimensional conformal field theory on an elliptic curve with a parameter q and one puncture.

Let's summarize the big idea. As explained previously, everything presented will be connected with everything else. Pairs of Young diagrams are related to the representation theory of $\hat{GL}_2$, and inside $\hat{GL}_2$, there's $\hat{SL}_2$, which is connected to the symmetry of the Liouville theory. So, this is the rough explanation. Next, we will introduce a new tool for studying these partition functions and expectation values, In the next section, we will introduce a new tool for studying these partition functions and expectation values. This new tool is called a QQ character.

As an even broader big idea, we're studying expectation values of some observables, where we defined a measure that can be noramlized on the set of multi-partitions

$$\frac{1}{Z} \mu^{\lambda^{(\alpha)}}$$

We want to study expectation values of these observables evaluated on Λ:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{(\lambda^{(\alpha)})} \mu_{(\lambda^{(\alpha)})} \mathcal{O}[\lambda^{(\alpha)}].$$

There exists observables $\mathcal{O}$ such that $\langle \mathcal{O} \rangle$ solve interesting equations:

- Knizhik-Samolodchikov $\hat{\mathfrak{sl}}_N$
- Belvain-Polyakov-Zamoldodchikov (Vir, W_3, etc)
- Wave functions of $A, D, E, \hat{A}, \hat{D}, \hat{E}$ spin chains

When we take $\epsilon_1, \epsilon_2 \to 0$,

$$Z \sim e^{\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(a, \epsilon_3, q)}$$

where $\mathcal{F}$ is the prepotential of some special Kähler geometry.

Often there is a family of curves with Krichever data that captures these asymptotics. The asymptotics at the level of these Young diagrams corresponds to the limit shape problem.

There are also other interesting limits. For example, when $\epsilon_2 \to 0$ with ϵ_1 fixed,

$$Z \sim e^{\frac{1}{\epsilon_2} \tilde{W}(a, \epsilon_1, \epsilon_3, q)}$$

where W describes the deformation of special geometry: sometimes it is quantization and sometimes it's a classical geometry but in a rotated complex structure.

3.6 From Generating Functions to qq-Characters

3.6.1 The Generating Function of Krichever Genera

Okounkov just talked about the generating function of Krichever genera

$$Z^{\text{inst}} = \sum_{k=0}^{\infty} q^k \text{Kr}(\mathcal{M}_k^{\text{framed}}(N))$$

or of the moduli spaces of rank n, charge $k = \text{ch}_2$ instantons, which are mathematically torsion-free sheaves on $\mathbb{P}^2$, trivial over $\mathbb{P}^1_\infty$.

Consider $(q_1, q_2) \subset \text{GL}(2)$ and $z = q_3 = e^{\epsilon_3}$. Yesterday, both q_1 and q_2 were crucial because they rotated the space on which the sheaves live, while q_3 is a more peculiar parameter used to weight the exterior powers of the tangent bundle. Today, we will remedy this and restore the "democracy" between q_1, q_2, q_3, q_4 by crossed isntantons.

But let's digress and continue the logic of the presentation. The logic so far was that localization provides a way to express the genera as a sum over fixed

points. Without much explanation, we saw fixed points presented as collections of partitions. So, this whole problem reduced to the partition function of some statistical model. However, we are not primarily interested in the partition function itself; we want to study correlation functions. So what are the good observables? As learned from the lecture by Smirnov, there are two types of interesting observables: order operators and disorder operators. We will study both for this model. These operators can also be called observables, as understood in complexified probability theory.

3.6.2 The Y-Observable

What's the difference between order and disorder operators? An order operator is something that can be easily defined. An original problem might involve a (complexified) measure on some measurable set and an operator is simply a function on that measurable set. It's easy to define, and its expectation value can be computed as an integral against the measure. It turns out that there are smarter ways of organizing these functions, and we want to have a way of assigning an interesting quantity to the collection of Young diagrams, using essentially the characters we introduced last time. So, let me begin by introducing the Y-observable:

Definition 3.22. *A **Y-observable** is*

$$Y(x)[\lambda^{(1)},\dots,\lambda^{(n)}] = \prod_{\alpha=1}^{N} \frac{\prod_{\square\in\partial_+\lambda^{(\alpha)}}(x-a_\alpha-c_{12}(\square))}{\prod_{\blacksquare\in\partial_-\lambda^{(\alpha)}}(x-a_\alpha-c_{12}(\blacksquare)-\epsilon_1-\epsilon_2)}$$

where $c((i,j)) = \epsilon_1(i-1)+\epsilon_2(j-1)$, valued in

- *rational functions of X for H*
- *degree N line bundles on E for Ell*

Proposition 3.23. *In the rational case,*

$$\lim_{x\to\infty} Y(x) = x^N \exp\sum_{i=1}^{\infty}\frac{(-1)}{i}x^{-i}ch_i(\epsilon),$$

where ϵ is the universal sheaf lcoalized to the point $O\in\mathbb{C}^2=\mathbb{P}^2\backslash\mathbb{P}^1_\infty$.

Unfortunately, it is unpleasant that Y is a meromorphic function, so it has poles. Y should be "thought of" as characteristic function $\det(x-\Phi)$. If E were a vector bundle, not a sheaf, the expression would sum up to a polynomial in x. Even in cohomology, we would get a holomorphic section of some line bundle over E, not a meromorphic one. This means that the presence of poles reflects the fact that it's not a good idea to restrict to sheaves at a point. However, there is a way to fix this problem.

Let $e^{\epsilon_1}=q_1, e^{\epsilon_2}=q_2$. Consider

$$\langle Y(x+\epsilon_1+\epsilon_2)\rangle := \frac{1}{Z}\sum_{\lambda^{(1)}\dots\lambda^{(N)}}\mu_{\lambda^{(1)}\dots\lambda^{(N)}}(\vec{a},\vec{\epsilon},q)\times Y(x+\epsilon_1+\epsilon_2)[\lambda^{(1)}\dots\lambda^{(N)}]$$

Proposition 3.24. *There are poles at* $x = a_\alpha + \epsilon_1(i-1) + \epsilon_2(j-1)$ *for some* $(i,j) \in \mathbb{N} \times \mathbb{N}$.

These poles come from the sum of all configurations of Young diagrams where $(i,j) \in \partial_- \lambda^{(\alpha)}$. If we remove (i,j) from $\lambda^{(\alpha)}$, we get $\tilde{\lambda}^{(\alpha)}$. That is, we have replaced $(\lambda^{(1)}...\lambda^{(\alpha)}...\lambda^N)$ with $(\lambda^{(1)}...\tilde{\lambda}^{\alpha}...\lambda^{(N)})$.

3.6.3 The qq-Character

Proposition 3.25.

$$Y(a_\alpha + \epsilon_1(i-1) + \epsilon_2(j-1))[\lambda^{(1)}...\tilde{\lambda}^{(\alpha)}...\lambda^{(N)}] = 0.$$

This means that if we add something proportional to $\frac{c}{Y(x)}$ to this expression for some value c, the poles might be removed. Indeed:

Proposition 3.26. *There are no poles for*

$$\mathfrak{X}(x) = \left\langle Y(x+\epsilon_1+\epsilon_2) + q\frac{(\epsilon_1+\epsilon_3)(\epsilon_1+\epsilon_4)}{\epsilon_3\epsilon_4}\frac{Y(x-\epsilon_3)Y(x-\epsilon_4)}{Y(x)} + ...q^2\,(...) + ... \right\rangle$$

which we call the ***qq-character****.*

Note that by adding each q^i term for some i, we have removed the poles of the previous expression. Unfortunately, the price we have paid is that new poles have appeared. This begins a process where we keep adding terms, each multiplied by smaller and smaller coefficients, and then we cross our fingers, hoping that we end up with the full formula for the entire series. It is fortunate that our series happens to be convergent, which is not guaranteed.

Proposition 3.27.

$$\mathfrak{X}(x) = \sum_{\square\in\lambda} q^{|\lambda|}\frac{\theta(\epsilon_3(arm_\square+1)-\epsilon_4 leg_\square+\epsilon_1)\theta(-\epsilon_3 arm_\square+\epsilon_4(leg_\square-1)+\epsilon_1)}{\theta(\epsilon_3(arm_\square+1)-\epsilon_4 leg_\square)\theta(-\epsilon_3 arm_\square+\epsilon_4(leg_\square+1))}$$
$$\times\frac{\prod_{\square\in\partial_+\lambda}Y(x+\epsilon_1+\epsilon_2+c_{34}(\square))}{\prod_{\blacksquare\in\partial_-\lambda}Y(x+c_{34}(\blacksquare))}$$

where $c_{34}(\square=(i,j)) = \epsilon_3(i-1)+\epsilon_4(j-1)$.

This is some expression built out of the Y-functions, constructed according to the shape of the Young diagram we are summing over. We use the contents of the boxes of that Young diagram to shift the arguments of the Y-observable, which was the observable on the ensemble of n Young diagrams. Importantly, there is a prefactor - this is precisely the prefactor of $\mathrm{Kr}(\mathrm{Hilb}^{[k]}(\mathbb{C}^2), z = e^{\epsilon_1} = q_1)$, which is $\mathbb{C}^\times \times \mathbb{C}^\times$ equivariant with weights (q_3, q_4).

3.6.4 Localization Formulas

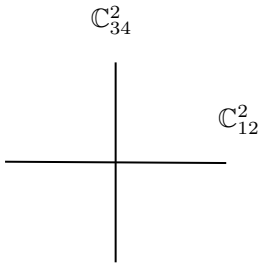
Notice that the roles of the variables have shifted: ϵ_3, which was just sitting in the numerator, not doing much, has now become quite important as it is all over the denominator of this localization formula. But why bother with all of this trouble?

Theorem 3.28.

$$\langle \mathfrak{X}(x) \rangle - \prod_{\alpha=1}^{N} \theta(x - A_\alpha)$$

where $A_\alpha(q) = a_\alpha + q(...)$ *is holomorphic in* x.

Now, we're going to generalize this problem and make it look like a problem that describes sheaves living on a cross



Horizontally, we can study rank N sheaves, whereas vertically, we study rank 1 sheaves. In between, there are some cross factors. This is a picture from string theory, where open strings are connected to stacks of branes.

Let's introduce the disorder operator, which are also known as surface defects, and are obtained by a procedure called orbifolding. Previously, we were studying sheaves on $\mathbb{P}^2$ which were trivial on the line $\mathbb{P}^1_\infty$. This was related to doing $\mathrm{U}(N)$ gauge theory on $\mathbb{R}^4$.

In gauge theory with some path integral $\int DAe^{-\frac{1}{q^2}\int \mathrm{Tr} F_A \wedge * F_A + \cdots}$, we can study observables as well. Typical observables people study in gauge theory are the holonomy of Wilson lines $\langle \mathrm{Tr} P \exp \oint_C A \rangle$. Physicists studying quantum chromodynamics use supercomputers to study them. Unfortunately, in our world, when everything is formulated in terms of holomorphic bundles or sheaves, we don't have access to those observables. However, invariants using Chern-Simons theory tell us that in three dimensions, such observables can also be understood as a disorder operator. This is because in Chern-Simons theory, given a link, an observable is associated with some contour in the gauge field and has a singularity such that the curvature of the gauge field has a delta function source: $F_A \sim J_R \delta_C$.

We can generalize this aspect of knots/links to four dimensions and define observables that will be codimension two defects. There are two main ways to do this:

- Algebraic geometry: Given a complex surface S and a complex curve C, we are interested in studying sheaves $\mathfrak{E}$ over S such that

$$\mathfrak{E}|_C \supset \mathfrak{E}_1 \supset \mathfrak{E}_2 \supset ... \supset \mathfrak{E}_n.$$

- Homological algebra: If C is defined by the equation $z_2 = 0$, wen can study the flag

$$\mathfrak{E} \supset \mathfrak{E}_1 \supset ... \supset \mathfrak{E}_n \supset z_2\mathfrak{E}.$$

This is the moral of the story we're going to follow. This is analogous to the well-studied problem in the world of holomorphic bundles on curves, where we fix a parabolic structure at points. We can think of curves as divisors, and we're fixing a parabolic structure on the divisors.

3.6.5 Orbifold Structures and Parabolic Sheaves

It turns out that there is a very convenient construction that realizes this parabolic structure via an orbifold story. Imagine that we're doing gauge theory not on the (z_1, z_2)-plane, but on a $(z_1, \tilde{z}_2)$ plane, where $\tilde{z}_2^N = z_2$. We will assume that n is the rank of the gauge group, which is not always the case.

Imagine that we are doing something simple on the covering space $(z_1, \tilde{z}_2)$, and then project it to (z_1, z_2), which is a complex manifold $\mathbb{C} \times \mathbb{C}/(\mathbb{Z}/N)$. But there's a defect at the origin...

Let's define a more refined measure. In addition to the characters with respect to the torus $\mathbb{C}^\times_{a_1} \times \mathbb{C}^\times_{a_2} \times \mathbb{C}^\times_{a_3}$, we will also keep track of the representations of the cyclic group $\Gamma = \mathbb{Z}/n$, which acts on $\mathbb{C}^4$ by rotating the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ coordinates in opposite directions, and is part of the Calabi-Yau 4-structure.

Definition 3.29. *The* Γ***-action*** *on* $\mathbb{C}^4$ *is given by*

$$(z_1, \tilde{z}_2, z_3, \tilde{z}_4) \mapsto (z_1, \Gamma\tilde{z}_2, z_3, \gamma^{-1}\tilde{z}_4)$$

where $\Omega^N - 1$.

Additionally, this group Γ acts in a framing space: $\mathbb{C}^N$, which is the fiber of the torsion free sheaf at ∞, becomes a regular representation of Γ. This is because we are taking the maximal flag of subsheaves.

Now we assign brackets

$$[a_\alpha] = c(\alpha) \in \mathbb{Z}/N = \mathrm{Rep}(\Gamma)[\epsilon_1] \qquad = 0[\epsilon_2] = 1[\epsilon_3] \qquad = 0[\epsilon_4] = -1$$

The map $c : \{1, \ldots, N\} \to \{0, \ldots, N-1\}$ should be a 1-to-1 correspondence. Let $R_\gamma \in \mathrm{Rep}(\Gamma)$ be given by $\Omega \mapsto e^{\frac{2\pi\sqrt{-1}\omega}{N}}$.

Definition 3.30. *The new measure is*

$$\mu^{new}_{\lambda^{(1)}\dots\lambda^{(N)}} = \prod_{\omega=0}^{N-1} q_\omega^{k_\omega} \prod \theta\left(\sum_\alpha a_\alpha + k_1\epsilon_1 + k_2\epsilon_2 + k_3\epsilon_3\right)$$

where $\sum n_\alpha c(\alpha) + k_2 = 0 \pmod N$ *and* $k_\omega = \sum_{\alpha=1}^N \#\{(i,j)|(i,j) \in \lambda^{(\alpha)}, c(\alpha) + j - 1 \equiv \omega \pmod N\}$.

For combinatorics, we have the following map:

Definition 3.31. *There's a projection map* $\pi : (\lambda^{(\alpha)})_{\alpha=1}^N \to (\Lambda^{(\alpha)})_{\alpha=1}^N$ *such that*

$$\Lambda_j^{\alpha(t)} = \lambda^{(\alpha)t}_{\alpha+N(j-1)}$$

This map corresponds to the projection from the moduli space of parabolic sheaves to the moduli space of sheaves, forgetting about the parabolic structure.

Proposition 3.32.

$$\sum_{\vec{\lambda}\in\pi^{-1}(\Lambda^{(1)},\dots,\Lambda^{(N)})} \mu^{new}_{\lambda^{(1)},\dots,\lambda^{(N)}} = q^{|\vec{\Lambda}|}\mu^{old}_{\vec{\Lambda}} \times \textit{observables}$$

where the observables are $\sum \prod_\omega q_\omega^{k_\omega} \int 1$ *and the integral is over all parabolic structures on a given* $\epsilon_{\vec{\Lambda}}$, *and* $q = \prod_{\omega=0}^{N-1} q_w$.

If we let $q_\omega = \frac{z_\omega}{z_{\omega-1}}$, we have $z_{\omega+N} = qz_\omega$, which is the start of an elliptic curve. The elliptic curve begins with the collection of particles, which forms an infinite set. When we shift the label of a particle by n, we obtain another copy of the same particle on the elliptic curve (quotiented by q). The resulting structure can Ω times the first Chern class of the associated quotient. $\Omega\times$ the first Chern class of the corresponding quivers. Despite its initial complexity, this formula is explicit and combinatorial in nature.

How big are the fibers in the map? The fibers are ∞-to-1, meaning the pre-image of any collection of partitions is an infinite set. However, this set has an interesting structure. This structure is related to maps, specifically quasimaps to flag varieties. Let me explain a few basic points to help clarify things.

If $\alpha = 1$, then $\Lambda_1^{(1)t} = \lambda_1^{(1)t}$. If $\alpha = 2$, then $\Lambda_1^{(2)t} = \lambda_2^{(2)t} \leq \lambda_1^{2(t)}$, so there is 1 integer freedom. If $\alpha = 3$, then $\Lambda_1^{(3)t} = \lambda_3^{(3)t} \leq \lambda_2^{3(t)} \leq \lambda_1^{(3)t}$, so there are 2 integers of freedom. More generally, for $\alpha = N$, there are $N-1$ degrees of freedom. This gives rise to a Gelfand-Zetlin table of numbers, which are degrees of quasi maps $\mathrm{QMaps}(\mathbb{C}_z, \mathbb{P}^1, T^*\mathrm{Flag})$. To understand why we have this triangular table of numbers, one way is to use the quiver variety representation of the flag variety.

This takes care of most of the fiber, but not all. There are also finite pieces, having to do with the possibility of inserting columns between N successive steps.

3.7 Connecting The Two Stories

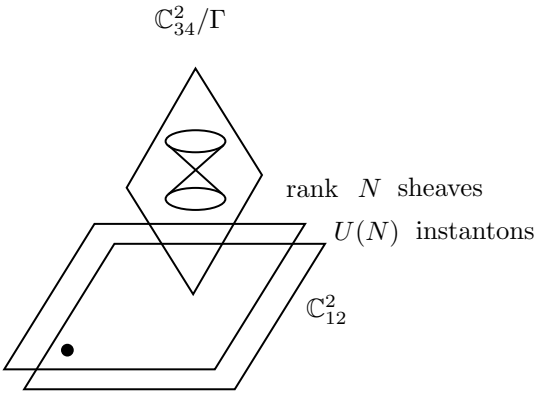
Our goal is to connect two stories: the story of the integration of instantons/moduli spaces of sheaves and the story of integral systems (classical and quantum). These are rich and vast topics, so we'll only touch on a few aspects. We will focus on understanding the origin of the Lax operators of integrable systems and how they come from the geometry of moduli spaces of instantons and their generalizations.

Before diving into the main discussion, we mention one remark about the terminology we used for the observables introduced in our previous sessions: why are they called qq-characters and how are they connected to characters? To clarify this, we will generalize both the problem and the observables in such a way that the classical Lie algebras and root systems associated with simple Lie algebras naturally emerge. In this context, we will see that these qq-characters are indeed related to the characters of representations of these Lie algebras. We will sketch the construction of this relationship, with the goal of providing a clearer understanding of the connection.

3.7.1 Linking Instantons and Integrable Systems

Previously, we discussed a problem that became a statistical mechanics problem through the process of localization. Initially, this problem involved calculating integrals over the moduli space of instantons, which were defined on a two-dimensional complex plane $\mathbb{C}^2_{12} \hookrightarrow \mathbb{C}^4_{1234}$, which is a Calabi-Yau 4-fold. The 4-fold intersects our plane at a single point, which corresponds to the $\mathbb{C}^2_{34}$ plane. Then, we introduced a defect by performing an orbifold action inside the two-dimensional plane such that the orbifold action, when observed from a certain point of view, looks like a complex line.

Let's change the orbifold action:



We want to modify the geometry so that the observer (denoted by the dot) observes nothing: the world is flat and in 3-dimensions, but the transverse space is now modified by quotienting out a finite subgroup $\Gamma \subset \mathrm{SU}(2)$. There

is an ADE choice for this subgroup, but we won't dwell on the details.

3.7.2 The Geometry of Instantons and Moduli Spaces

Example 3.33. $\Gamma = \mathbb{Z}_{r+2}$ *acts by*

$$(z_3, z_4) \mapsto (\omega z_3, \omega^{-1} z_4)$$

where $\omega^{v+2} = 1$. *The framing space is important because the rank-N sheaves describe surfaces with multiplicities, and these multiplicities are spread out in the transverse direction. So we can think of N-points moving on the transverse plane, and the group acts on the collection of these points. One simple way to let the group act is to say that the N points form orbits of the group: the framing* $\mathbb{N} =$ *functions on N copies of* Γ *orbits.*

Algebraically, let the manifold $\mathbb{C}^2_{34}$ *be described by the equation* $AB = C^{r+2}$, *where* $A = z_3^{r+2}, B = z_4^{r+2}$, *and* $C = z_3 z_4$. *On the quotient space, we have N points which leads to an orbit. Then on the covering space, we have* $N(r+2)$ *sheaves with* Γ*-actions.*

Then

$$Ch\mathbb{N} = \sum_{\alpha=1}^{N} \sum_{s=0}^{r+1} e^{a_{\alpha,s}} \hat{\mathfrak{R}}_S$$

with $Rep(\Gamma) = \{\hat{\mathfrak{R}}_0, \hat{\mathfrak{R}_{r+1}}\}$, *where* $\hat{\mathfrak{R}}_s$ *is the irreducible representation corresponding to* $\omega \mapsto \omega^s$. *For each representation, we have an N-dimensional vector space, which is the fiber of the trivialization of my shift at infinity in that space.*

The framing group is $(GL(N))^{\times(r+2)} = GL(N(r+2))^{\mathbb{Z}_{r+2}}$. *Now, each irreducible representation of* Γ *has its own frame. Therefore, if we formulate the problem of instanton counting as an integral over R-products, we can proceed with this framework.*

The moduli space of quiver instantons is $\mathcal{M}$, *which satisfies* $\sqcup_{\vec{k}} \mathcal{M}_{\vec{k}}$*(derived moduli space of framed rank* $= N(r+2)$ *sheaves on* $\mathbb{P}^2$, *with trivializations over* $\mathbb{P}^1_\infty$ $\bigoplus_{s=0}^{r+1} N_s \otimes \hat{\mathfrak{R}}_s)^{\mathbb{Z}_{r+2}} = \{(\mathcal{E}_0, \ldots, \mathcal{E}_{r+1})\}$ *such that* $\mathcal{E}_s$ *is torsion free on* $\mathbb{P}^2$,$\mathcal{E}_s|_{\mathbb{P}^1_\infty} \cong N_s \otimes G$, *rank* $\mathcal{E}_s = N$, *and* $ch_2(\mathfrak{E}_s) = k_s$.

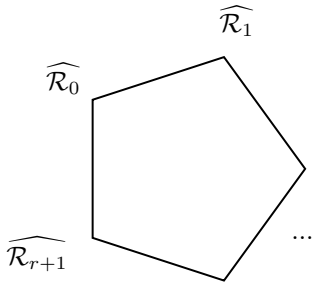
Now, we replace $q \mapsto (q_0, \ldots, q_{r+1})$. *Then*

$$Z^{inst}(a_{\alpha,s}; \vec{q}; \epsilon_1, \epsilon_2, \epsilon_3) = \sum_{\vec{k}} \prod_{s=0}^{r+1} q_s^{k_s} \sum_{\lambda^{(\alpha,s)}} \prod$$

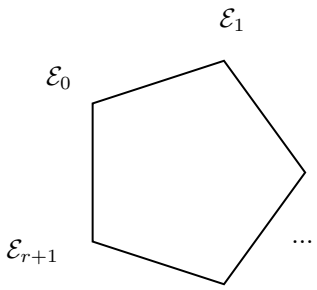
where $\lambda^{(\alpha,s)}$ *is a* $N(r+2)$*-tuple of Young diagrams,* $k_s = \sum_{\alpha=1}^{N} (\lambda^{(\alpha,s)})$, $[\sum n_{\alpha,s} a_{\alpha,s} + m_a \epsilon_a] = (\sum n_{\alpha,s} s + m_3 - m_4) \pmod{r+2}$, $[a_{\alpha,s}] = s, [\epsilon_1] = [\epsilon_2] = 0, [\epsilon_3] = 1, [\epsilon_4] = -1$.

We organize the representations of $\mathbb{Z}_{r+2}$ into a McKay graph, where the vertices are the representations and the edges represent the decomposition

$$\hat{\mathfrak{R}}_s \otimes \mathbb{C}^2_{34} = \bigoplus_{S^1} \mathfrak{R}_{s'} \otimes \mathbb{C}_{m_{ss'}}$$



Then, we promote the vertices into sheaves of rank N, and study the moduli spaces of these sheaves:



But on top of these sheaves, we should think of these moduli spaces as abstract spaces with obstructions coming from

$$\int_{\mathcal{M}_0^{\text{framed}} \times \mathcal{M}_1^{\text{framed}} \times \ldots \times \mathcal{M}_{r+1}^{\text{framed}}} c_{\epsilon_3} \left(\bigoplus_S \text{RHom}(\mathfrak{E}_s, \mathfrak{E}_{s+1}) \right).$$

Although it looks like we are complicating things, things will become simpler soon.

3.7.3 Example: A_r Type Theory

Let $q_0 = 0 = q_{r+1}$, so the chain of sheaves become fixed vector spaces. Then the measure becomes

$$\mu_{A_1}(\lambda^{(\alpha)}) = \prod_{\alpha,\alpha'} \frac{\prod_{\square \in \lambda^{(\alpha)}} \theta(a_\alpha - m_s^- + c_{12}(\square)) \prod_{s=1}^N \theta(a_\alpha - m_s^+ + c_{12}\square)}{\prod_{\square \in \lambda^{(\alpha)}} \theta(...) \prod_{\square \in \lambda^{(\alpha')}} \theta(...)}$$

where $a_{\alpha,r+1} = m_\alpha^-, a_{\alpha,r+1} + \epsilon_1 + \epsilon_2 = m_\alpha^+$.

The observables are

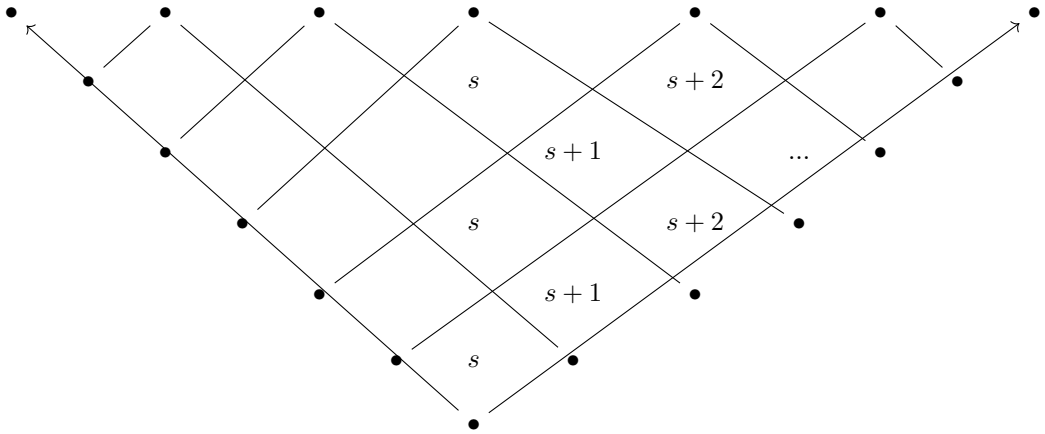
$$Y_s(x) = \prod_{\alpha=1}^{N} \frac{\prod_{\square \in \partial_+ \lambda^{(\alpha,s)}} (x - a_{\alpha,s} - c_{12}(\square))}{\prod_{\blacksquare \in \partial_- \lambda^{(\alpha,s)}} (x - a_{\alpha,s} - \epsilon_1 - \epsilon_2 - c_{12}(\blacksquare))}$$

and the fundamental qq-characters are

$$\mathfrak{X}_s(x) = Y_s(x + \epsilon_1 + \epsilon_2) + q_s \frac{Y(x + \epsilon_1 + \epsilon_2) Y_{s+1}(x)}{Y_s(x)} + ...$$

This is the same as the summation over the Young diagram

z_4 z_3



The weight of this daigram is $q_s^3 q_{s+1}^2 q_{s+2}^2$, and in the A_r situation ($q_0 = 0, q_{r+1} = 0$), the weight is $q^{|\lambda|} \to \prod_{(i,j)\in\lambda} q_{s+i-j}$, where we are considering only the λ's for which $(i,j) \in \lambda$ with $0 < s + i - j < r + 1$, so there are $\binom{r+1}{s}$ choices for λ.

Then

$$Y_0(x) = \prod_{\alpha=0}^{N} (x - m_\alpha^-)$$

$$Y_{r+1}(x - \epsilon_1 - \epsilon_2) = \prod_{\alpha=1}^{N} (x - m_\alpha^+)$$

and

$$\mathfrak{X}_s(x) = Y_0(x + \epsilon(1-s)) \sum_{0 \le i_1 < ... < i_\ell \le r+1} \prod_{\beta=1}^{\ell} \frac{\Lambda_{i_\beta}(x + \epsilon(\beta - 1))}{z_{\beta-1}}$$

where

$$\Lambda_i(x) = z_i \frac{Y_{i+1}(x + \epsilon)}{Y_i(x)}$$

for $i = 0, \ldots, r+1$

3.8 The Limit Shape Phenomenon

3.8.1 The Limit Shape Phenomenon

Let's take the limit as $\epsilon_1, \epsilon_2 \to 0$. In this case, the expectation values we were computing factorize.

$$\langle Y(x) \rangle$$
$$\langle Y^{-1}(x) \rangle = \frac{1}{\langle Y(x) \rangle}$$

This is called the **limit shape phenomenon**. This is because $\langle \mathfrak{X}_s(x) \rangle = T_s(x)$ has no poles in x and it's a holomorphic section of a degree N line bundle $\mathcal{L}|_E$, where

$$T_s(x) = \frac{Y_0(x)}{z_0 \dots z_{s-1}} e_s(\Lambda_0, \dots, \Lambda_{r+1})$$

where $\Lambda_i = \lim_{\epsilon_1, \epsilon_2 \to 0} z_i \langle \frac{Y_{i+1}(x)}{Y_i(x)} \rangle$.

Now, we have a system of algebraic equations. To solve it, we need to fix some poles or theta functions, and recover the Λ's by inverting the system. To do this, compute the generating function

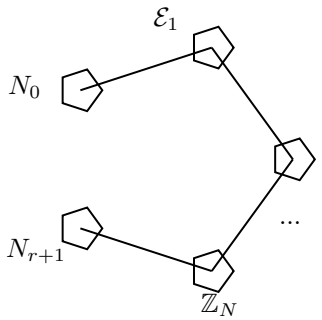
$$0 = \sum_{s=0}^{r+1} z_0 \dots z_{r-1} \Gamma_s(x) (-z^{-1})^s = Y_0(x) \prod_{i=0}^{r+1} (1 - z^{-1} z_i \frac{Y_{i+1}(x)}{Y_i(x)})$$

where the sum, denoted $\mathfrak{R}(x, z)$, is an algebraic curve in $E \times \mathbb{P}^1$ and $z^{(i)} = z_i \frac{Y_{i+1}}{Y_i(x)}$.

3.8.2 Spectral Curve Construction

Furthermore, we will describe this algebraic curve as a spectral curve, but in order to do this we first need to define an operator satisfying $\det(\hat{\mathfrak{R}}(x, z)) = \mathfrak{R}(x, z)$. To build this operator, we need to combine the orbifold structure we had before and introduce an additional orbifold in the $\mathbb{C}^2_{24}$ plane. This adds a surface defect by introducing a $\mathbb{Z}_N$ orbifold with respect to a bigger group $\mathbb{Z}_N \times \mathbb{Z}_{r+2}$.

The representations of this larger group are now labeled by two integers. This results in a more complex quiver structure with an additional dimension:



Now, each representation will create a Z_n-cyclic structure, and the resulting diagram resembles a lattice model on the 2-dimensional torus.

Concretely, $Y_s(x)$ becomes

$$Y_{s,\omega}(x) = \theta(x - a_{s,\omega}) \prod_{\beta=1}^{N} \prod_{\square \in \lambda^{(\beta,s)}} \left(\frac{\theta(s - a_{s,w} - c_{12}(\square) - \epsilon_2)}{\theta(x - a_{s,\omega} - c_{12}(\square) - \epsilon_1 - \epsilon_2)} \right)$$

if $\beta + j \equiv s \pmod N$ and

$$Y_{s,\omega}(x) = \theta(x - a_{s,\omega}) \prod_{\beta=1}^{N} \prod_{\square \in \lambda^{(\beta,s)}} \left(\frac{\theta(s - a_{s,w} - c_{12}(\square) - \epsilon_1)}{\theta(x - a_{s,\omega} - c_{12}(\square))} \right)$$

if $\beta + j - 1 \equiv s \pmod N$

3.8.3 Lax Operators, Residues, and Integrability

Now, we can organize this collection of observables into an $N \times N$ matrix:

$$\hat{Y}_s(x) = \mathrm{diag}(Y_{s,0}, \ldots, Y_{s,N-1}).$$

As $\epsilon_1, \epsilon_2 \to 0$,

$$\hat{Y}_0(x) \prod \left(1 - \hat{C}_z \hat{Z}_i \frac{\hat{Y}_{i+1}(x)}{\hat{Y}_i(x)} \right) = \sum \prod_{i=0}^{s-1} (\hat{C}_Z \hat{Z}_i) \hat{X}_s(x)$$

where

$$\hat{C}_z = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 1 & \\ & & \ddots & 1 \\ z^{-1} & & & 0 \end{pmatrix},$$

$\hat{Z}_i = \mathrm{diag}(Z_{i,\omega})$, and $\frac{Z_{i,\omega+1}}{Z_{i,\omega}} = q_{i,\omega}$. By expanding the brackets and reordering on the right hand side, we can see that coefficients are the collection of the qq-characters.

Theorem 3.34. *$\langle \mathfrak{X}_{s,\omega}(x) \rangle$ is a holomorphic section of degree 1 $\mathfrak{L}|_E$.*

The excitement lies in the fact that the degree is 1, meaning that it has a simple structure.

Now, I organize them in a diagonal matrix operator

$$\hat{\mathfrak{D}}(x, z) = \sum \prod_{i=0}^{s-1} (\hat{C}_Z \hat{Z}_i) \hat{X}_s(x).$$

As a function of x, this is a linear function; it's a section of a degree-one line bundle. As a function of z, it is a polynomial.

If we want to reconstruct the entries of the matrix, we just need to solve the equation $\hat{\mathfrak{D}}(x,z)\psi = 0$.

Let's bring the Lax operator back into the story. In the rational case, $\hat{\mathfrak{D}}(x,z)$ is a degree 1 polynomial in x such that

$$\begin{aligned}\hat{\mathfrak{D}}(x,z) &= x\hat{\mathfrak{D}}_0(z) + \hat{\mathfrak{D}}_1(z) \\ &= \hat{\mathfrak{D}}_0(z)(x - \hat{L}(z))\end{aligned}$$

where $\hat{\mathfrak{D}}_0(z) = \prod_{i=0}^{r+1}(1 - \hat{C}_z\hat{Z}_i)$ and the Lax operator $\hat{L}(z) = \hat{\mathfrak{D}}_0(z)^{-1}\hat{\mathfrak{D}}_1(z)$. Since $\det(1 - \hat{C}_z\hat{A}) = 1 - z^{-1}\det\hat{A}$, $\det\hat{\mathfrak{D}}_0(z) = 1 - z^{-1}z_i$. The poles of $\hat{L}(z)$ are at $z = z_0, \ldots, z_{r+1}$, so we can write $\hat{L}(z)$ in the form

$$\hat{L}(z) = \sum_{i=0}^{r+1} \frac{\hat{L}_i}{z - z_i}$$

which is the Lax operator on the Gaudin-Garnier system on genus 0 curve with $r+3$ punctures (the additional ones being at 0 and ∞).

Theorem 3.35. *The residues are rank-1 matrices.*

This is because when we invert an operator that's a polynomial in z, the residues correspond to projections onto the kernel and the dual kernel of the operator.

These $\hat{L}(Z)$ matrices contain information about the theory both with and without the surface defect. From the perspective of an integrable system, they can be seen as integrals of motion. The surface defect adds extra data, such as the Z_i parameters, which are not visible in the bulk theory. These Z_i values are degrees of freedom that change the dynamics in the Lax flow. In total, there are $2(N-1) \times r$ dynamical parameters.

The Higgs operator is $\phi(z) = \frac{dz}{z}\hat{L}(z)$.

Theorem 3.36. *The eigenvalues of the residues of ϕ at 0 and ∞ are m_α^- and m_α^+.*

Where does the Poisson structure come from? Let Ψ be the surface defect expectation value. We have

$$q_{s,\omega}\frac{\partial\Psi}{\partial q_{s,\omega}} = \langle a_{s,\omega} + \epsilon_1 k_{s,\omega}\rangle.$$

As $\epsilon_1, \epsilon_2 \to 0$,

$$\psi = e^{\frac{1}{\epsilon_1}S(z_{s,\omega}, a_{s,\omega}, \ldots)}$$

and $\frac{\partial S}{\partial Z_{s,\omega}}$ are the entries of u_i, v_i. This relation is a Hamiltonian-Jacobi potential where the momenta are related to the coordinates, allowing us to derive a symplectic structure for the residues of the Lax operator. Ths enumerative geometry creates a Lagrangian subvariety in the phase space of the integral system.

We've build part of the finite A-type theory. Let's finish with one last point: there's a similar but more complicated formulas for the $\hat{A}_0$ involving infinite product in both directions, where we replace $\hat{Y}_i(x)$ with $\hat{Y}(x + i\epsilon_3)$ and $\hat{X}_s(x)$ with $\hat{X}(x + s\epsilon_3)$.

4 Andrei Okounkov: From Elliptic Genera to Elliptic Quantum Groups

Abstract

This course will be an example-based introduction to elliptic cohomology, Krichever elliptic genera, rigidity, and related topics. We will work our way towards the geometric construction of elliptic quantum groups.

Contents

4.1 Introduction

The speakers at this conference have already covered various aspects of Krichever's work. In this session, we will focus on his early contributions to topology. This follows the work of Sergei Novikov, who also passed away recently. Krichever made significant strides in topology, and these contributions will be the central theme of my lectures. Later, we'll explore how his work evolved toward integrable systems. Even in his early topological papers, however, we can observe connections to Baker-Akhiezer functions and other areas. For Krichever, these subjects were always closely intertwined. In fact, today's leading research in integrable systems often engages with topological questions.

We will begin by explaining the concept of the elliptic genus, discussing its rigidity and other key properties. We'll also touch on how these ideas have evolved and highlight some of the current directions in the field. Those working in integrable systems should be particularly interested in this work, as we can apply representation theory and geometry to gain deeper insights into these systems. Geometry, in particular, offers a powerful way to simplify and prove complex identities, especially those involving elliptic functions. For instance, when dealing with a complicated multivariable expression of elliptic functions, it can be difficult to discern how it simplifies. The traditional approach to proving elliptic function identities involves verifying that the function transforms correctly when shifted by periods. Then, for a rational expression, one would aim to show that it has no poles. Typically, this involves proving that all poles cancel out. However, geometry provides a much more robust method to demonstrate that a rational expression is regular. Even for those not directly interested in geometry, this type of argument can still be of significant interest.

We will begin our discussion with genera, starting with the classical example of the Gauss map.

4.2 The Gauss Map

4.2.1 Definition and The Degree

Let C be an orientable surface of genus g embedded in $\mathbb{R}^3$. At each point on the surface, there is a unique normal vector and a corresponding tangent plane. Using this information, we can define a map that sends each point on the surface to a point on the unit sphere S^2, where the position on the sphere corresponds to the direction of the normal vector at that point on the surface, preserving the orientation. This defines the **Gauss map**:

$$\gamma : C \to S^2 = \mathrm{Gr}_+(1, 3, \mathbb{R}) = \mathrm{Gr}_+(2, 3, \mathbb{R}),$$

where Gr refers to the Grassmannian. What is the degree of this map?

Proposition 4.1. *The degree of the Gauss map is given by* $\deg \gamma = 1 - g(C)$*, where* $g(C)$ *is the genus of the surface* C*.*

Proof. There are several ways to prove this result. One approach involves examining the preimages of the north and south poles of S^2. These correspond to points on C where the tangent plane is horizontal, which are the critical points of the height function on C. By analyzing these critical points, we can derive the degree. Even if the surface is embedded differently, Morse theory tells us that there may be additional critical points, but since the cohomology of the surface can be computed using the critical points, the degree remains unchanged.

Another proof uses vector fields on S^2 and the Hopf index theorem, which relates the sum of the indices of a vector field to the Euler characteristic of the surface. Let v be a vector field on S^2, which is a section of the tangent bundle TS^2. Then the pullback $\gamma^* TS^2$ is the tangent bundle of C, and $\gamma^* v$ defines a vector field on C. This gives an index:

$$\mathrm{ind}(v) = \sum_{v(p)=0} \mathrm{ind}_p(v) = C \cap C \text{ inside } TC = \mathrm{Euler}(TC) = \chi(C),$$

where Euler denotes the Euler class, and

$$\mathrm{ind}(\gamma^* v) = \sum_{v(p)=0} (\deg \gamma) \cdot \mathrm{ind}_p(v) = \deg \gamma \cdot \mathrm{ind}_v.$$

Combining these, we obtain:

$$\chi(C) = \deg \gamma \cdot \chi(S^2),$$

where $\chi(C) = 2 - 2g$ is the Euler characteristic of C and $\chi(S^2) = 2$ is the Euler characteristic of the sphere. Thus, we find that

$$\deg \gamma = 1 - g,$$

as desired.

□

4.2.2 Characteristic Classes and Pullbacks

For a vector bundle, there are characteristic classes that capture important topological information. When the bundle is pulled back by a map, the characteristic classes also pull back accordingly. The Euler class is one such characteristic class - it encodes how many times a subsection of the bundle vanishes, counting with multiplicity.

The topological computations we are performing here have an analytic counterpart, which is often encountered in applications. Although we won't focus on this analytic side, its important to note that many problems can be viewed through this lens. When a bundle is pulled back by a map, not only can we pull back the characteristic classes, but we can also pull back other structures like connections and curvature.

For example, the sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ has an invariant metric. Since the metric is invariant under the action of the group, parallel transport with respect to this metric is simply given by the group action. If we want to translate along a geodesic, we can achieve this by acting with an element of the group that moves us along that geodesic. A concrete example involves a geodesic triangle: if we rotate it by an angle θ, the computation of the rotation's effect reduces to the commutator in the group, which, in this case, equals the area of the triangle.

In particular, when we pull back a metric or connection, we can also pull back the associated data, including the curvature. While some might think of curvature as merely a number, it is, in fact, a 2-form that tells us how much the geometry deviates as we traverse a small parallelogram. This leads to the following integral expression:

$$\int_C \text{curvature}(TC) = (1-g) \int_{S^2} \text{Area}$$

In our case, the area of a sphere is 4π and we get the Gauss-Bonnet theorem.

In modern high-energy physics, many integrals are written in the form of Riemann or Lebesgue integrals. However, these integrals often have a deeper meaning related to the characteristic classes of vector bundles. While we can compute these integrals using standard calculus techniques, they can also be understood from a topological perspective.

We won't dive deeply into connections and curvature here, as these topics are well-covered in many texts. Instead, we will focus on two analytic aspects of this story. The first relates to integrals, as illustrated in this example, and the second concerns the indices of (pseudo)-differential operators. These are defined analytically, but in the end, they reduce to integer values.

The topological computations we're working with today have corresponding analytic counterparts. While we won't emphasize this aspect, it's worth noting that these connections are often encountered in practical applications. When a bundle is pulled back by a map, we can pull back all its structures - connections, curvature, and so on.

Take the sphere, for example. While it may seem too large or imprecisely drawn, the sphere is a homogeneous space, and this property makes it particularly well-behaved in the context of pullbacks. Specifically, the sphere is the quotient of the rotation group by $\mathrm{SO}(2)$, and it carries an invariant metric under this group action. Since the metric is invariant, parallel transport with respect to it is simply given by the action of the group.

To illustrate this, consider translating along a geodesic on the sphere: we can achieve this by acting with an element of the rotation group that moves us along the desired geodesic. For example, imagine we have a geodesic triangle. If we take a tangent vector at one point, translate it along the geodesic to another point, and then repeat the process, eventually returning to the original point, the vector will have rotated by some angle, say θ. The amount of rotation can

be computed as the commutator in the group, which, in this case, is equal to the area of the triangle.

When we pull back a metric or connection, we also pull back the associated data, including curvature. This allows us to derive the property that integrating the curvature of the tangent bundle over a curve C gives an expression of the form

$$\int_C \text{curvature}(TC) = (1-g)\cdot \deg(\gamma)$$

where g is the genus of the surface C and $\deg(\gamma)$ is the degree of the map. For sphere, the curvature is directly related to the area, which in this case is π

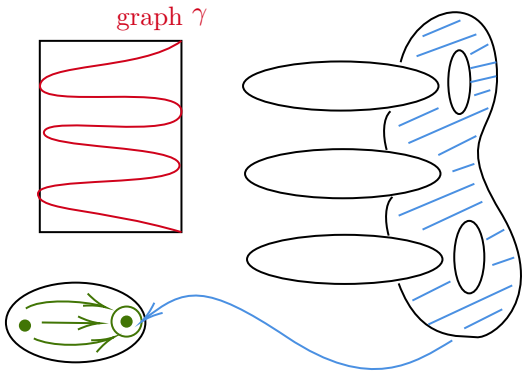
In modern high-energy physics, many integrals are written in the form of Riemann integrals, but they often have a deeper meaning tied to the characteristic classes of vector bundles. While we can compute these integrals using standard calculus, they also have a topological interpretation.

We won't focus on connections and curvature here, as these are well-covered in many books. Instead, we'll explore two key analytic aspects of this story: one involving integrals, as we've just illustrated, and the other concerning the indices of differential operators. These are defined analytically, but ultimately, they are integers.

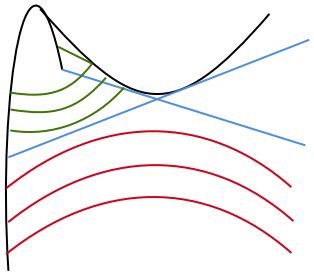
4.2.3 Interpreting The Degree

Now, let's interpret the degree of the map more concretely. Consider the graph of the map. On the sphere, we can choose a point and "stretch" the space from this point toward another. For instance, we might use a Morse flow to move everything from the north pole to the south pole. What happens if we continue this process? We obtain a result where we can split the space into subsets consisting of a curve and a bunch of spheres. The curve is collapsed to a point, and the spheres are mapped with degree ± 1 (counting with orientation) to $1-g$. Thus, we end up with a Gauss map:

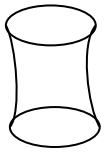
$$C \to \vee S^2 \to S^2.$$

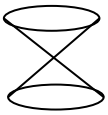


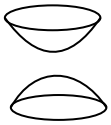
If we're computing a Riemann integral or something pulled back from here, that's good enough because whatever lives over that point doesn't matter. However, if we're computing indices of differential operators, this is not sufficient because it's not smooth. But there's a way to write a smooth cobordism between the Gauss map and the union of $1-g$ spheres. Consider the graph of a function $f(x) = x_1^2 + x_2^2 - x_3^2$. The graph of this function is a cone roughly looks like the follows:



with the following inverse maps

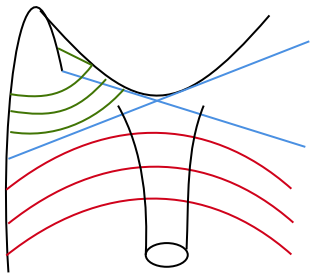
$f^{-1}(1) =$ 

$f^{-1}(0) =$ 

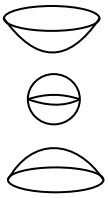
$f^{-1}(-1) =$ 

So we can smoothly pinch any handle on this surface, make a surface a bunch of spheres, and turn it into nothing because the surface in the 3-space bounded by 3-manifolds is cobordant to 0.

The problem is at the origin since the Gauss map is $\gamma(x) = \frac{1}{\|x\|}(x_1, x_2, -x_3)$, but we can fix this by turning the function into $f(x) = x_1^2 + x_2^2 - x_3^2 - \frac{\epsilon}{\|x\|}$ which would make a hole in the graph:



which will change add a sphere for one of the inverse functions:

$f^{-1}(-1) =$ 

So we have $\sqcup_{1-g}$spheres. In particular, we have

$$\text{ind}(\overline{\partial} : C^\infty \to C^\infty \overline{dz})$$

and

$$\text{ind}_C\left(\overline{\partial}\right)(1-g)\text{ind}_{S^2}\left(\overline{\partial}\right).$$

Now, we can see that the kernel is holomorphic functions, and the cokernel $\sim$ holomorphic differentials. So, not only can we compute some Riemann statements topologically, but we can also compute statements about indices of elliptic operators in this topological fashion.

In this course, we will not deal with Riemann integrals or elliptic differential operators. Instead, we will focus on situations where the manifold is complex, and we are dealing with the sheaf of holomorphic functions $\chi(\mathcal{O}_C)$ where we have $\mathcal{O}_C \to C^\infty \xrightarrow{\overline{\partial}} C^\infty \overline{dz} \to 0$ and we can tensor every term by any holomorphic bundle $\mathcal{F}$, allowing us to compute $\chi(\mathcal{F})$.

We will always work in the case where we have this sheaf and we can compute this, but in principle we can also phrase it as an index of a differential operator, namely the Dolbeault operator.

To summarize this discussion, there are is an important notion of a vector bundle. With vector bundles, particularly in the context of a manifold, which by definition has a tangent bundle (a very important bundle), we can associate to it characteristic classes. These classes can be related to connections, curvature, and other differential-geometric concepts, but ultimately, they can be computed

purely topologically. Some involve more sophisticated topology, but in the end, the computations remain completely topological.

4.3 Elliptic Genus and K-theory

This leads us to the discussion of elliptic genus and K-theory. Elliptic genus is a kind of elliptic function associated with a bundle on a manifold, and K-theory involves associating a certain group to a topological space, which we'll discuss in more detail.

4.3.1 K-Theory of Vector Bundles

Let X be a compact Hausdorff topological space, such as a smooth manifold or a simplicial complex. For such a space, we will define the **K-group of vector bundles**, denoted $K^0(X)$, which is an algebraic structure that encodes the classification of vector bundles over X. Elements of this group correspond to isomorphism classes of vector bundles, and the group is equipped with a ring structure.

Definition 4.2. *A **vector bundle** is a topological space V together with a projection map $p : V \to X$, where for each point $x \in X$, the fiber $V_x = p^{-1}(x)$ is a vector space.*

In this context, we primarily focus on **complex vector bundles** since complex vector spaces offer richer structural properties. Given a vector bundle V with projection $p : V \to X$, for any open set $U \subset X$, we have an isomorphism $p^{-1}(U) \cong U \times \mathbb{C}^n$ for some n, where n is the rank of the bundle over U. Moreover, the transition maps between different neighborhoods must be linear in the fiber $\mathbb{C}^n$.

Given two vector bundles V_1 and V_2 over X, we can construct their **direct sum** $V_1 \oplus V_2$, which is also a vector bundle over X. The transition functions for $V_1 \oplus V_2$ arise from those of V_1 and V_2, and can be expressed through the general linear group $\mathrm{GL}(n)$. Specifically, if the transition functions for V_1 and V_2 lie in $\mathrm{GL}(n_1)$ and $\mathrm{GL}(n_2)$, respectively, then the transition functions for $V_1 \oplus V_2$ lie in $\mathrm{GL}(n_1 + n_2)$. Similarly, the **tensor product** $V_1 \otimes V_2$ of two vector bundles is another vector bundle, and the direct sum and tensor product together generate a semiring of vector bundles.

In a semiring, subtraction is not typically defined. However, we can introduce a formal notion of subtraction by defining an operation $\ominus$ as follows: for two vector bundles V_1 and V_2,

$$V_1 \ominus V_2 = W_1 \ominus W_2 \leftrightarrow V_1 \oplus W_2 = W_1 \oplus V_2.$$

Thus, $\ominus$ serves as a formal subtraction that respects the structure of the semiring.

Proposition 4.3. *For any vector bundle V, there exists an integer N such that $V \subset \mathbb{C}^N$.*

Alternatively, we can say that V is a quotient of a trivial bundle or there is a surjection $\mathbb{C}^n \twoheadrightarrow V$:

Proposition 4.4. *For any vector bundle V, there exists an integer $N \gg 0$ such that $V \oplus V^\perp = \mathbb{C}^N$.*

Using the formal subtraction operation, we can express the result as:

$$\ominus V = \ominus \mathbb{C}^N \oplus V^\perp.$$

This formalism allows for more algebraic manipulation of vector bundles within the framework of K-theory.

4.3.2 Stably Isomorphic

In K-theory, vector bundles are considered equal if they are **stably isomorphic**. Although we primarily work with complex vector bundles, we simplify the concept by considering the tangent bundle of a sphere. Specifically, if we take the tangent bundle of a sphere and add a trivial bundle, the resulting bundle is trivial. This is because the normal bundle to the sphere is trivial, and since the sphere is orientable, the tangent bundle, when combined with the trivial bundle, gives us the ambient space $\mathbb{R}^3$ restricted to the sphere. Thus, while the two bundles may not be isomorphic initially, they become isomorphic once we add a trivial bundle. This construction shows that even non-trivial bundles can become equivalent when supplemented with additional structure, a concept that is widely applied in complex algebraic geometry.

This idea of stable isomorphism allows us to extend the framework of vector bundles, and more importantly, it facilitates the definition of an analog of the Gauss map. Let X be a topological space, $\mathbb{C}^N$ a trivial bundle, and V a sub-bundle of $\mathbb{C}^N$. In this context, we can form a map to the Grassmannian $\mathrm{Gr}(\text{rank } V, N, \mathbb{C})$, which parametrizes all the subspaces of rank rank V in $\mathbb{C}^N$. This map allows us to identify the sub-bundle V with a point in the Grassmannian. The Grassmannian is a well-behaved manifold, a homogeneous space with a rich geometry, and this allows us to apply this analogy in complex settings.

Moreover, there exists a tautological bundle Taut over the Grassmannian, whose fibers correspond to the subspaces themselves. By pulling back this tautological bundle via the map γ, we obtain a bundle $V = \gamma^*(\text{Taut})$, which is the pullback of the tautological bundle over X. This construction shows how the Gauss map extends naturally to more general spaces.

Working with complex vector bundles is particularly advantageous because the cohomology of the complex Grassmannian is significantly more intricate than that of the real Grassmannian. While the cohomology of real Grassmannians can be studied, it is often characterized by mod 2 torsion, which renders it less interesting from a topological perspective. In contrast, the cohomology of complex Grassmannians is more flexible, allowing for operations such as adding

trivial bundles. Furthermore, we can take limits over increasing dimensions:

$$\mathrm{Gr}(\text{rank } V, N, \mathbb{C}) \hookrightarrow \bigcup_N \mathrm{Gr}(r, N, \mathbb{C}),$$

and the cohomology of these Grassmannians behaves well under such limits. This provides a means to extend the cohomology in a natural way.

4.3.3 Characteristic Classes

Next, we examine characteristic classes in the context of cohomology. These are represented by pullbacks of cohomology classes from the Grassmannian. For instance, when $r = 1$, the Grassmannian $\mathrm{Gr}(1, n)$ is isomorphic to the projective space $\mathbb{P}^{n-1}$, which has a nice cell decomposition. Each cell in this decomposition is complex and of even dimension, meaning there are no boundary maps between them. In particular, we can identify the cohomology ring of $\mathbb{P}^{n-1}$ as $H^\bullet(\mathbb{P}^{n-1}) = \mathbb{Z}[x]/\langle x^n = 0\rangle$, where x is the generator, corresponding to the Poincaré dual to the hyperplane. As $n \to \infty$, this cohomology extends to $H^\bullet(\mathbb{P}^\infty) = \mathbb{Z}[x]$.

he infinite projective space $\mathbb{P}^\infty$ can be expressed as:

$$\mathbb{P}^\infty = \frac{\mathbb{C}^\infty \backslash 0}{\mathrm{GL}(1)} = \frac{S^\infty}{\mathrm{U}(1)},$$

where $\mathbb{C}^\infty = \bigcup_N \mathbb{C}^N$, and S^∞ is contractible. This construction is the classifying space $\mathrm{BU}(1)$, which serves as the base for understanding line bundles over complex spaces.

For higher ranks, we consider the Grassmannian $\mathrm{Gr}(r, N)$, which can be described as the space of $r \times N$ matrices of full rank, modulo the action of $\mathrm{GL}(r)$. As $N \to \infty$, this Grassmannian converges to the classifying space $\mathrm{BU}(r)$ of r-dimensional complex vector bundles. The cohomology of these spaces is governed by the characteristic classes, which are typically expressed as symmetric polynomials in the Chern roots of the corresponding vector bundles. For example, the Chern classes can be written as elementary symmetric polynomials in the roots $x_1, x_2, \ldots, x_r$, leading to expressions like:

$$e_r = x_1 x_2 \ldots x_r,$$

which lies in H^{2r}, the cohomology class corresponding to the real dimension of $\mathbb{C}^r$.

When a section of a vector bundle vanishes, it imposes constraints on the manifold. In the context of K-theory, the principle is that every vector bundle can be pulled back from the universal bundle. To compute sections of the universal bundle, we consider homomorphisms $\mathrm{Hom}(\mathbb{C}^r \to \mathbb{C}^N)$. In this case, $\mathbb{C}^r$ becomes the tautological bundle, and the vanishing of any row in the matrix representing the homomorphism corresponds to a hyperplane condition. These sections, and

the invariance under certain operations, provide the foundation for computing characteristic classes, such as the Chern classes.

More generally, we can consider the case where c_{r-i} represents the locus where $i+1$ sections of the bundle are linearly dependent. In this case, we can write the bundle V as $V = \mathbb{C}^i \oplus V'$, where V' is a sub-bundle associated with the vanishing sections.

In cohomology, the characteristic classes are expressed as:

$$c_K(v) = \gamma^* e_k,$$

where $e_k \in H^\bullet(\mathrm{Gr}(\cdot,\cdot))$ are the cohomology classes of the Grassmannian. These classes can be multiplied and integrated over the manifold, as demonstrated in previous examples involving the Euler class.

In K-theory, we focus on operations such as tensor products and wedge products of bundles, which serve as analogs of cup products in cohomology. For example, in cohomology, we compute integrals like:

$$\int_X \prod c_{k_i}(V),$$

whereas in K-theory, we compute the Euler characteristic of the tensor product:

$$\chi(\otimes \wedge^{k_i} V),$$

which serves as the topological invariant of the vector bundle in the K-theoretic context.

4.4 Towards Equivariant Cohomology Theories

Equivariant cohomology theory extends the classical cohomology theory by incorporating group actions on spaces. This theory is a powerful tool in both topology and geometry, enabling the study of spaces with symmetries. To understand equivariant cohomology, we begin by discussing the foundational elements of cohomology and then consider how group actions influence the theory.

4.4.1 Foundations of Cohomology

Cohomology is a topological invariant that can be defined for a wide range of spaces. Let X be a topological space, and we are interested in understanding its cohomological properties. Typically, one studies spaces like cell complexes or manifolds, and for our purposes, we often focus on complex manifolds or algebraic varieties defined over fields.

To define cohomology, we consider maps $f : X \to Y$ between topological spaces, where these maps are identified up to homotopy equivalence. For the purposes of equivariant cohomology, we equip these spaces with group actions. Let G be a compact group acting on X, and suppose there is a map $f : X \to Y$ that also

respects a group action by G on Y, with a compatible map $\varphi : G \to H$ between the group actions on X and Y.

Given this data, cohomology theory is typically formalized through a functor. The cohomology groups $h^\bullet(X)$ form a complex of abelian groups or modules over a ring, with the map $f^* : h^\bullet(Y) \to h^\bullet(X)$ induced by the map f. These constructions are functorial, and cohomology satisfies important properties, such as the preservation of long exact sequences.

Example 4.5. *Consider a simplicial complex, or the de Rham complex:*

$$\Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \Omega^2 \xrightarrow{\delta} \cdots,$$

where each Ω^n represents the space of functions on oriented n-cells. The differential δ acts like a discrete version of the de Rham differential, capturing the change in function values between adjacent vertices of a cell. This differential satisfies $\delta^2 = 0$, forming a well-defined complex.

A deeper understanding of cohomology is achieved by introducing derived categories and long exact sequences. When studying topological spaces, we often encounter different ways of constructing cohomology complexes for the same space. For example, one might subdivide a simplicial complex by adding more vertices, resulting in a new complex. Although the individual complexes may differ, the maps between them can be shown to induce isomorphisms in cohomology.

This leads to the concept of **derived categories**, where maps that induce isomorphisms in cohomology are considered equivalent. This is a finer notion than simply declaring two spaces to be isomorphic based on their cohomology groups. The notion of exact sequences also plays a crucial role in the study of cohomology. The main property that we wish for the functor to satisfy is that in the category of topological spaces, every map is the start of a long exact sequence:

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to ...$$

The condition for a functor to be a cohomology theory is that it should take cones of topological maps to cones of complexes.

Remark 4.6. *Every map between topological spaces can be replaced, up to homotopy, by a nice embedding. This is achieved by using the cylinder construction: we attach a cylinder to Y by gluing it along the map f, which satisfies $Cyl(f) \sim Y$. After this attachment, we can contract the non-glued side of the cylinder to a point, yielding the cone of f. If Y is contracted to a point, we obtain the **suspension** ΣX of X.*

Here is a crucial (and non-obvious) exercise:

Exercise 4.7. *The next map in the long exact sequence is $\sum X \xrightarrow{\sum f} \sum Y$.*

Now, we present a slightly technical point.

Remark 4.8. *It is often more convenient to work with a base point in each space. By choosing a base point in X and ensuring that all maps respect this base point (i.e., they map the base point of X to the base point of Y), we can apply the cylinder or cone construction. This ensures that when we contract the cylinder, we obtain what is called the* ***reduced suspension*** *of X.*

There are several consequences:

1. For the identity map $\mathrm{id} : \mathrm{pt} \to \mathrm{pt}$, we have $\mathrm{Cone}(\mathrm{id}) = \mathrm{pt}$, and hence $h^\bullet(\mathrm{pt}) = \mathrm{Cone}(\mathrm{id}) = 0$.
2. For any space X, define $X_+ = X \sqcup \mathrm{pt}$, where pt is a marked point. Then $h^\bullet(X_+)$ is an "old unreduced cohomology," while $h^\bullet(S^0)$ is the old cohomology of a point, which is nice.

By considering the sequence:

$$X \to CX \to \Sigma X \to \cdots$$

where CX is the cone of X, we observe that $h^\bullet(\Sigma X)$ is simply the shifted cohomology of X, specifically $h^0(\Sigma X) = h^0(X)$ and $h^i(\Sigma X) = h^{i-1}(X)$ for $i \geq 1$. This shifting property is important because it suggests that the suspension operation only shifts the cohomology groups. Consequently, for large enough n, the cohomology of $\Sigma^n X$ behaves as $h^{n-1}(X)$.

We can consider the following sequence of spaces:

$$X \to CX \to \Sigma X \to \cdots$$

where CX denotes the cone of X, and ΣX represents the suspension of X. The cone of the suspension ΣX is contractible to a point. This gives rise to a shift in cohomology: $h^\bullet(\Sigma X)$ with the cone shift becomes $h^0(X)$, and for $i \geq 1$, we have $h^i(\Sigma X) = h^{i-1}(X)$. This shift property is important because it shows that the suspension operation only shifts the degree of the cohomology groups, and this shift is independent of the specific details of the space as $n \to \infty$.

For long exact sequences of spaces, such as

$$\cdots \to \Sigma^n X \to \Sigma^n Y \to \cdots,$$

it suffices to focus on $n \gg 0$. While these spaces have additional structure, we observe that the space of maps $\mathrm{Map}(\Sigma^n X, Z)$ becomes a group for $n \geq 1$, and this group is abelian for $n > 1$. The reasoning behind this is analogous to the argument for $\pi_2(Z)$. The addition of maps in this context corresponds to the addition of maps in a category of complexes, which can be verified by showing that any homomorphism between two abelian groups, when restricted to the identity on one side and zero on the other, must be an addition map.

This insight reveals that the functorial behavior we observe is not limited to topological spaces but also preserves the structure of map addition.

Next, let's turn to an example from equivariant K-theory.

4.4.2 Equivariant K-Theory and Bott Periodicity

Consider a compact Hausdorff space X and a vector bundle V over X. A vector bundle over X consists of a collection of vector spaces V_x parameterized by the points $x \in X$. If we have a map $g : x \mapsto g(x)$ from X to itself, this map induces a group action on the fibers of the vector bundle. Specifically, for each x, the group action lifts to the fibers $V_x \to V_{g(x)}$, which is a linear map. This situation is analogous to studying a group representation, where the group acts on a vector space. However, in this case, the group is acting on a family of vector spaces (the fibers), and the group action permutes the indices associated with each vector space.

Consider $K_G(\text{pt})$, which represents a semiring of group representations (with operations $\oplus$ and $\otimes$). When we add the operation $\ominus$, we obtain the representation ring of G. If $T \subset K_G(\text{pt})$ is a maximal torus, the ring takes the form $\mathbb{Z}[z_1^{\pm 1}, \ldots, z_r^{\pm 1}]^W$, where W is the Weyl group. In the case of GL_n, this is closely related to the Laurent polynomial ring, because the representation is determined by a character, and a character can be expressed as a sum of Laurent polynomials. By allowing subtraction of these polynomials, we recover the full set of Laurent polynomials. Thus, we are left with a ring $K_G^0(X)$ along with a homomorphism $K_G^0(\text{pt}) \to K_G^0(X)$.

Next, define $K_G^{-1}(X) := K_G^0(SX)$, where SX is the suspension of X and the group action does not extend to SX. Intuitively, this construction allows us to perform an operation known as the suspension of a vector bundle by a nontrivial representation. While we won't dive further into this here, we can think of this operation as adding a "point at infinity" to the space, which induces a suspension by a nontrivial representation of the group. This in turn enables the definition of functors that operate at infinity and at negative indices.

Theorem 4.9 (Bott Periodicity)**.**

$$K_G^0(S^2X) = K_G^0(X).$$

Remark 4.10. *If we introduce a new fixed point to the space, we have* $\tilde{K}_G^0(X \sqcup pt) := K_G^0(X)$*, where the symbol* $\sim$ *indicates that when we restrict the vector bundle to the point, we obtain a trivial bundle.*

This implies that K^i exhibits 2-periodicity, meaning that for all i, we have $K^{i+2} = K^i$.

4.4.3 The Thom Isomorphism

Now, we move towards a more general result. But before we get there, we need to present another definition.

Definition 4.11. *Given a space X and a complex vector bundle V, the **Thom space** $Thom(V)$ is constructed by introducing a metric on V, taking a ball of fixed radius within each fiber, and then collapsing the spheres to a point.*

With this definition in place, we can state the following result:

Theorem 4.12 (Thom Isomorphism)**.**

$$\tilde{K}_G(Thom(V)) \cong \tilde{K}_G(X).$$

Remark 4.13. *If $V = \mathbb{C}$, this results in Bott periodicity.*

Now consider a pair $Y \subset X$, and let $K(X/Y)$ denote the space of complexes of vector bundles over X that become exact when restricted to Y. Specifically, for vector bundles V_1 and V_2 over X, suppose there is a map $\varphi : V_1 \to V_2$ such that when restricted to Y, φ becomes an isomorphism. In this context, $\tilde{K}_G(\text{Thom}(V))$ can be interpreted as complexes of vector bundles over the total space of V, which remain exact away from the zero section of V. Another way to think of this is that the infinity, S^∞, is homotopically equivalent to the complement of the zero section of the vector bundle.

In general, elements of a K-theory group are of the form $V_1 \ominus V_2$. In the case of $K(X/Y)$, this means that the map $\varphi : V_1 \to V_2$ gives rise to an exact complex:

$$0 \to V_1 \to V_2 \to 0.$$

Thus, $\tilde{K}_G(\text{Thom}(V))$ forms a module over $K_G(X)$. This is because if we have an exact complex, we can tensor it with any object pulled back from X, and exactness is preserved. Thom periodicity implies the following corollary:

Corollary 4.14. *The module is free, with rank 1, and is generated by the Koszul complexes.*

Next, we observe that in the context of elliptic cohomology, the Thom isomorphism does not hold. Specifically, in elliptic cohomology, $\tilde{K}_G(X)$ corresponds to a scheme, while $\tilde{K}_G(\text{Thom}(V))$ corresponds to a line bundle over that scheme. Although the line bundle is rank 1 and locally free, it is not trivial. This distinction is crucial because it is the failure of the Thom isomorphism in elliptic cohomology that contributes to its richness and uniqueness.

All cohomology theories share a common feature when we apply functorial maps (such as pullbacks under maps). However, the real richness of a cohomology theory emerges when we consider pushforward operations. The Thom isomorphism plays a key role in defining the pushforward in $K^\bullet$, since it provides a basic example of the pushforward, namely the pushforward from the zero section to the

ambient space. We have the following sequence:

$$K(X) \xrightarrow{\text{Thom}} K(\text{Thom}) \to K(\text{Total Space of } V),$$

where the second map is functorial. Note that the Thom isomorphism is not functorial in all cases; for example, it does not hold in the case of elliptic cohomology.

Example 4.15. *Consider $G = S^1 = U(1)$. In this case, $Ell_G(pt)$ corresponds to an elliptic curve E. Suppose we have a bundle $\mathbb{C}$ over the point, with a representation acting on it. Then Thom$(0 \hookrightarrow \mathbb{C})$ corresponds to the line bundle on the elliptic curve E with the origin 0, which is represented by $\mathcal{O}_E(-[0])$, the sheaf of functions vanishing at the origin 0.*

Now, let's discuss a specific case of the Thom isomorphism, which will be very instructive.

Problem 4.16. *Let $V = \mathbb{C}^n$ be a vector space, and consider the defining action of $G = GL(n)$ on V. Determine $K_G(\mathbb{P}(V))$, where*

$$\mathbb{P}(V) = (V \setminus \{0\})/GL(1),$$

with $GL(1)$ denoting the center of $GL(n)$.

Proposition 4.17. *If a group G acts freely on a space X, then*

$$K_G(X) \cong K(X/G).$$

Thus,

$$K_G(\mathbb{P}(V)) \cong K_{G\times \mathrm{GL}(1)}(V \setminus \{0\}),$$

where GL(1) represents the center of GL(n). From this, we derive the sequence:

$$K_{G\times \mathrm{GL}(1)}(\text{Thom}(0 \to V)) \to K_{G\times \mathrm{GL}(1)}(V) \to K_{G\times \mathrm{GL}(1)}(V \setminus \{0\}),$$

where V can be treated as a single point. Consequently, the middle term simplifies to

$$K_G(\mathrm{pt})[u^{\pm 1}],$$

where u is a formal parameter corresponding to the grading induced by the GL(1)-action.

The term $K_{G\times \mathrm{GL}(1)}(\text{Thom}(0 \hookrightarrow V))$ consists of complexes of vector bundles that are exact away from the origin. Since the group $\mathrm{GL}(n) \times \mathrm{GL}(1)$ acts equivariantly, the resulting maps are homogeneous and can be interpreted as polynomials.

Rather than viewing these as complexes of vector bundles, we can reinterpret them as graded equivariant complexes of modules over the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, where the x_i are the coordinates in V. This reformulation emphasizes the algebraic structure of the problem while retaining the essential geometric information.

4.4.4 Coherent Sheaves

In algebraic geometry, vector bundles are modules over the ring of functions on an algebraic variety that are locally free. However, we can generalize this notion by considering coherent sheaves:

Definition 4.18. *A **coherent sheaf** is a finitely generated module that does not need to be locally free.*

This broader perspective allows for greater flexibility in describing K-theory.

In algebraic geometry, vector bundles are modules over the ring of functions on an algebraic variety that are locally free. But in this case, there is no need to restrict ourselves to modules that are locally free, and we can consider any finitely generated modules, ie. coherent sheaves. Coherent sheaf generalizes the concept of a vector bundle by allowing modules that are not necessarily free but are still finitely generated.

Example 4.19. *Consider $\mathcal{O}_0 = \mathbb{C}[x_1, \ldots, x_n]/m_0$, where $m_0 = (x_1, \ldots, x_n)$. This module represents the structure sheaf of the origin in V. The following exact sequence illustrates its behavior:*

$$K^{Coh}(0) = K_G^{Coh}(pt)[\mathcal{O}_0] \to K_G^{Coh}(V) \to K_G^{Coh}(V \setminus \{0\}) \to 0.$$

In algebraic geometry, it is evident that coherent sheaves supported only at the origin are all multiples of $\mathcal{O}_0$. This is because being zero outside the origin implies that the coordinates act nilpotently. In other words, at least some of the coordinates must vanish to ensure they are not invertible. This nilpotent action creates a flag or filtration structure where the coordinates effectively act as zero.

Next, let's introduce an essential tool for describing coherent sheaves:

Definition 4.20. *A **Koszul complex** is a resolution of $\mathcal{O}_0$ by free modules.*

Let's dive into an example:

Example 4.21. *If $R = \mathbb{C}[x_1, x_2]$, the Koszul complex takes the form:*

$$0 \to x_1x_2R \to x_1R \oplus x_2R \to R \to \mathcal{O}_0 \to 0.$$

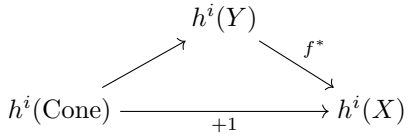
This exact sequence describes the resolution of $\mathcal{O}_0$ by free R-modules, illustrating how coherent sheaves are constructed algebraically.

4.4.5 Long Exact Sequences

Consider the group cohomology $h_G^i(X)$ in the context of K-theory. In practice, these quantities are not always expressed as the cohomology of some complex. For instance, when we study K-groups, we often consider them individually, rather than as part of a larger complex. However, they still adhere to a general pattern, which can be described via a long exact sequence:

$$X \xrightarrow{f} Y \to \text{Cone}(f),$$

where we have the following diagram:



This structure arises naturally when we examine K-groups, where maps between spaces induce relations between their cohomology groups.

We are still discussing the question:

Problem 4.22. *Let $U(V) \hookrightarrow \mathbb{P}(V)$, where V is a complex vector space of dimension n. What is $K_{U(V)}(\mathbb{P}(V))$?*

Consider the projective space $S^{2n-1}/\mathrm{U}(1)$ and its associated disc. When we take the disc and collapse the sphere, we obtain a topological space. The question arises: what is the relationship between the cohomology of the projective space and that of the sphere?

In general, we have the following result:

Proposition 4.23. *If G acts freely on X, then*

$$K_G(X) = K(X/G).$$

From this, we can deduce the following corollary for the specific case of projective spaces:

Corollary 4.24.

$$K_{U(V)}(\mathbb{P}(V)) = K_{U(V)\times U(1)}(S^{2n-1}).$$

Thus, for G-equivariant K-theory of a homogeneous space, understanding the behavior of vector bundles under group actions is crucial. For example, when considering a 3-dimensional sphere and a vector bundle that is equivariant under all rotations, it suffices to understand how the stabilizer of a point acts on the fiber. The stabilizer's action uniquely determines the group action on the entire bundle:

Proposition 4.25.

$$K_G(G/H) = K_H(pt).$$

In this context, the G-equivariant cohomology of a point corresponds to the representation ring of G, denoted $\mathbb{Z}[h/\sim]$, which is the $\mathbb{Z}$-linear span of the irreducible representations of G. For a circle, this becomes the Laurent polynomial ring with integer coefficients.

To further explore the relation between the cohomology of projective spaces, we consider the long exact sequence associated with the inclusion of the sphere into the disc:

$$\cdots K_G^0(\text{Thom}(0 \to V)) \to K_G^0(D^{2n}) \to K_G^0(S^{2n-1}) \cdots$$

where the terms correspond to:

- $K_G^0(S^{2n-1})$ is the K-group of a sphere where the group G acts transitively. Thus, $K_G^0(S^{2n-1}) = \text{Rep}(G_V)$, where G_V is the stabilizer subgroup of a nontrivial vector.
- The action on $K_G^0(D^{2n})$ corresponds to the action on a point in the center of the space, so $K_G^0(D^{2n}) = \text{Rep}(G)$, which is the ring of functions on the group G: $R = \mathbb{Z}[G/\sim]$.
- $K_G^0(\text{Thom}(0 \to V))$ consists of representations of G that are trivial when restricted to G_V, the stabilizer subgroup. These are functions that vanish when restricted to the subgroup, and can be described as $R \cdot \det(1 - tg)$.
- $K_G^0\left(S^{2n-1}\right)$ is a sphere where the group acts transitively, so $K_G^0\left(S^{2n-1}\right) = \text{Rep}(G_V)$, where G_v is the stabilizer subgroup of a nontrivial vector.
- The action on $K_G^0\left(D^{2n}\right)$ is the same as the action on the point on the center of the space, so $K_G^0\left(D^{2n}\right) = \text{Rep}(G)$. This is functions on the group G: $R = \mathbb{Z}[G/\text{cong}]$
- The group $K_G^0(\text{Thom}(0 \to V))$ consists of representations of G that become trivial when restricted to G_V. More specifically, consider the pair (g, t), where $g \in \text{U}(V)$ is represented by a block matrix of the form

  $$g = \begin{pmatrix} a_1 & * \\ 0 & g' \end{pmatrix} \in \text{U}(V),$$

  and $t \in \text{U}(1)$. The condition $a_1 t = 1$ ensures that the pair (g, t) fixes a vector in the vector bundle. In other words, the corresponding functions vanish when restricted to the subgroup G_V. These functions can be described as elements of the ring $R \cdot \det(1 - tg)$, where R represents the ring of functions on the group.

  Thus, we obtain the following exact sequence:

  $$0 \to K_G^0(\text{Thom}(0 \to V)) \to K_G^0(D^{2n}) \to K_G^0(S^{2n-1}) \to 0,$$

  with the odd cohomology of a point being zero.

Finally, we observe the following important result:

Proposition 4.26.

$$K_{U(V)} = K_{U(V)}(pt)\left[t^{\pm 1}\right] / \det(1 - tg) = 0,$$

where $t \in \mathcal{O}_{\mathbb{P}}(1)$ *and* $K_{U(V)}(pt) = \mathbb{Z}[g]/conj.$

If we decompose $V = V_1 \oplus V_2 \oplus \dots$ and compute $\prod \mathrm{U}(V_i) \supset \mathrm{U}(V)$ by performing the same computation in smaller steps, we will obtain the same answer but with conjugation taken with respect to the smaller group.

Example 4.27.

$$K_T(\mathbb{P}(V)) = \mathbb{Z}[a_1^{\pm 1}, a_2^{\pm 1}, \dots]\left[t^{\pm 1}\right] / \prod_{i=1}(1 - ta_i) = 0.$$

4.4.6 Kozsul Complexes

From the perspective of Koszul complexes, $\det(1 - t^{-1}g^{-1})$ is related to the Koszul complex. As before, we have the following exact sequence:

$$0 \to x_1x_2\mathbb{C}[x_1, x_2] \to x_1\mathbb{C}[x_1, x_2] \oplus x_2\mathbb{C}[x_1, x_2] \to \mathbb{C}[x_1, x_2] \to \mathcal{O}_0 \to 0,$$

where $\mathcal{O}_0 = \mathbb{C}[x_1, x_2]/m$ and $m = (x_1, x_2)$. The terms in this complex are elements of $K_{\mathrm{U}(2)\times\mathrm{U}(1)}(\mathbb{C}^2)$.

Term by term, we obtain the representation:

$$1 - V^*t^{-1} + \wedge^2 V^* t^{-2}.$$

The character is given by:

$$1 - \mathrm{tr}(g^{-1}) \cdot t^{-1} + \det(g^{-1}) \cdot t^{-2} = \det(1 - t^{-1}g^{-1}).$$

Remark 4.28. *There's a funny relationship between topological and algebraic K-theory — sometimes they are very closely related, and other times they diverge. However, for a point, they are very closely related. Any representation of a compact group is holomorphic, so any spaces built from points in simple ways inherit the relationship between topological and algebraic K-theory.*

4.5 Elliptic Cohomology

4.5.1 An Introduction

Let G be a compact group, for example, $G = \mathrm{U}(1)$. Then,

$$K_{\mathrm{U}(1)}(\mathrm{pt}) = \mathrm{Rep}(\mathrm{U}(1)) = \mathbb{Z}[t^{\pm 1}],$$

which can be interpreted as the ring of trigonometric polynomials on $\mathrm{U}(1) = S^1 = \mathbb{R}/\mathbb{Z}$, or equivalently, as the ring of Laurent polynomials on $\mathbb{C}^\times = G_{\mathbb{C}}$. We aim to understand

$$\mathrm{Ell}_{\mathrm{U}(1)}(\mathrm{pt}) \approx \text{elliptic functions on } \mathrm{U}(1),$$

which corresponds to the set of holomorphic functions on $E = \mathbb{C}^*/q^{\mathbb{Z}}$, an elliptic curve.

However, it turns out that there are no non-constant holomorphic functions on this elliptic curve, which is a crucial observation when considering certain properties. One significant property, proven by key results in the theory, is rigidity, which eventually reduces to the fact that there are no such non-constant holomorphic functions on this curve.

Recall that the cohomology groups of a space X can take values in vector spaces, but more generally, they should take values in some category where exact sequences are well-defined. Thus, instead of merely saying that the cohomology groups are modules over a ring, we need to ask: what does it mean for the cohomology to be "equivariant K-theory"?

Every equivariant K-theory group is indeed a module over the K-theory of a point. However, we might extend this by allowing $h^\bullet_G$ to take values in coherent sheaves over some space. For example, in the case of elliptic cohomology, we have

$$h^\bullet_{G,\mathrm{ell}} = \text{sheaf of } \mathrm{Ell}_G(\mathrm{pt}),$$

which for $G = \mathrm{U}(1)$ corresponds to a sheaf on the elliptic curve E.

To clarify, K-theory is a module over a ring. But in the context of elliptic cohomology, this is just a special case of a sheaf on an algebraic variety. When dealing with non-affine algebraic varieties, such as an elliptic curve, it becomes natural for cohomology to be represented as a sheaf rather than simply as a vector space.

Thus, cohomology doesn't necessarily take values in vector spaces. It can instead take values in more general structures, such as bundles over other spaces or sheaves on spaces. This is an important distinction, especially when studying non-affine varieties like elliptic curves, where cohomology is often more naturally represented by sheaves.

4.5.2 Elliptic Analogs

Let's look at some elliptic analogs of our previous work: Let G be a compact group, e.g., $G = \mathrm{U}(1)$. We define the elliptic cohomology of a point as:

$$\mathrm{Ell}_T(\mathrm{pt}) = E^r,$$

where E^r denotes a certain space (often associated with the cohomology of an elliptic curve or a related structure). Here, we are not merely referring to rings, but to the objects that these rings define. Specifically, a ring can be thought of as the ring of functions on an algebraic variety, and this variety is an algebraic object that we consider.

Additionally, we have:

$$\mathrm{Ell}_T(\mathbb{P}(V)) = \{\text{one of the } a_i\text{'s is equal}\} \subset E^r \times E,$$

where $a_i \in E^r$ and $t \in E$. This structure is invariant under the action of a group, specifically the Weyl group $S(1)$ in this context. These correspond to ordered r-tuples; when we consider unordered r-tuples, we obtain:

$$\mathrm{Ell}_{\mathrm{U}(V)}(\mathbb{P}(V)) = \{\text{one of the } a_i\text{'s is equal}\} \subset S^r E \times E.$$

We can think of

$$\mathrm{Ell}_{\mathrm{U}(R)}(\mathrm{pt}) = S^r E$$

as a definition, but we still want to think of an element $g \in S^r E$ as somewhat analogous to an element in a group. Hence, there should be some kind of map from the group element to this structure.

To motivate this, consider identifying the unit circle with another circle, where $|z| = 1$ is mapped to a circle of radius $|q|$, where q is typically less than one. The identification is achieved by multiplication by q. This process defines our elliptic curve.

If g is an element of a group G, we can construct a vector bundle over the elliptic curve by considering sections of a vector bundle that satisfy the equation:

$$f(qz) = g \cdot f(z),$$

where f is a section of the bundle. Thus, each group element defines a vector bundle over the elliptic curve. The classification of these bundles is up to conjugation, and the degree of the bundle is zero because the transition function is constant. Therefore, $S^r E$ can be viewed as the space of semistable $\mathrm{GL}(r)$-degree 0 bundles on E.

In the lectures by Nikita Nekrasov, we have encountered pictures where one traverses a circle and experiences a monodromy represented by g. This idea is analogous, though in a lower-dimensional context. For K-theory, we can express it as:

$$K_G(X) \approx H^0_{\mathrm{eq}}(\mathrm{Maps}(S^2 \to X/G)),$$

which is equivariant with respect to the parameter q. If we move to one higher dimension, we have elliptic cohomology:

$$\text{Elliptic Cohomology} \approx H^0(\mathrm{Maps}(E \to X/G)).$$

Although a precise treatment of these objects is outside the scope here, we can interpret computations in elliptic cohomology as calculations in this framework. However, it is often more convenient to work with the direct definitions.

To proceed, we need to address some general constructions, including the general statement of the Thom isomorphism.

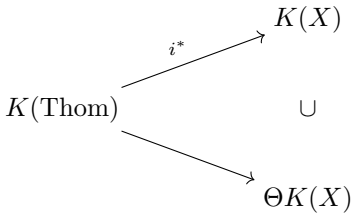
4.5.3 The Thom Isomorphism

Let X be a space and V a complex vector bundle. The Thom isomorphism states:

$$K_G(\text{Thom}(X \hookrightarrow V)) \simeq K_G(X),$$

where the left-hand side represents the K-theory of vector bundles over a neighborhood of the zero section of V, which are exact away from the zero section. Within this framework, we can introduce an element, namely the Koszul complex $\Theta = \sum(-1)^i \wedge^i V^*$. The Thom isomorphism asserts that this Koszul complex is a generator.

To express this geometrically, consider the inclusion map:



We have a map from complex vector bundles of rank r to $\text{BU}(r) = \text{Gr}(r, \infty)$, where r is the rank of the tautological bundle $= \mathbb{C}^r$. Therefore, $\Theta(V) = \gamma^*\Theta_{\text{universal}}$, where the universal Theta class is given by:

$$\Theta_{\text{universal}} = \prod_{i=1}^{r} \left(1 - a_i^{-1}\right),$$

and a_i are the Chern roots of the universal bundle.

This expression suggests that a matrix fixes a vector if the vector is an eigenvector of the matrix. In other words, if a matrix has an eigenvalue, the equation implies that the matrix fixes the corresponding eigenvector.

4.5.4 The Theta Divisor

In the context of elliptic cohomology, we encounter the concept of the Theta divisor inside the symmetric powers of an elliptic curve $\Theta \subset S^r E$. This divisor is distinguished by the property that one of the variables (or points) is zero, or lies at the origin. In terms of the corresponding bundle over the elliptic curve, this condition implies that the bundle has a subsection.

This concept plays a crucial role in integrable systems. Many spaces in integrable systems are moduli spaces of bundles or similar objects, and these moduli spaces often contain canonical divisors. While these divisors are not canonical in the traditional sense, they are distinguished because they correspond to objects with an extra subsection — this often occurs when a determinant vanishes or a similar condition holds. Such divisors are referred to as Theta divisors. For example, on an elliptic curve, the Theta divisor is the only degree zero bundle

that has a subsection. Theta divisors are also encountered in the context of degree $g-1$ bundles that have a subsection, as discussed in earlier lectures. These subsections are typically characterized by determinants or Theta functions.

The key takeaway here is that the elliptic analog is both similar and distinct from other cases. For instance, there is a Thom isomorphism in elliptic cohomology. However, unlike the standard Thom isomorphism, which is an actual isomorphism, the elliptic version indicates that the object in question is a rank-one module that is locally free but not globally trivial.

Practically speaking, when we consider the pullback in elliptic cohomology, it behaves differently from the standard case. Instead of being globally trivial, it becomes a line bundle, and the global structure introduces non-trivial factors. For example, if we take a product of Theta functions:

$$\prod_{i=1}^{r} \theta(a_i),$$

this product vanishes precisely when one of the variables is equal to 1 or 0, depending on how we choose to represent the origin on the elliptic curve. Importantly, this product is not just a function — it is actually a subsection of a line bundle, and this subsection changes when we shift the periods or adjust the variables by a parameter q.

To clarify further, if we alter the periods or shift a variable by q, this introduces a non-trivial factor. This is a significant point: formulas in elliptic cohomology often involve Theta functions, but these functions are not ordinary functions. Rather, they represent subsections of a line bundle. The meaning of these subsections and line bundles arises from the fact that they define a locus where one of the variables is zero, which corresponds to a non-trivial divisor in the product of the elliptic curve that defines a non-trivial line bundle.

4.5.5 Koszul Complex and Pushforward Map

Next, we define the Koszul complex in the context of Theta functions. For example, we may encounter an expression like:

$$\prod_{n=1}^{\infty} (1 - q^n z)(1 - q^n z^{-1}),$$

which can be interpreted as analogous to the Koszul complex, albeit with some correction factor arising from the context of loops. Essentially, when q acts like a rotation of a loop by some parameter, this structure naturally emerges.

With the Koszul complex defined, we can proceed to define the pushforward map. This leads us to the concept of a "wrong-way" map in algebraic geometry. Suppose we have a variety X embedded in some ambient space Y. If we consider the normal bundle of X inside Y, and this normal bundle has a complex

structure (not necessarily holomorphic), we can collapse certain points in Y to map it to the Thom space of X:

$$Y \to \text{Thom}(X \hookrightarrow Y).$$

This construction allows us to define a push-forward map in K-theory:

$$K(Y) \overset{\leftarrow}{i_*} K(X).$$

In algebraic geometry, if X is a holomorphic submanifold within a holomorphic manifold, we can take the structure sheaf of X and push it forward. However, in the topological setting, this is handled as described above.

he key property of this pushforward map is:

$$i^* i_* = \text{multiplication by } \Theta(V).$$

This means that when we pushforward using the Koszul complex and then pull it back, we obtain the class of the Koszul complex.

4.5.6 Complex Oriented Maps

When considering a general map, the concept of "complex oriented" comes into play.

Definition 4.29. *A map is said to be **complex oriented** if we can embed X into a trivial bundle $\mathbb{C}^N \times Y$, and subsequently collapse it into the Thom space of X within its normal bundle:*

$$\mathit{Thom}(Y \to Y \times \mathbb{C}^N) \to \mathit{Thom}(X \to N).$$

This map's pullback is functorial, and by applying the Thom isomorphism, we obtain the identities:

$$\text{Thom}(Y \to Y \times \mathbb{C}^N) = K(Y) \quad \text{and} \quad \text{Thom}(X \to N) = X,$$

resulting in a map $K(X) \to K(Y)$.

In the context of algebraic geometry, this notion enables us to pushforward objects. For instance, given a subvariety, it defines a coherent sheaf on the ambient space, which can then be pushed forward. Similarly, if a sheaf is defined on a variety that maps to another variety or point, we can also push the sheaf forward.

Let's unravel this a bit more.

Proposition 4.30. *Let $N_f = N_X - \mathbb{C}^N$. Then we have the following map:*

$$\Theta(-N_f) \to \Theta_{\mathit{Ell}(r)} V,$$

where $\Theta(-N_f)$ is a sheaf on $\mathit{Ell}(X)$, assuming that N_{X/X^A} exists.

Remark 4.31. *In general, if X is a topological space, and we consider the inclusion of a subspace into a larger space followed by the contraction of its neighborhood, the resulting space will typically not be the total space of a bundle. While the space might still be manageable, it would not necessarily have a description via a Thom isomorphism. For example, consider the inclusion of a singular space into another singular space, and then contracting the complement. The resulting space could behave unpredictably. Hence, it is crucial for the space to have a normal bundle. When this condition is satisfied, the structure behaves as expected. Otherwise, a more detailed study is required.*

4.5.7 Examples

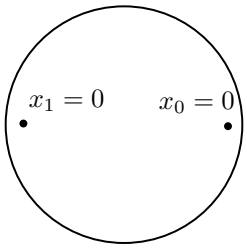
Let's discuss the equivariant K-theory of projective spaces:

Example 4.32. *Consider the projective space $\mathbb{P}^n = \{[x_0 : x_1 : \cdots : x_n]\}$ with an action given by $A = diag(a_0, a_1, \ldots, a_n)$. The equivariant K-theory of this projective space is:*

$$K_A(\mathbb{P}^n) = \mathbb{Z}[a_i^{\pm 1}] / \prod (1 - a_i^{-1} t^{-1}).$$

Let's look at another example: the Riemann sphere, where $n = 1$.

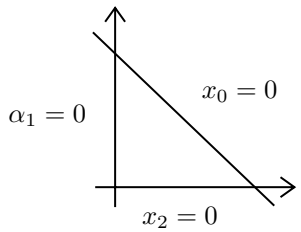
Example 4.33. *When $n = 1$, the projective space is a Riemann sphere with two points: one at 0 and the other at ∞.*



The equivariant K-theory class corresponding to $x_1 = 0$ vanishes when $1 - a_1^{-1} t^{-1} = 0$, while the class corresponding to $x_0 = 0$ vanishes when $a_0 - a_0^{-1} t^{-1} = 0$. The vanishing of the product tells us that these two points do not intersect.

Let's discuss the projective plane, when $n = 2$:

Example 4.34. *For $n = 2$, the projective plane consists of two perpendicular lines, one of which intersects both at infinity:*



Similarly, each of these lines corresponds to a divisor in the projective plane. The class of $\alpha_1 = 0$ is $1 - a_1 t$. Thus, the equation becomes:

$$\prod_{i=0}^{\infty} \left(1 - a_i^{-1} t^{-1}\right) = 0.$$

By analogous reasoning, instead of computing the class of a line, we can compute the class of a point. For instance, at the origin, the class is given by:

$$\left(1 - a_1^{-1} t\right) \left(1 - a_2^{-1} t^{-1}\right).$$

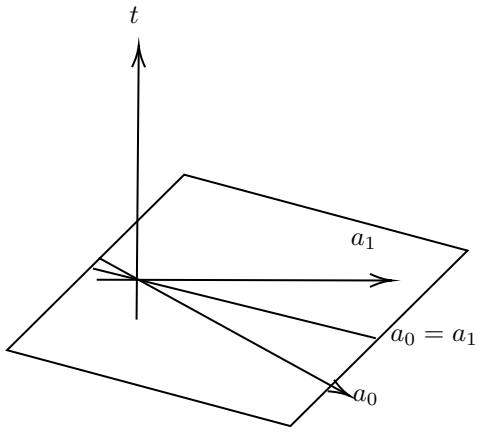
We can verify this result by examining the Koszul complex, which serves as the resolution of the structure sheaf at that point. The expression above corresponds to the character of this complex, confirming its correctness.

4.5.8 Geometric Objects

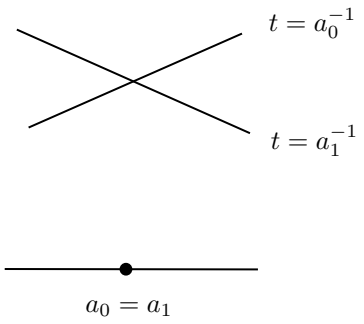
Since we want to study equivariant elliptic cohomology, we don't have a ring, but rather a geometric object. Therefore, we aim to consider the variety that arises from the equation:

$$K_A\left(\mathbb{P}^n\right) \mathbb{Z}\left[a_i^{\pm 1}\right] / \prod \left(1 - a_i^{-1} t^{-1}\right) = 0.$$

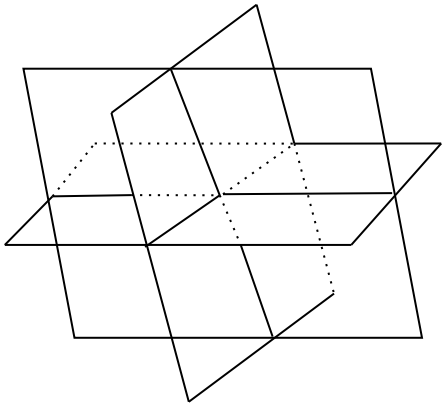
This gives us Spec$K_A(\mathbb{P}^1)$:



If we imagine this in terms of an additive group instead of a multiplicative group, we have two planes that intersect along a line where $a_0 = a_1$. This is the projection Spec$K_A(\mathbb{P}^1) \to$ SpecK_A(pt):



If we do the same picture for 3 planes, we get:



For instance, in the $n = 2$ case, the origin corresponds to $t = \mathcal{O}(1)$, the functions on the tautological line. Here, a_0 acts on $(\mathbb{C}, 0, 0)$, so a_0^{-1} acts on a_0^{-1}, and the origin corresponds to when $t = a_0^{-1}$. Similarly, the bottom right of the triangle corresponds to $(0, \mathbb{C}, 0)$ where $t = a_1^{-1}$, and so on.

This process of gluing and intersecting planes is a familiar concept in toric geometry, where the points correspond to planes, and the geometric structure can be interpreted in terms of fans, with the dimension of the objects being preserved.

Several important lessons emerge from this:

- The derived fiber over $a \in A$ is given by $K(X^a)$. A standard example is the case where $a = 1$, corresponding to $K^0_{S^1}(S^1)$. This is significant because it illustrates how equivariant geometry encapsulates the ordinary geometry of the fixed loci of all subgroups.
- Away from certain subvarieties of the form $a_i/a_j = 1$ (which can be generalized to character$(A) = 1$), it holds that $K_A(X) = K_A(X^A)$. This follows from the earlier statement, as a generic element of the torus will

have a fixed locus that corresponds to the generic fiber. We thus have

$$K_A(X^A) \underset{i_*}{\overset{i^*}{\rightleftarrows}} K_A(X)$$

where the top part is the pullback, which is functorial, and the bottom part is the pushforward. Concretely, we have:

$$i^* : f(t, a) \to (f(a_0^1, a), f(a_1^{-1}, a), \dots),$$

and

$$i^* : (1, 0, \dots, 0) \to \prod_{i>0} (1 - t^{-1} a_i^{-1}).$$

Furthermore, by computing $i^* i_* = \Theta(N_{X/X^A})$, where N represents the normal bundle, we observe that the weight of the action is given by $\prod_{i>1}(1 - a_0/a_i^{-1})$.

4.5.9 The Localization Theorem

Theorem 4.35 (The Localization Theorem)**.** *This map is an isomorphism away from some subvariety. More algebraically, the kernel and cokernel of both maps are torsion, meaning they are annihilated by some nontrivial element in A.*

Proof. Consider the sequence:

$$K_A(\mathrm{Thom}(X^A \hookrightarrow X)) \to K_A(X) \to K_A(X \setminus X^A).$$

The first map, i_*, is injective because $i^* i_*$ is defined using multiplication by $\Theta(N)$, which has torus weights, resulting in a nontrivial polynomial that does not divide zero. Hence, this map is injective.

The term $K_A(X \setminus X^A)$ is constructed from A/A', where $A' \neq A$. However, $K_A(A/A') = K_{A'}(\mathrm{pt})$, which is torsion. This concludes the proof. □

This result is particularly powerful because it tells us that, if we can compute the K-theory for a vector bundle map to a point, we can compute the pushforward to the point. Specifically, if $V \in K_A(X)$, we have

$$\begin{array}{ccc} K_A(X) & \ni & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ K_A(\mathrm{pt}) & \ni & \chi(X, V) \end{array}$$

In topology, the Euler characteristic is defined earlier. Algebraically, the ordinary Euler characteristic can be expressed using the $\overline{\partial}$-complex, which is a

complex of differential operators. In the realm of pseudo-differential operators, we can deform at infinity as long as we have some kind of deformation. To understand what it means to apply the $\overline{\partial}$-operator, we need a notion of multiplication by I.

We can expand the diagram into the following:

$$\begin{array}{ccccc} K_A(X^A) & \xrightarrow{i_*} & K_A(X) & \ni & V \\ & \searrow & \downarrow{\scriptstyle p} & & \downarrow \\ & & K_A(\mathrm{pt}) & \ni & \chi(X,V) \end{array}$$

Proposition 4.36.

$$\chi(X,V) = \chi(X^A, \frac{i^*V}{\Theta(N)}).$$

This result is crucial, particularly in the context of Nekrasov's lectures, when the fixed locus is isolated. Although the expressions involved can be intricate, their geometric interpretation is clear: the moduli space has a group action, and its isolated fixed points correspond to partitions. While the computations can be performed over the entire space, the most significant insight comes from focusing on the fixed points.

At this point, one might think that geometry is no longer necessary, as we can simply substitute expressions and apply the formula. However, geometric insight is still essential, especially when analyzing the underlying structure of the space. When working with algebraic expressions, it is important to consider aspects like poles and support. The geometric framework provides the necessary tools to handle these considerations effectively. For instance, geometry helps us conclude that certain meromorphic functions are actually regular functions.

4.6 Elliptic Quantum Groups

4.6.1 Krichever Genera

Let X be a manifold, and consider its tangent bundle TX. If X is a complex manifold, then TX is a complex tangent bundle. However, X does not need to be a complex manifold for its tangent bundle to be complex. All that is required is some method of multiplying by I within the bundle.

Since adding a trivial bundle does not affect the structure, it is often more convenient to consider $TX \oplus \mathbb{P}^N$ with a complex structure, which possesses Chern classes. These manifolds are referred to as **stable almost complex manifolds**. In particular, we may consider $\Theta(TX)$.

The goal of Krichever genera is to generalize expressions like $\frac{\theta(xz)}{\theta(x)\theta(z)}$. To achieve this, we set $z = \exp(2\pi i \epsilon_3) \in \mathrm{U}(1)$ or $\mathbb{C}^\times$, which acts on the bundle by multiplication. We then tensor the tangent bundle with the defining representation,

yielding a new bundle $\Theta(TX \otimes z)$. Since scalars act on any vector space, they similarly act on any vector bundle. Consequently, if we take any bundle, we can act on it by multiplying by a scalar from U(1), a group of scalars. Thus, the characteristic classes of the bundle are now multiplied by z.

Definition 4.37. *The **Krichever genus** is*

$$Kr(X) = (X \to pt)_* \Theta(TX \otimes z).$$

This genus depends on z and the group already acting on X. In Nekrasov's lectures, we encountered objects of the form:

$$\sum \prod_{\square \in \dots} \frac{\theta(\dots)}{\theta(\dots)},$$

which correspond to the Krichever genus of the moduli space of framed rank N instantons on $\mathbb{C}^2$.

This applies to more specific scenarios, such as the framing of a vector bundle. The key distinction here is that the underlying space is non-compact, making the function particularly interesting. According to the Krichever rigidity theorem, under certain conditions, the Krichever genus no longer depends on the group already acting on X.

Next, let us discuss the K-theory analog. In K-theory, we have:

$$\Theta(TX \otimes z) = 1 - z^{-1}T^* + z^{-1}\Lambda^2 T^* - \dots$$

and

$$\chi(\Theta) = \sum_k (-z^{-1})^k \chi(X, \Lambda^k T^*),$$

where $\Lambda^k T^*$ are the holomorphic k-forms. If X is a complex Kähler manifold, we can compute its topological cohomology using the topological cohomology of the holomorphic k-forms. Thus, we have $X \hookrightarrow H_{\text{Top}}(X, \mathbb{C})$.

Let's look at an example:

Example 4.38. *Consider $H^0_{Top}(\mathbb{P}^1, \mathbb{C}) = \mathbb{C}^0 \oplus \mathbb{C}^2$, where $\mathbb{C}^0$ corresponds to $H^0(\mathcal{O}_{\mathbb{P}^1})$ and $\mathbb{C}^2$ corresponds to $H^1(T^*_{\mathbb{P}^1}) = \mathcal{O}(-2)$, which has a 1-dimensional first Chern class. For any automorphism of $\mathbb{P}^1$, the action on this bundle cannot change the topological cohomology. Specifically, for a connected group acting on the manifold, the topological cohomology must remain trivial because the action simply permutes cycles. Therefore, $H^1(T^*_{\mathbb{P}^1})$ is the trivial module for $Aut(\mathbb{P}^1)$.*

A simpler proof follows: to show that a function is constant on a group, it suffices to prove this for any one-parameter subgroup. Similarly, to demonstrate that a representation is trivial, it is enough to show that it is trivial for any one-parameter subgroup. In this case, we have a trivial action. Let $S^1 \hookrightarrow X$, and let a be a coordinate ring in $A = \mathbb{C}^*$ and a component of S^1. To compute the Euler

characteristic of X, we apply localization, which gives the Euler characteristic of the fixed locus:

$$\chi(X, \Theta(TX \otimes z)) = \chi\left(X^A, \Theta(TX^A \otimes z)\frac{\Theta(N \otimes z)}{\Theta(N)}\right).$$

The tangent bundle of X, when restricted to the fixed locus, decomposes as:

$$TX|_{X^A} = TX^A \oplus N.$$

Notice that a only acts on the fraction. This implies that in K-theory, the fraction becomes:

$$\prod \frac{1 - w_i^{-1} \cdot a^k \cdot z}{1 - w_i^{-1} a^k},$$

for $k \neq 0$. Since X is compact, it follows that X is a finite-dimensional representation of A, and more specifically, the character is a Laurent polynomial in a. Furthermore, we observe that:

$$\prod \frac{1 - w_i^{-1} \cdot a^k \cdot z}{1 - w_i^{-1} a^k}$$

has a finite limit as $a \to 0$ and $a \to \infty$. These facts together imply that the product is constant, and the proof is complete.

We can draw further insights from this proof. For the limit:

$$\lim_{a \to 0} \prod \frac{1 - w_i^{-1} \cdot a^k \cdot z}{1 - w_i^{-1} a^k} = \begin{cases} 1 & \text{if } k > 0, \\ z & \text{if } k < 0. \end{cases}$$

and the product becomes $z^{\#\text{attracting directions in } N}$.

In the limit:

$$\lim_{a \to \infty} \prod \frac{1 - w_i^{-1} \cdot a^k \cdot z}{1 - w_i^{-1} a^k} = \begin{cases} z & \text{if } k > 0, \\ 1 & \text{if } k < 0. \end{cases}$$

and the product becomes $z^{\#\text{repelling directions in } N}$. Thus, in the normal bundle, some directions will be attracted to the fixed point, while others will be repelled. This mirrors the theory of moment maps, where critical points and the structure of the descent manifold help us understand the topology of the manifold. Despite knowing that the function is constant in a, we can still use this action to gain valuable insights into the function's behavior.

Theorem 4.39 (Krichever, 1990; Höhn, 1991)**.** *Suppose that $c_1(TX) = 0$ in $H^2(X, \mathbb{Z})$ or $N|c_1(TX)$ and $z^N = 1$. Let X be a compact stably almost complex manifold. Then $Kr(X)$ is a function of z only.*

Proof. We begin by considering the theta bundle, which is associated with the tangent bundle of the manifold X of dimension n. The theta bundle is represented by the following diagram:

$$\begin{array}{ccc} TX & & \text{taut} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\gamma} & \mathrm{BU}(n) \\ & \searrow & \\ & & \mathrm{BSU}(n) \end{array}$$

where γ is a Gauss map. The theta bundle is pulled back from elliptic cohomology, as shown below:

$$\begin{array}{ccccc} & & \Theta(TX) & & \\ & & \overset{\gamma^*\Theta}{\nwarrow} & & \\ E = \mathrm{Ell}_{\mathrm{U}(1)}(\mathrm{pt}) & & \Theta(\mathrm{Taut}) & = & \mathcal{O}(\text{one point is the origin}) \\ \uparrow & \swarrow & & & \\ \mathrm{Ell}_{\mathrm{U}(n)}(\mathrm{pt}) & & = & & S^n E \\ \uparrow & & & & \cup \\ \mathrm{Ell}_{\mathrm{SU}(n)}(\mathrm{pt}) & & = & & n\text{-tuples that sum to 0} \\ \uparrow & & & & \\ 0 & & & & \end{array}$$

ere, the $\mathrm{BU}(n)$ theory corresponds to $\mathrm{Ell}_{\mathrm{U}(n)}(\mathrm{pt})$, and the $\mathrm{BSU}(n)$ theory corresponds to $\mathrm{Ell}_{\mathrm{SU}(n)}(\mathrm{pt})$.

We are pulling back from the locus of n-tuples that sum to 0, which is a projective space $\mathbb{P}^{n-1}$. Explicitly, this map is given by:

$$\mathbb{P}^{n-1} \hookrightarrow S^n E \xrightarrow{\sum p_i} E \longrightarrow 0$$

where $\mathbb{P}^{n-1}$ represents divisors linearly equivalent to n[origin], and $S^n E$ represents the divisors of degree n (denoted as $p_1, \ldots, p_n$).

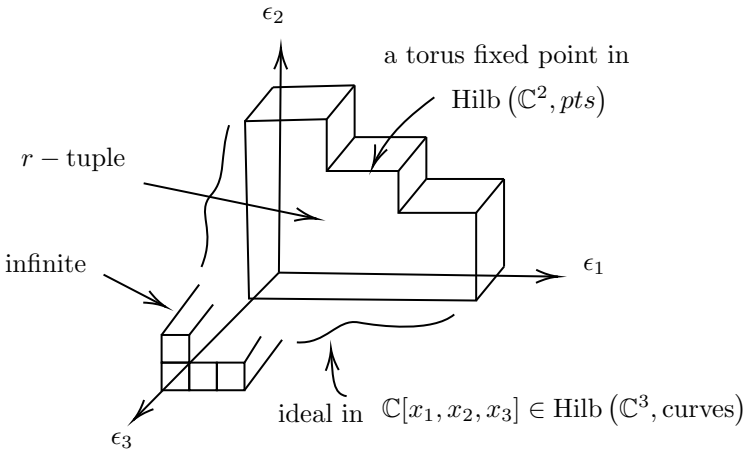
A key property of projective space is that all line bundles on projective space are discrete families. If we have a continuous family of bundles of the same degree, they are actually the same bundle. This implies that the bundle $\Theta(TX \otimes z)$, pulled back from elliptic cohomology, is a specific bundle that cannot be deformed by z, because it is a bundle over projective space. Since bundles on projective space cannot be continuously deformed into distinct bundles, we conclude that $\Theta(TX \otimes z)$ is the same as Θ_Z, as the structure does not change in projective space.

Thus, the bundle Θ_{TX} for some space X is equivalent to the bundle Θ_Z, and the map $(X \to \mathrm{pt})_*(TX \otimes z)$ is a section of a trivial bundle, which means it is constant. This completes the proof.

□

4.6.2 Partitions

In Nekrasov's lecture, we discussed r-tuples of a partition that was related to do with the moduli spaces of bundles, instantons, shifts, or in general just something that lives on a plane with two coordinates, ϵ_1, ϵ_2. Then, we introduced a defect along one of the lines. However, there is another direction in which one can proceed, ϵ_3:



The r-tuples of such objects correspond to torus fixed points in m-sheaves of rank 2 on $\mathbb{C}^3$. This framework arises within the context of Donaldson-Thomas theory, which is primarily concerned with the enumeration of geometric objects—such as curves, points, and vector bundles—on threefolds.

4.6.3 R-Matrices

Consider

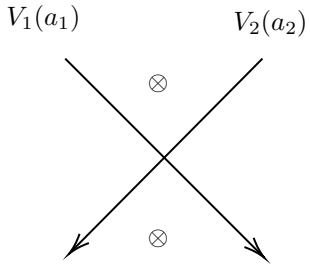
$$Z\left(\ydiagram{3,1},\ydiagram{2,2}, z, t_1, t_2, t_3, a_1, \ldots, a_r\right) = \sum z^{\text{vol}} \frac{\cdots}{\cdots}$$

where $\ydiagram{2,2}$ represents an r-tuple of partitions within the space

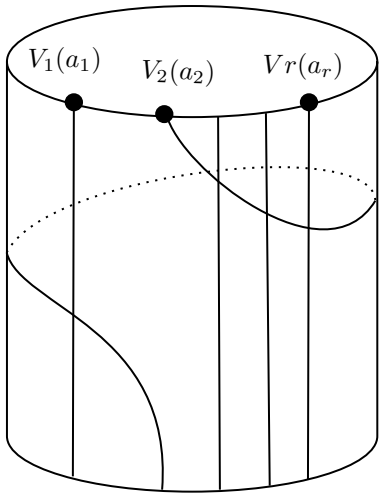
$$\text{Fock}(a_1) \otimes \text{Fock}(a_2) \otimes \cdots \otimes \text{Fock}(a_r),$$

which is a solution to the quantum Knizhnik-Zamolodchikov (qKZ) equations for $U_\hbar(\hat{\hat{\mathfrak{gl}}}(1))$, with $\hbar = t_1 t_2$, and the corresponding dynamical equation in z.

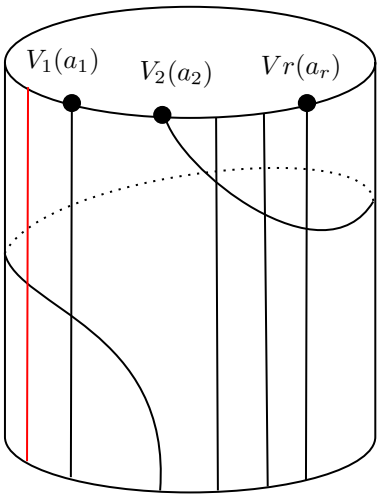
Writing out the qKZ equations in full detail would occupy many pages, even in the simplest of examples. However, the situation becomes much more manageable when viewed through a graphical lens. For instance, one can consider the R-matrix



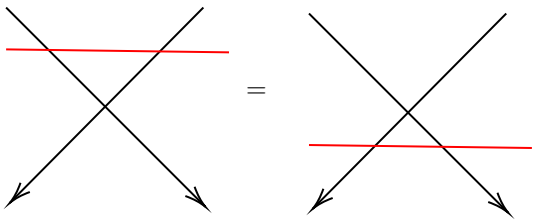
Now, imagine we have a cylinder and a collection of representations $V_1(a_1), V_2(a_2), \ldots, V_r(a_r)$. While almost of these representations remain unaffected, one of them undergoes a transformation:



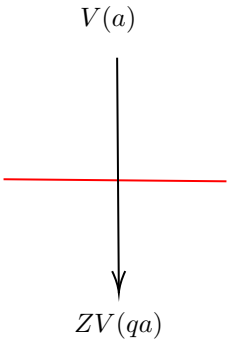
When working with R-matrices, the orientation of the system plays a crucial role. While symmetries may exist that permit changes in orientation, the system is typically fixed with a specific orientation. In our case, all lines are directed downwards. This leads to the concept that in such systems, there can be an additional operator that governs the transformation of the space as it crosses a boundary:



This operator governs the behavior of the space as it crosses a line. The goal is to express this operator in a form that mirrors the structure of the Yang-Baxter equation. The core idea is that the crossing can be shifted either before or after an action, which corresponds to performing the action in the tensor product of two representations. Crucially, this operator must commute with the R-matrix.



In general, the R-matrix could be an arbitrary function. But in our case, R only depends on a_1/a_2. For example, we can take the operator to be a shift:



We can apply a shift using a difference operator, as the matrix depends solely on the differences between elements. Acting on this space with any element, denoted by Z (which represents an element of a quantum group), we obtain the

condition

$$[Z \otimes Z, R] = 0.$$

For example, we can choose Z to be an element of the Cartan subgroup of the quantum group. The Cartan subgroup is analogous to the maximal torus in Lie groups and consists only of diagonal matrices. In the case of a quantum group, the Cartan subgroup behaves like a one-dimensional torus, represented by diagonal matrices. For our purposes, z, an element of the Cartan subgroup of $\mathcal{U}_\hbar(\hat{\hat{\mathfrak{gl}}}(1))$ (which depends on t_1, t_2), acts by z^{vol}.

If we expand this equation, it can be viewed as a linear q-difference equation:

$$\Psi(a_1, qa_2, a_3, \dots) = \mathrm{Matrix}(\dots)\Psi(a_1, a_2, a_3, \dots),$$

where $q = t_3$. This equation corresponds to a genus of disjoint union spaces, summed with weights z^{vol}. Consequently, objects with different volumes belong to distinct moduli spaces, and we sum over discrete data. The moduli space in question consists of connected components, each associated with the degree of the curve. This degree is determined by the sizes of the partitions and the volume of the object. In a manner similar to Nekrasov's work, partitions of different sizes belong to different moduli spaces. Here, we partition the genus associated with the space.

One can imagine decomposing the space into smaller pieces and computing the genus for each piece.

Theorem 4.40. *The q-KZ equations are monomial in z.*

This suggests that a vast number of conditions must hold simultaneously for the result to be valid. The ratio of the left and right Ψ's should correspond to the ratio of one matrix series divided by another. However, upon closer inspection, each matrix coefficient turns out to be a monomial. Regardless of the form of the R-matrix, when expressed in this manner, every coefficient is indeed a monomial. Although this vanishing may initially appear trivial, it reflects an underlying conservation law and is of significant importance from a geometric perspective. This vanishing stems from rigidity.

What type of rigidity can we invoke? For K-theory, AH rigidity is always effective, as it applies to the construction of the R-matrix. However, this is not always the case for elliptic cohomology. Krichever rigidity requires $c_1 = 0$ or $N|c_1$, with $(\text{parameter})^N = 1$.

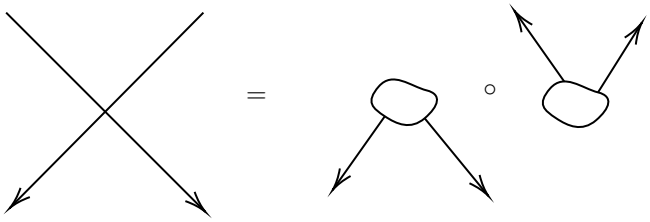
As demonstrated in Nekrasov's lecture, it is more fruitful to view the application of the Krichever genus not as a process on a moduli space, but as an integration of 1 over a derived moduli space. In the lecture, the moduli space was smooth, but upon adding a cotangent bundle, it became a zero-dimensional derived object, and we compute its fundamental class. This situation is unavoidable in our current context.

ere, we face a moduli problem, and the tangent space to this problem is not a typical vector space. Instead, it is a more singular object that lacks a well-defined tangent or cotangent bundle. Instead, we have the difference between these structures, which leads to two terms: deformations and obstructions. Nekrasov discussed a similar scenario, where the Krichever genus can be understood as follows: every bundle or sheaf over a manifold has automorphisms; it is a linear object, and scaling is always possible. There is a parameter æ associated with this process (related to the canonical class of c_3):

$$T \text{ Moduli} = \text{Deformations} - æ \cdot (\text{Deformations})^*.$$

In the case under consideration, $æ = t_1, t_2, t_3$, as these correspond to the canonical class. In a more general scenario for a 3-fold, æ corresponds to the canonical class. By examining this space, we can deduce the first Chern class: $c_1 = 2c_1(\text{Deformations})$. While no bundle exists over the deformation, if one did, it would take the form given above. Moreover, we know that $c_1 \neq 0$, but $2|c_1$. A full-fledged version of the Krichever genus for this space is not feasible, but we can still proceed if we require that $æ^2 = 1$.

The R-matrix in this context marks the beginning of a lengthy and intricate story. If we consider the framework of quantum field theory, where two particles possess internal degrees of freedom in two spaces, one may interact by coming together. This interaction suggests the presence of a more fundamental operator governing the particles' behavior when they meet. Specifically, there exists an operator that describes the interaction when these particles collide, after which the particles may decay into other entities:



where the two objects on the right are transposes of one another. Geometrically, the following equivalence holds:

$$\text{Fock}(a_1) \otimes \cdots \otimes \text{Fock}(a_r) = K_{\text{eq}}(\text{Hilb}(\mathbb{C}^2, \text{pts})),$$

where $K_{\text{eq}}(\text{Hilb}(\mathbb{C}^2, \text{pts}))$ represents the equivariant K-theory of the Hilbert scheme of points on $\mathbb{C}^2$. More generally, we have the map

$$K_{\text{eq}}(M(r_1)) \otimes K_{\text{eq}}(M(r_2)) \to K_{\text{eq}}(M(r_1 + r_2)),$$

where $K_{\text{eq}}(M(r_i))$ denotes sheaves of rank r_i on $\mathbb{P}^2$, and $M(r_i)$ represents the moduli space of sheaves with rank r_i. This is reminiscent of a scenario in which two particles might interact but do not, emphasizing that to ensure something meaningful occurs in this map, the operation must possess additional structure or non-triviality that makes it interesting.

$$M(r_1) \times M(r_2) \overset{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}}{\hookrightarrow} M(r_1 + r_2),$$

where the map is given by a direct sum of bundles. This operation effectively combines two bundles into one, resulting in an object that is fixed under the group action. This situation mirrors the discussion in Nekrasov's lecture, where the action of $\mathrm{GL}_{\mathbb{R}}$ on the moduli space of sheaves involves a diagonal action. Though the diagonal action might initially seem trivial, it is, in fact, crucial. The action is non-trivial on each sheaf, as the matrix

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

is fixed by the group element

$$\begin{pmatrix} u & & & & & \\ & u & & & & \\ & & u & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \subset \mathrm{diag}(a_1, a_2, \dots),$$

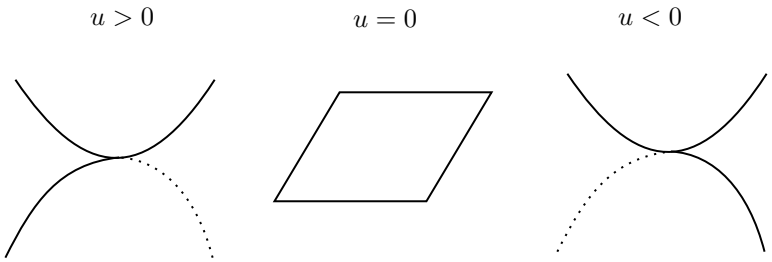
where the diagonal matrices $\mathrm{diag}(a_1, a_2, \dots)$ form a subgroup of GL_n, and the matrix represents the fixed locus of the group action. The direct sum of these sheaves is therefore preserved by the critical moment map associated with this group.

In mathematical physics, there is often a distinction between those who focus on the dynamics of a system and those who study its equilibrium properties. In particular, some scholars focus on the critical loci of functions, which describe equilibrium states. For instance, one might study a system by introducing a moment map associated with a torus action. This arises naturally when kinetic energy terms are added to systems exhibiting rotational symmetry, thus introducing the corresponding moment map for the torus action.

To clarify, let us consider a situation where we have a function, with u representing an element of the Lie algebra associated with the moment map. The moment map in this case might be expressed as the function

$$u(|x_1|^2 - |x_2|^2 \dots),$$

which is defined in terms of the difference of the squares of the coordinates. As we vary the parameter u, we observe that the critical locus of the function can change in a dramatic manner, leading to significant alterations in the system's behavior:

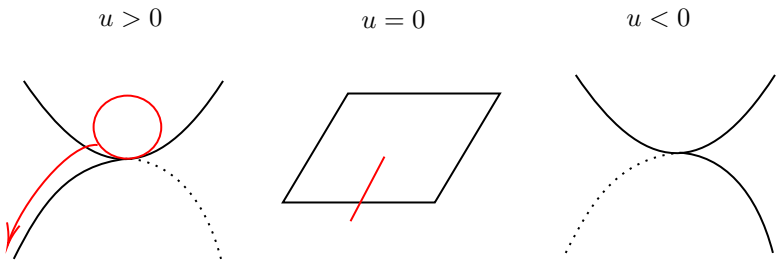


We are interested in defining a map

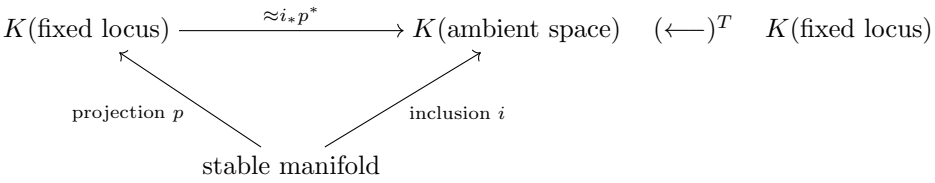
$$K(\text{fixed locus}) \to K(\text{ambient space}) \xleftarrow{T} K(\text{fixed locus}),$$

which carries significance both mathematically and physically. Mathematically, this map describes a transition between the K-theory of the fixed locus and that of the ambient space. Physically, this map can be interpreted as a potential term that undergoes a sign change, resulting in a shift of the critical locus. Such a shift has important consequences, as it alters the nature of the critical points of the function. Specifically, it transforms minima into maxima and vice versa, fundamentally changing the system's behavior.

To illustrate this concept, imagine placing a ball at the center of the parabola when $u > 0$. As the ball rolls downward along the left side of the parabola, it follows a path dictated by the critical points of the function. In the special case where $u = 0$, the trajectory of the ball becomes a straight line. This scenario underscores how variations in the parameter u influence the geometry of the system and shift the critical locus, thereby changing the system's dynamics.



Mathematically, we want a descending manifold that projects onto the fixed locus, since by definition, a descending manifold is a set that evolves towards a fixed locus. From this, we can deduce the stable manifold:



4.6.4 Interpolation Problems

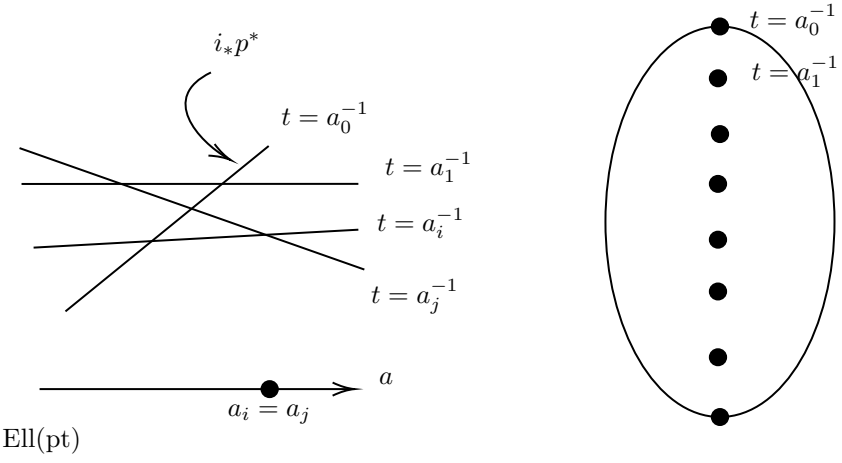
Now, let us examine the geometric and physical challenges inherent in this definition. The main issue lies in the fact that the descending or stable manifold is not closed within the ambient space. The inclusion map is not closed, and consequently, there is no strict pushforward. While the manifold may appear closed locally, if we allow the ball to continue rolling, it could end up in an entirely different configuration. In algebraic geometry, the process of taking closures is often acceptable for abstract reasoning, but it does not always provide practical insight. This is because we may not be able to ascertain what lies in the closure, and even if we could, it could contain singularities that render the map either non-smooth or undefined. Therefore, instead of directly working with closures, it is more fruitful to approach the problem from a different perspective.

Historically, this difficulty has prompted the development of alternative definitions. A more practical strategy, however, is to begin with a more concrete concept, such as elliptic cohomology. In this context, we consider the map

$$\text{Ell(fixed locus)} \overset{\approx i_* p^*}{\to} \text{Ell(ambient space)},$$

where we must clarify that we are not using the traditional elliptic cohomology ring that the notation might suggest. Rather, we are working with a map between two line bundles, specifically a map that involves a section of some line bundle (often a Θ-bundle) for which we already have information about one component of the fixed locus and wish to extend it.

In a previous discussion, we introduced $\text{Ell}_{\text{eq}}(\mathbb{P}^n)$, where elliptic cohomology was associated with a variable a and a collection of abelian varieties given by the equations $t = a_i^{-1}$.



LLet us assume we start at the top of a projective space. This space contains several fixed points, and in the case of a projective space P_n, there are $n + 1$ such fixed points. These fixed points are where the parameters $t = a_i^{-1}$ attract

the system. The map of interest in this context is the interaction of sections of line bundles. We can track how these sections evolve as we move along the trajectory within the space. It is important to note that the section of the line bundle is supported on the full attracting locus.

Fixing a starting point, we can observe the behavior of a section of the line bundle from that point. Using this information, we can push forward or pull back along the fixed points, reconstructing the sections of the line bundle by examining the intersections and behaviors at these fixed points. As we move along, we gain more information about the section. For instance, by induction, we know the value of the section at the intersection of $t = a_0^{-1}$ and $t = a_1^{-1}$. Once we gather sufficient information about the section of the line bundle on the abelian variety, we can reconstruct it uniquely.

Let Y be an algebraic variety, $\mathcal{L}$ a line bundle, and D a divisor on Y. We can examine the following cohomological sequence:

$$H^0(Y, \mathcal{L}(-D)) \to H^0(Y, \mathcal{L}) \to H^0(D, \mathcal{L}) \to H^1(Y, \mathcal{L}(-D)).$$

- **The second map**: The main interpolation problem is to reverse the second map, i.e., given a section on a divisor, we aim to lift it to a section defined on the whole variety. This is analogous to the problem of reconstructing a polynomial in two variables from its restriction to a curve. In essence, we are trying to reverse the map induced by this restriction.
- **The last map**. The cokernel of this map corresponds to the obstructions to the interpolation. These obstructions represent the difficulties encountered in lifting the section from the divisor to the variety.
- **The first arrow**. The first map represents ambiguities in the interpolation. Specifically, there may be cases where a section vanishes on the divisor, leading to the trivial (zero) section.

For the interpolation problem to have a unique solution, we require the following exactness in the cohomological sequence:

$$0 \to H^0(Y, \mathcal{L}(-D)) \to H^0(Y, \mathcal{L}) \to H^0(D, \mathcal{L}) \to H^1(Y, \mathcal{L}(-D)) = 0.$$

Here, Y is an abelian variety, and both $\mathcal{L}$ and D correspond to Θ-bundles. If $\deg \mathcal{K}(-D) = 0$, which reflects a balance from the symplectic form on X, then either $\mathcal{L}(-D)$ is trivial, or $H^0(\mathcal{L}(-D)) = 0$.

This result implies that the interpolation problem will generally have a unique solution except in the presence of special resonances. These resonances are critical because if encountered, they may render the problem ill-defined. This is illustrated by formulas such as $\frac{\theta(xz)}{\theta(x)\theta(z)}$, which are used to interpolate a function defined on an elliptic curve. In such cases, we attempt to lift a function defined at 1 to the entire elliptic curve. The term $\theta(z)$ in the denominator introduces a subtle issue: when the line bundle is trivial, a resonance arises that can lead to division by zero.

Thus, while the interpolation problem for a fixed line bundle usually has a well-defined solution, there is a potential risk of encountering a resonance if we venture too far. To avoid this, we introduce a variable Z from the outset. This allows us to vary the line bundle and bypass the resonance. By twisting by $\mathrm{Pic}_0(\mathrm{Ell}(X)) \supset \mathrm{Pic}(X) \otimes E$, where Z and Z^{vol} are the dynamical variables in elliptic quantum groups and Kähler variables, we ensure that the interpolation problem remains well-posed. This transformation does not change the structure of the problem, but rather stabilizes it.

The introduction of these variables allows us to solve the interpolation problem in the context of varying line bundles. By integrating over elliptic functions, we can obtain explicit solutions to the interpolation problem while avoiding the pitfalls of resonance. The elliptic functions involved are q-constant, and their integration provides explicit solutions. In the framework of elliptic quantum groups, the introduction of these variables is indispensable for ensuring the smooth resolution of the problem.

With these variables in place, we arrive at the Fock $\otimes\, r$ function:

$$\Psi(a_1, \ldots, a_r, t_1, t_2, t_3, z) = \int \left(\text{rational function} \cdot \text{elliptic function} \cdot \prod \frac{\Gamma_q}{\Gamma_q} \right) d\text{Haar}$$

In this expression:

- The integral is taken over the maximal torus of the gauge group.
- The rational function corresponds to the off-shell Bethe arising from $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$.
- The elliptic function represents the analog of the contour integral coming from Ell.
- The term $\prod \frac{\Gamma_q}{\Gamma_q}$ represents the Bethe equation.

This formulation underscores the importance of integrating over elliptic functions to resolve the interpolation problem, avoiding the resonance issues that arise when the line bundle is fixed. By varying the line bundle and incorporating elliptic functions, we effectively "solve" the interpolation problem while ensuring that the solution remains well-defined and smooth.

Part II

Week 1 Talks

There were seven talks in week 1. There were no talks on Wednesday, June 26th.

Week 1

Monday, June 24th

- Mikhail Bershtein: Chiralization of Cluster Structures

  The chiralization in the title denotes a certain procedure which turns cluster X-varieties into $q-W$ algebras. Many important notions from cluster and $q-W$ worlds, such as mutations, global functions, screening operators, R-matrices, etc. emerge naturally in this context. In particular, we discover new bosonizations of $q-W$ algebras and establish connections between previously known bosonizations. If time permits, I will discuss potential applications of our approach to the study of 3d topological theories and local systems with affine gauge groups. This talk is based on a joint project with J. Shiraishi, J.E. Bourgine, B. Feigin, A. Shapiro, and G. Schrader.

- What is... a Riemann-Hilbert problem?

  In its classical setting, the Riemann-Hilbert problem refers to Hilbert's 21st problem of constructing a Fuchsian ODE system with prescribed poles and a given monodromy group. Using singular integral equation techniques, Plemelj presented a solution to this problem in 1908 which became widely accepted. However, Kohn, Arnold and Ilyashenko noticed in the mid 1980s that Plemelj had actually worked on a problem similar to Hilbert's 21st for so-called regular ODE systems rather than Fuchsian ones. These new investigations resulted eventually in a negative answer to Hilbert's original problem given by Bolibruch in 1989 with further developments by Bolibruch and Kostov soon after. Tangentially to the solution of Hilbert's classical problem, the singular integral equation techniques used therein, a.k.a. analytic factorizations of given functions defined on curves, gave rise to a class of modern Riemann-Hilbert factorization problems. In fact nowadays we view such problems as part of a broad analytical toolbox that is useful in the analysis of problems in mathematics and physics, for instance the Wiener-Hopf methods in hydrodynamics and diffraction. The goal of this talk is to first review some facts of the classical Riemann-Hilbert theory and then present a few recent developments of its modern counterpart. Special attention in the second part will be given to matrix-

and operator-valued Riemann-Hilbert problems that arise in random matrix theory and integrable probability.

Tuesday, June 25th

- Anton Zabrodin: Integrability and Time Discretization of the Deformed Ruijsenaars-Schneider Model

  We will discuss the recently introduced deformed Ruijsenaars—Schneider (RS) many-body system. One the one hand, it is the dynamical system for poles of elliptic solutions to the Toda lattice with constraint of type B. On the other hand, equations of motion for this system coincide with those for pairs of RS particles which stick together preserving a special fixed distance between the particles. We prove integrability of the deformed RS system by finding the integrals of motion explicitly. We also obtain Backlund transformations and integrable time discretization of the deformed RS system.

- Henry Liu: Invariance of Elliptic Genus Under Wall Crossing

  Elliptic genus, and its various generalizations, is one of the simplest numerical invariants of a scheme that one can consider in elliptic cohomology. I will present a topological condition which implies that elliptic genus is invariant under wall-crossing. It is related to Krichever—Höhn's elliptic rigidity. Many applications are possible; I will focus on elliptic Donaldson—Thomas theory for this talk.

Thursday, June 27th

- Alexei Borodin: Geometry of Dimer Models

  Random dimer coverings of large planar graphs are known to exhibit unusual and visually apparent asymptotic phenomena that include formation of frozen regions and various phases in the unfrozen ones. For a specific family of subgraphs of the (periodically weighted) square lattice known as the Aztec diamonds, the asymptotic behavior of dimers admits a precise description in terms of geometry of underlying Riemann surfaces. The goal of the talk is to explain how the surface structure manifests itself through the statistics of dimers. Based on joint works with T. Berggren and M. Duits.

- Alexander Bobenko: Dimers and M-curves

  We develop a general approach to dimer models analogous to Krichever's scheme in the theory of integrable systems. This leads to dimer models on doubly periodic bipartite graphs with quasiperiodic positive weights. Dimer models with periodic weights and Harnack curves are recovered as a special case. This generalization from Harnack curves to general M-curves, which are in the focus of our approach, leads to transparent algebro-geometric structures. In particular, the Ronkin function and surface tension are expressed as integrals of meromorphic differentials on

M-curves. Based on Schottky uniformization of Riemann surfaces, we compute the weights and dimer configurations. The computational results are in complete agreement with the theoretical predictions. The talk is based on joint works with N. Bobenko and Yu. Suris.

Friday, June 28th

- Youjin Zhang: Bihamiltonian Integrable Systems and their Classification

  Bihamiltonian structure plays an important role in the theory of integrable systems. For a system of evolutionary PDEs with one spatial variable which possesses a bihamiltonian structure, one is able to find, under a certain appropriate condition, infinitely many conservation laws of the system from the bihamiltonian recursion relation and to arrive at its integrability. In the case when the bihamiltonian structure of the system of evolutionary PDEs possesses a hydrodynamic limit, one can further obtain from it a flat pencil of metrics, and relate it to Frobenius manifold structures or their generalizations under a certain condition, such a relationship may help one to find applications of the integrable system in different research areas of mathematical physics. In this talk, we will recall the notion of bihamiltonian integrable systems, explain their relationship with Frobenius manifold structures or their generalizations, and review the results on the classification of bihamiltonian integrable hierarchies which possess semisimple hydrodynamic limits.

5 Mikhail Bershtein: Chiralization of Cluster Structures

Abstract

The chiralization in the title denotes a certain procedure which turns cluster X-varieties into $q-W$ algebras. Many important notions from cluster and $q-W$ worlds, such as mutations, global functions, screening operators, R-matrices, etc. emerge naturally in this context. In particular, we discover new bosonizations of $q-W$ algebras and establish connections between previously known bosonizations. If time permits, I will discuss potential applications of our approach to the study of 3d topological theories and local systems with affine gauge groups. This talk is based on a joint project with J. Shiraishi, J.E. Bourgine, B. Feigin, A. Shapiro, and G. Schrader.

Contents

5.1 Introduction

The purpose of this talk is to discuss the following connection:

$$\{\text{representation theory of } \infty\text{-dim algebras}\} \leftrightarrow \{\text{cluster structures}\}$$

On the right-hand side, the term cluster structures refers to cluster varieties and cluster algebras. On the left-hand side, examples of ∞-dimensional algebras include vertex algebras, $\mathcal{W}$-algebras, and affine Lie algebras $\hat{\mathfrak{g}}$. Since cluster structures possess a multiplicative nature, we focus on the q-deformed setting, considering $W_{q,t}$, $\mathcal{U}_g(\hat{\mathfrak{g}})$, and toroidal algebras.

5.2 Quantum Cluster Algebras

What is a quantum cluster algebra? A quantum cluster algebra begins with an oriented graph, or quiver. Consider the simple quiver with two vertices and a single arrow between them:

$$A \longrightarrow B$$

The arrows in the quiver represent commutation relations between the associated variables, in this case $ab = q^2ba$. More generally, the relation for two

variables x_i and x_j is given by

$$x_i x_j = q^{2b_{ij}} x_j x_i,$$

where b_{ij} is the skew-symmetric integer matrix determined by the number of arrows between the vertices. Specifically, b_{ij} is given by the difference between the number of arrows directed from vertex i to vertex j and the number of arrows directed in the opposite direction.

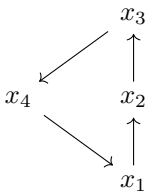
In practice, the algebra

$$\mathbb{C}\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}\rangle / x_1 x_\alpha = q^{\#} x_\alpha x_1$$

is often referred to as the **quantum torus**. These algebras are relatively simple, and their connection to quantum groups lies in the fact that representations of quantum groups can be realized using these algebras.

5.3 The Quantum Group Realization

Example 5.1. *Consider the quiver*



We can define the following expressions:

$$\begin{aligned} E &= x_1 + q x_1 x_2 \\ K &= x_1 x_2 x_3 \\ F &= x_3 + q x_3 x_4 \\ K' &= x_3 x_4 x_1. \end{aligned}$$

These expressions satisfy the relations of the quantum group, specifically $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}))$, the double of the universal enveloping algebra of the Borel subalgebra. When the relation $KK' = 1$ holds, we recover the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$.

This quantum group realization can be viewed as a representation of q-difference operators. For example, by setting $x_1 \mapsto x$, $x_2 \mapsto D^{-1}$, $x_3 \mapsto x^2 D$, and $x_4 \mapsto q^{-2}x^{-1}x$, we obtain a representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ using q-difference operators.

This realization of a quantum group in terms of difference operators is analogous to the classical representation of a Lie algebra using differential operators. For example, for $\mathfrak{sl}_2$, we have:

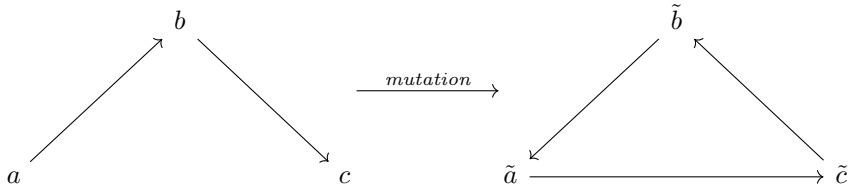
$$\mathfrak{sl}_2 \to \mathrm{Diff}(\mathbb{A}^1), \quad E \mapsto \partial_x, \quad H \mapsto -2x\partial_x + \lambda, \quad F \mapsto -x^2\partial_x + \lambda_x.$$

5.4 Cluster Variables and Mutation Formulas

The q-deformation replaces these differential operators with q-difference operators, which enables the introduction of **cluster variables** and the study of their algebraic properties.

At this stage, we have defined quantum tori and, more geometrically, cluster varieties corresponding to a single quantum torus. However, one of the most important concepts in cluster varieties is the notion of mutation, which describes the transformation of one cluster variety into another, typically by separating or "gluing" charts. We will illustrate this with an example rather than providing the full formal definition.

Example 5.2. *Consider the mutation of a quiver:*

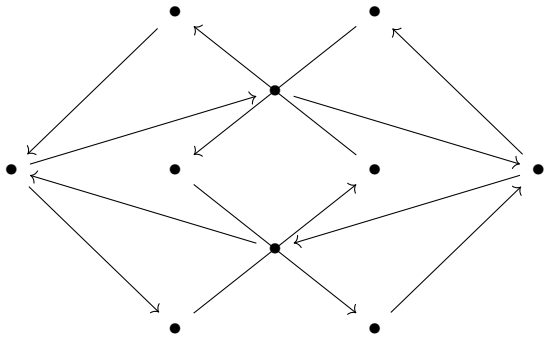


The mutation yields the following relations:

$$\tilde{a} = a(1 + qb), \quad \tilde{b} = b^{-1}, \quad \tilde{c} = c(1 + qb^{-1})^{-1}.$$

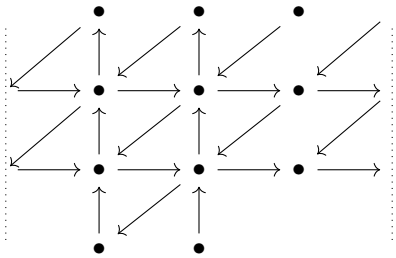
Proposition 5.3. *The generators of the quantum group remain Laurent polynomials after mutation.*

For more complicated quantum groups, such as $\mathcal{U}_q(\mathfrak{sl}_3)$, larger quivers are required. For example, consider the following quiver:



The squares in the diagram arise from the geometry of local systems on surfaces, as explored by Goncharov. The quiver corresponds to a disc with a puncture and two marked points on the boundary. This configuration can be thought of as encoding the geometric structure of local systems on the disc, where the

marked points represent boundary conditions. There is a natural second chart, corresponding to a different type of surface:



This new chart represents a cylinder, which serves as an alternative realization of the quiver. The transformation from the punctured disc to the cylinder is part of a more general transition between charts, though it is non-trivial to compare quivers in different charts directly. The topology of the surface influences the structure of the quiver and the mutation process.

What is the size of the quiver? The number of vertices in the quiver corresponds geometrically to the dimension of the manifold associated with the system. This means that in the case of a punctured disc, for instance, the number of vertices reflects the underlying geometric structure of the surface, which in turn dictates the algebraic properties of the quantum torus or cluster variety in question. For higher-dimensional surfaces, the number of vertices in the corresponding quiver increases, reflecting the more complex geometry involved.

Another way to think about the mutation formulas is in terms of conjugation by the quantum dilogarithm

$$\varphi(b) = \prod_j (1 + q^{2j+1} b)$$

Geometrically, one can think about this as a quantization of the algebra of global functions, but instead of $\mathbb{A}^1$ we have the additional restraint $\mathbb{P}^1$.

5.5 Chiralization

We now turn to the main point of this talk: chiralization. We begin similarly to the previous discussion, with a quiver:

$$A \xrightarrow{\quad P \quad} B$$

but this time, the arrow is assigned a weight P. In this case, we work with current equations such as:

$$A(z) = a_0 \exp\left(\sum \frac{a_n}{n} z^n\right),$$

where the a_n's satisfy the relations of a Heisenberg algebra. The commutation relations for the operators $A(z)$ and $B(w)$ are given by:

$$A(z)B(w) = \frac{qpz - q^{-1}w}{pz - w} : A(z)B(w) :, \quad B(z)A(w) = \frac{q^{-1}z - qpw}{z - pw} : B(z)A(w) : .$$

The operator product expansion is bosonic, and the negative-indexed terms are shifted to the left, while the positive-indexed terms are shifted to the right. Additionally, the commutator relations for the vertex operators are:

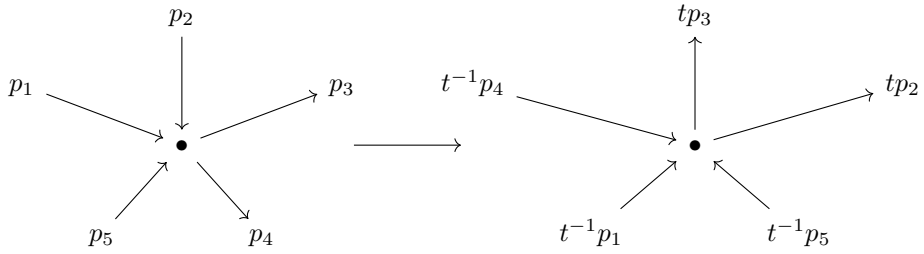
$$A(z)A(w) = \frac{(z-w)^n}{(z-qw)(z-q^{-1}w)} : A(z)A(w) : .$$

This can be viewed as a deformation of the commutation relations for the vertex algebra, where instead of assigning a single variable to each vertex, we assign an infinite sequence of variables. The quantization of these variables removes the commutative structure, and we obtain the relations:

$$[a_n, b_m] = np^{-n}(q^{-2n} - n)\delta_{m+n,0},$$

$$[a_n, a_m] = n(-q^{-2n} + 2 - q^{2n})\delta_{m+n,0}.$$

How crucial are the weights P? First, it's important to note that there are gauge transformations acting on these weights. Specifically, for each vertex, we can assign a current $A(z) \to A(tz)$. Since we are already working in the q-deformed world, we are comfortable with having poles on shifted diagonals. This means there's no intrinsic preference for a particular choice of the variable. Thus, the gauge transformation takes the following form:



Here, the outgoing edges are multiplied by a factor of t, while the incoming edges get a factor of t^{-1}.

A consequence of this observation is that the precise choice of the weights is not that important. What matters more is the monodromy around closed loops - at least, for our purposes, this is sufficient. Weights may vary, but the essential features of the system are encoded in the monodromies, which are invariant under gauge transformations.

5.6 The Geometric Perspective

There are additional examples we should consider that come from geometric perspectives: in the context of the works of Fock and Goncharov, quivers are often drawn on surfaces. On these surfaces, we impose a condition that the product of the monodromies around a local cycle equals q^{-2}. This condition implies that the essential monodromies are related to the homology group of the surface, thereby tying the quiver's algebraic properties to the topology of the underlying surface.

Example 5.4. *In the case of the quantum group $\mathcal{U}_q(\mathfrak{sl}_3)$, we can assign weights as follows: assign a weight of 1 to the vertical vectors, q_1 to the horizontal vectors, and q_3 to the diagonal vectors. The corresponding relation between these weights is:*

$$q_1 q^2 q_3 = 1,$$

where $q_2 = q^2$. This illustrates how the weights interact and the corresponding relations that govern the structure of the quiver in this example.

Theorem 5.5. *We have two important properties:*

1. *The formulas in the example give representations of $\mathcal{U}_g(\ddot{\mathfrak{gl}}_3)$ in the Wakimoto representation.*
2. *This representation is mutation-equivalent to the known one.*

One might ask: what happens to mutations when quivers are weighted? It's easy to suspect that the mutation process is no longer purely combinatorial. Indeed, algebraically, we encounter a complication for $\tilde{c}$, since the term $1+qb^{-1}$ transforms into an infinite sum. However, rather than focusing on this directly, we can take a more conceptual approach and view the mutation as a system of equations. We won't explicitly state the system, but its existence is crucial to understanding the mutation process.

In this alternate perspective, mutation becomes an intertwining operator on the Fock space corresponding to the Heisenberg algebra B, mapping the space to itself: $F_B \to F_B$. This action arises from the R-matrix associated with $\mathcal{U}(\ddot{\mathfrak{gl}}_1)$.

When considering a loop that returns to itself, we recognize that this corresponds to the Maulik-Okounkov q-deformed R-matrix. More specifically, it's the q-deformed analog of the Liouville reflection operator, as introduced by Maulik and Okounkov.

We mention two final remarks:

- There exist screening operators. The main thing we should know about them is the combinatorics of the screening operators can betray the combinatorics of the cluster varieties.
- We did not discuss geometry (such as surfaces), and it makes sense to ask whether the pentagon identitfy holds for this mutations. The answer is that it holds in projective space, leading to the Fock-Rosly brackets.

Finally, we can ask whether we expect quantized cluster varieties to possess a chiralization. While this is still an open question, there is reason to be hopeful. In a related story, non-quantized cluster varieties that exhibit a chiralization arise as Coulomb branches of four-dimensional supersymmetric theories. A similar narrative holds for three-dimensional theories, where these varieties correspond to associated varieties for certain vertex algebras. Given these parallels, it is reasonable to expect that quantized cluster varieties may also admit a chiralization in the future.

6 Thomas Bothner: What is a Riemann-Hilbert Problem?

Abstract

In its classical setting, the Riemann-Hilbert problem refers to Hilbert's 21st problem of constructing a Fuchsian ordinary differential equation system with prescribed poles and a given monodromy group. Using singular integral equation techniques, Plemelj presented a solution to this problem in 1908 which became widely accepted. However, Kohn, Arnold and Ilyashenko noticed in the mid 1980s that Plemelj had actually worked on a problem similar to Hilbert's 21st for so-called regular ordinary differential equation systems rather than Fuchsian ones. These new investigations resulted eventually in a negative answer to Hilbert's original problem given by Bolibruch in 1989 with further developments by Bolibruch and Kostov soon after. Tangentially to the solution of Hilbert's classical problem, the singular integral equation techniques used therein, a.k.a. analytic factorizations of given functions defined on curves, gave rise to a class of modern Riemann-Hilbert factorization problems. In fact nowadays we view such problems as part of a broad analytical toolbox that is useful in the analysis of problems in mathematics and physics, for instance the Wiener-Hopf methods in hydrodynamics and diffraction. The goal of this talk is to first review some facts of the classical Riemann-Hilbert theory and then present a few recent developments of its modern counterpart. Special attention in the second part will be given to matrix- and operator-valued Riemann-Hilbert problems that arise in random matrix theory and integrable probability.

Contents

6.1 Origins

The first Riemann-Hilbert problem was introduced 125 years ago, in 1900, at the 2nd International Congress of Mathematics in Paris. During one of the sessions, David Hilbert delivered his address and presented a list of problems that he anticipated would shape the mathematics of the 21st century. Among these, 23 problems were particularly significant, many of which have become famous and have had a profound influence on the development of mathematics. We will focus on **Problem 21**.

Problem 21 involves the proof of the existence of linear differential equations with a prescribed monodromy group. At the time, Hilbert stated that this problem was one that Riemann had "probably already thought about." The problem asks for the construction of a linear differential equation belonging to the Fuchsian class, with prescribed singular points and a prescribed monodromy group. Specifically, the task is to demonstrate the existence of n functions of the complex variable z, which are regular and complex analytic in the z-plane, except at certain singular points. At these singularities, the functions may become infinite, but only to a finite order. Moreover, as one traverses a loop around a singularity, the function changes according to the specified monodromy group.

This is often considered the first Riemann-Hilbert problem because Hilbert formulated it and explicitly noted that Riemann had likely pondered it as well. Although no direct evidence from Riemann's papers addresses this particular type of problem, we continue to refer to it as the Riemann-Hilbert problem, following the convention established by Anosov and Bolibruch. These scholars, whose names will frequently appear in the first half of this discussion, co-authored a book on Hilbert's 21st problem and have been instrumental in shaping our understanding of it. We adopt their terminology in referring to it as the original Riemann-Hilbert problem, as it is indeed rooted in Riemann's broader mathematical ideas, particularly his work on constructing global functions from local data, a theme central to Riemann's contributions to complex analysis.

Since the problem's introduction, 125 years have passed, making it natural to ask whether it has been resolved. A quick search of "21st problem" on Wikipedia yields the answer: partially resolved. Moreover, the resolution depends on the specific formulation of the problem, allowing for various interpretations with different answers: yes, no, or still open.

Hilbert's formulation of the problem was notably vague, raising several questions about its precise nature. Specifically, given a monodromy group, one must ask whether it is realized by:

1. A Fuchsian linear nth-order differential equation? This interpretation is possible because Hilbert never explicitly refers to a matrix system in his statement.

2. A linear system with only regular singularities? Hilbert described the functions he sought as having singularities of finite order, but he did not use the term "Fuchsian singularity," which has since become the standard term. Instead, he referred to regular singularities, which differ significantly from Fuchsian singularities, as we will see later.

3. Fuchsian system on the entire Riemann sphere $\mathbb{CP}^1$? This interpretation is also plausible, as Hilbert mentions the Fuchsian class in his problem statement.

These three interpretations lead to different conclusions:

1. No. By the time of the conference, it was already known (to Poincaré) that the answer was no, meaning Hilbert was likely not considering this option. Prior to the conference, Poincaré had determined that the number of parameters in a finite linear scalar differential equation was strictly less than the dimension of the admissible monodromy group.

2. Yes. This interpretation was highly influential in the development of the modern theory of Riemann-Hilbert problems. In 1908, Plemelj published two papers that demonstrated a positive answer to this question. Plemelj also claimed that his methods solved the third interpretation as well, but this claim was later refuted after his death in the late 1950s by Ilyashenko and Treibich (1983-1988). Nevertheless, Plemelj's techniques were seminal, and we will revisit them later in this discussion.

3. More subtle. The resolution of this interpretation is credited to Bolibruch and Kostov (1992)

6.2 Background terminology

Consider the $p \times p$ system

$$\frac{d\Psi}{dz} = A(z)\Psi(z) \tag{1}$$

with

1. $A(z)$ is analytic in a disc $\mathbb{D}_r(z_0) \subset \mathbb{C}$. By the **Picard-Lindelöf theorem**, any fundamental solution to (1) is analytic in $\mathbb{D}_r(z_0)$. However, we are interested in differential equations of singularities so we have to move to the case with n punctures.

2. $A(z)$ analytic in $S := \mathbb{CP}^2 \backslash \{a_1, \ldots, a_n\}$. By the **monodromy theorem**, any fundamental solution to (1) can be continued along any path in S.

3. The **monodromy** of (1) is defined as follows: fix $a_0 \in S$ and continue $\Psi(z)$ along $\gamma \in \pi_1(S, a_0)$ to obtain $\tilde{\Psi}(z)$. Since $\Psi(z) = \tilde{\Psi}(z)G_\gamma$, the map $[\gamma] \mapsto G_\gamma$ defines a representation $\chi : \pi_1(S, a_0) \to \mathrm{GL}(\rho, \mathbb{C})$.

4. The **monodromy matrix** at a singular point a_i equals $G_i = \chi([\gamma_i])$ for $i = 1, \ldots, n$ and $\{G_1, \ldots, G_n\}$ generate the **monodromy group**.

5. Strictly speaking, monodromy is defined up to conjugation equivalence, so it is really an element of the space

$$\mathcal{M} := \mathrm{Hom}(\pi_1(S, a_0), \mathrm{GL}(\rho, \mathbb{C}))/\mathrm{GL}(\rho, \mathbb{C}).$$

   Note that

$$\mathcal{M} \cong \{(G_1, \ldots, G_n) : G_1 \cdot \ldots \cdot G_n = \mathbb{I}\}/\mathrm{GL}(\rho, \mathbb{C})$$

   so $\dim(\mathcal{M}) = (n-1)p^2 - (p^2-1) = (n-2)p^2 + 1$.

6. A system (1) is called **Fuchsian** if all of its singular points $\{a_1, \ldots, a_n\}$ are first order poles of $A(z)$, ie. without loss of generality we have

$$A(z) = \sum_{i=1}^{n} \frac{B_i}{z - a_i}, \sum_{i=1}^{n} B_i = 0$$

   In particular, the point at infinity is not a singularity.

Now we have everything in place to give a precise formulation of the original Riemann-Hilbert problem in the third setting:

Problem 6.1 (The Original Riemann-Hilbert, 1900)**.** *Is the monodromy map $\mu : \mathcal{M}^* \to \mathcal{M}$ from the space*

$$\mathcal{M}^* := \left\{ (B_1, \ldots, B_n) : B_i \in Mat(\rho, \mathbb{C}), \sum_{i=1}^{n} B_i = 0 \right\} / GL(\rho, \mathbb{C})$$

of Fuchsian systems with fixed singularities $a_1, \ldots, a_n$ into $\mathcal{M}$ surjective?

6.3 Plemelj's Solution

In 1908, Plemelj published what he claimed be a solution, and it was accepted by the community for almost 80 years. Plemelj tackled this problem by reducing the original problem to a **Hilbert boundary value problem** in the theory of integral equations. This was an analytic solution using state-of-the-art methods at the time - Fredholm had developed the theory of integral equations 10 years before.

Plemelj did the following: join all singularities $a_1, \ldots, a_n \in \mathbb{C}$ by a simple closed contour Γ. Now define the piecewise constant matrix-valued invertible function $G(z) := G_i G_{i-1} \ldots G_1, z \in [a_i, a_{i+1}), i = 1, \ldots, n-1$ and let C^- denote the domain in $\mathbb{C}$ bounded by Γ, as well as C^+ the complement of $\overline{C^-}$ in $\mathbb{CP}^1$.

Let's state the Hilbert boundary value problem.

Proposition 6.2 (Hilbert Boundary Value Problem)**.** *Find all pairs (Y_+, Y_-) of matrix-valued $p \times p$ functions such that*

1. *$Y_\pm(z)$ are analytic in $C^\pm$ and $Y_\pm$ are continuous up to the contour $\Gamma\backslash\{a_1,\dots,a_n\}$*
2. *On (a_i, a_{i+1}), the functions are connected by the relation*

$$Y_+ = Y_- G(z)$$

1. *$Y_\pm(z)(z-a_k)^\kappa \to 0$ as $z \to a_k$ with some $0 \le \kappa < 1$ and $z^D Y_+(z) \to \mathbb{I}$ as $z \to \infty$ with some integral diagonal matrix D.*

This problem is equivalent to a system of singular integral equations, which we won't write out. Using the theory of singular integral equations, Plemelj showed in 1908 that the above BVP is solvable (but not necessarily uniquely). This relates back to the Riemann Hilbert problem because the matrix product

$$\Psi(z-a_1)^D Y_\pm(z), z \in \mathbb{CP}^1\backslash\Gamma$$

satisfies the linear equation

$$\frac{d\Psi}{dz} = R(z)\Psi(z) \qquad (2)$$

where $R(z)$ is rational with poles at $\{a_i\}_{i=1}^n$ and whose monodromy group is generated by $(G_1,\dots,G_n)$. However, (2) in general is not a Fuchsian system, but rather a regular system!

It's a good time to explain regular vs Fuchsian singularities.

Definition 6.3. *A singularity a_i of (1) is called **regular** if any solution of the system has at most polynomial growth in a vicinity of a_i (solutions have either a removable singularity or a pole at a_i)*

The difference is that a Fuchsian singularity of a linear system is always regular (Sauvage, 1886), but the converse is not true for $p > 1$:

$$\frac{d\Psi}{dz} = \left(-\frac{3}{16z}\right)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\Psi$$

has a fundamental solution of the form

$$\Psi(z) = \frac{1}{4}\left[\begin{pmatrix}4z & 4z\\ 1 & 3\end{pmatrix}\right] z^{-\frac{1}{4}\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}}\Psi, z \in \mathbb{C}\backslash(-\infty, 0]$$

Thus {Fuchsian systems} $\subsetneq$ {regular systems} but Plemelj wanted to solve Hilbert's 21st problem for Fuchsian systems, so he applied a procedure that takes him from the regular system to another system with equal monodromy and same singular points. Plemelj's error was that he believed he could do this procedure for all singularities, which is incorrect because this transformation

is valid for all but one point. However, if we add an additional constraint, Plemelj's argument does work:

Theorem 6.4 (Plemej's Theorem, 1908 + Ilyashenko, Treibich, 1983-1988)**.** *If at least one of the generators $G_1, \ldots, G_n$ is diagonalizable, then Hilbert's 21st problem has a positive solution.*

As a corollary to Plemelj's work, we have a positive solution to Hilbert's 21st problem for regular systems! Although parts of Plemelj's original work was wrong, the following (correct) theorem did later come out of it:

Theorem 6.5 (Röhrl-Plemelj Theorem, 1957)**.** *Any matrix group with n generators $G_1, \ldots, G_n$ satisfying the constraint $G_1 \cdot \ldots \cdot G_n = \mathbb{1}$ can be realized as the monodromy group of a regular system on $\mathbb{CP}^1$ having all singularities Fuchsian with at most one exception.*

6.4 Further Down the Timeline

The community had accepted Plemelj's argument as a positive affirmative solution to Hilbert's 21st problem for Fuchsian systems for a long time, and henceforth the community focused on other aspects of this problem.

1. In 1913, Birckhoff simplified Plemelj's argument, generalizing into systems of difference equations, but we will not focus on this for this talk.
2. In 1929, Lappo and Danilevskii gave an effective construction (using power series related techniques) of Fuchsian systems with given mondromy matrices, but only for monodromy matrices "close" to the identity matrix.
3. In 1956, Krylov provided an effective construction (using hypergeometric functions) of Fuchsian system for $p = 2, 3$ singular points. Erugin followed this up for four singular points in 1983 (using ideas from the Painlevé equations).
4. In addition to making things more concrete, people started generalizing. In particular, in 1957, Röhrl reformulated and generalized Hilbert's 21st problem to holomorphic vector bundles over Riemann surfaces. This was a very geometric approach, and nowadays all of these beautiful generalizations to surfaces are attached to the name Riemann-Hilbert correspondence. We will not talk about this side of the story any further.
5. Between 1989-1992, Bolibruch and Kostov showed the irreducibility of the mondromy group. In these years, they gave the necessary and sufficient conditions for the solvability of Hilbert's 21st problem on the Riemann sphere. Here is one of their theorems:

Theorem 6.6 (Bolibruch-Kostov Theorem, 1989)**.** *Any irreducible mondromy group can be realized by a Fuchsian system on $\mathbb{CP}^1$.*

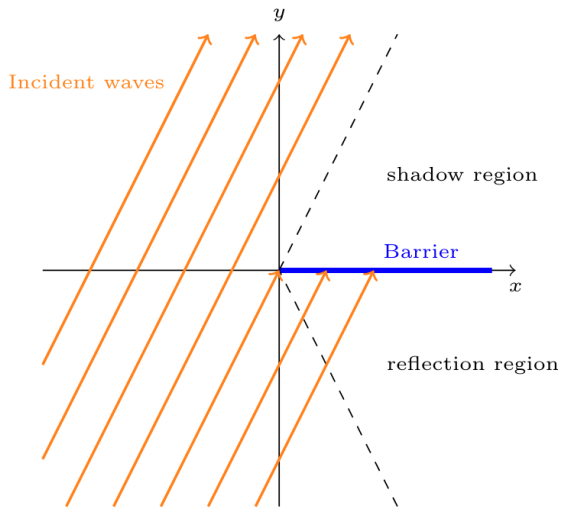
But certain reducible monodromy groups cannot be realized by Fuchsian systems. Furthermore, they gave a recipe to construct counter examples, with the

first example found by Bolibruch for $n = 3, p = 4$. So we can finally answer the original Riemann-Hilbert problem in the third case: we cannot always realize a given monodromy group by a Fuchsian system on $\mathbb{CP}^1$. Bolibruch first lectured these lectures at the 1994 International Congress of Mathematics, but sadly soon after he passed away.

Now, let's see some developments tangential to Plemelj's work. The analysis techniques he used in his "almost solution" continued to reappear in problems related to integrable systems. Let's walk through some of that history and show the whole toolbox around this problem.

6.4.1 The Wiener-Hopf method in linear elasticity, hydrodynamics, and diffraction

There is this famous **Sommerfeld diffraction problem** in 1896 in wave mechanics. Sommerfeld came up with a solution but we will use a different approach requiring the Hilbert BVP.



The question is as follows: we have incoming plain waves as indicated by the orange arrows and an infinite barrier along the positive x-axis, creating a boundary that the waves cannot pass through. Consequently, we will have a reflection region where the waves are reflected and a shadow region that is in perpetual darkness since the rays cannot penetrate it. Now, we take the 2d wave equation and try to solve the associated BVP. How do we do that?

The Sommerfield problem (electromagnetic wave diffraction by a perfectly conducting half plane) is analyzed as a BVP for

$$\Delta\phi + k_0^2\phi = 0, \phi = \phi(x, y) \oplus \text{BC}$$

and then reduced via Fourier techniques to a Hilbert boundary value problem or Wiener-Hopf integral equations. We leave a out a lot of details, such as the boundary condition, but we mention it because his problem was the first problem in mathematical physics where Hilbert BVP can be used efficiently.

6.4.2 The integrable systems revolution

By Gardner, Green, Kruskal, Miura, 1967; Lax, 1968; Faddeev, Zakharov, 1971; and Shabat, Zakharov, 1971.

In order to solve the Cauchy problem for the defocusing nonlinear Schröodinger equation, $y = y(x,t) : \mathbb{R}^2 \to \mathbb{C}$,

$$iy_t + y_{xx} - 2|y|^2 y = 0, \quad y(x,0) = y_0(x) \in \mathcal{S}(\mathbb{R});$$

one first computes the reflection coefficient $r(z) \in \mathcal{S}(R)$ associated to y_0 through the **direct scattering transform**.

There's not much wiggle room in this because if you start with Cauchy data from $\mathcal{S}(\mathbb{R})$, there is a bijection to the class of reflection coefficients in the same function space, and moreover, reflection coefficients with $||\sup|| < 1$. Very rarely we will have an actual formula for $r(z)$ but that's ok because we have a bunch of nice analytic properties.

After that we solve the following BVP:

Proposition 6.7 (Zakharov-Shabat Problem)**.** *For any $(x,t) \in \mathbb{R} \times (0,\infty)$, determine $X(z) = X(z;x,t) \in \mathbb{C}^{2\times 2}$ such that*

1. *$X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{C}_+}$.*
2. *The limits $X_\pm(z) := \lim_{\epsilon\to 0} X(z \pm i\epsilon)$, $z \in \mathbb{R}$, satisfy*

$$X_+(z) = X_-(z) \begin{pmatrix} 1-|r(z)|^2 & -r(z)e^{-2i(2tz+xz)} \\ r(z)e^{2i(2tz+xz)} & 1 \end{pmatrix};$$

3. *As $z \to \infty$, we require $X(z) = I + X_1 z^{-1} + \mathcal{O}(z^{-2})$, $X_i = X_i(x,t)$.*

How does this relate to the partial differential equation intial value problem? Provided this problem is solvable, its (unique) solution solves dNLS with $y(x,0) = y_0(x)$ via $y(x,t) = 2iX_1^{12}(x,t)$.

6.4.3 The isomonodromy method

This part was developed by [Jimbo, Miwa, Ueno, 1981; Flachka, Newell, 1980; and Its, Novokshenov, Kapaev, Kitaev, 1986].

Besides partial differential equations, there are ordinary differential equations, many of which are very important in mathematical physics. One of these famous

ordinary differential equations is the Painlevé equation, and we will focus on the second one.

How do we "solve" the second Painlevé equation

$$u_{xx} = xu + 2u^3, \overline{u}(x) = u(\overline{x})?$$

We added the constraint that says the solution is real valued on the real axis. "Solve" is in quotation marks because $u = 0$ solves this of course, but there's a hard theorem that says any other nontrivial solution cannot be constructed in terms of classical special and a finite number of contour integrals. Here, we will replace it in terms of it's Hilbert BVP.

Given

$$\mathcal{M} = \{(s_1, s_2, s_3) \in \mathbb{C}^3 : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, s_1 = \overline{s_3}, s_2 = \overline{s_2}\}$$

we set with $s_{k+3} = -s_k$ for $k = 1, \ldots, 6$, the Stokes' matrices

$$S_k = \begin{pmatrix} 1 & 0 \\ s_k & 1 \end{pmatrix}, k \equiv 1 \pmod 2; \qquad S_k = \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix}, k \equiv 0 \pmod 2$$

and look at the following Riemann-Hilbert factorization problem:

Problem 6.8 (Painlevé-II Problem)**.** *For any $(x, t) \in \mathbb{R} \times (0, \infty)$ determine $X(z) = X(z; x, t) \in \mathbb{C}^{2\times 2}$ such that*

1. *$X(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{C}}_-+$.*
2. *The limits $X * \pm(z) := \lim * \epsilon \to 0 X(z \pm i\epsilon)$, $z \in \mathbb{R}$ satisfy*
$$X_+(z) = X_-(z) \begin{bmatrix} 1 - |r(z)|^2 & -r(z)e^{-2i(2tz+xz)} \\ r(z)e^{2i(2tz+xz)} & 1 \end{bmatrix};$$
1. *As $z \to \infty$, we require $X(z) = I + X_1 z^{-1} + \mathcal{O}(z^{-2}), X_i = X_i(x, t)$.*

Note that the $s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$ constraint makes $\mathcal{M}$ a 2d manifold, which is fine because we are looking at a second order differential equation. Furthermore, the $s_1 = \overline{s_3}, s_2 = \overline{s_2}$ restraints are added because we are looking for the solutions which are on the real axis, giving us a real 2d manifold.

For any $s = (s_1, s_2, s_3) \in \mathcal{M}$ the above problem for $X(z)$ is meromorphically (wrt x) solvable and

$$u(x) = u(x|s) = 2X_1^{12}(x|s)$$

solves the Painlevé-II equation (3); in addition $\overline{u}(x) = u(\overline{x})$ and

$$\{\text{solutions of Painlevé-II equation}\} \tilde{\to} \mathcal{M}$$

is a bijection.

6.4.4 The study of quantum integrable systems

This part was developed by [Jimbo, Miwa, Mori, Sato, 1981 and Bogoliubov, Izergin, Korepin, Its, 1990]. Consider the spin-$\frac{1}{2}XY$ model in a magnetic field with Hamiltonian

$$H = -\frac{1}{2}\sum_{\ell\in\mathbb{Z}} \left(\sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^z\right)$$

for which we can compute the so-called autocorrelation function $\chi(t)$ of the first spin component as

$$\chi(t) = e^{-\frac{1}{2}t^2} \det(1 - K_t \restriction_{L^2(-1,1)}), K_t(x,y) = \phi(x)\frac{\sin it(x-y)}{\pi(x-y)},$$

where det is the Fredholm determinant. Fredholm determinants are very highly complicated objects, and we can calculate them efficiently by recasting them in terms of a Hilbert BVP:

Problem 6.9 (XY Correlation Function Problem)**.** *For any $t > 0$ determine $X(z) = X(z;t) \in \mathbb{C}^{2\times 2}$ such that*

1. *$X(z)$ is analytic for $z \in \mathbb{C}\backslash[-1,1]$.*
2. *The limits $X_\pm(z) := \lim_{\epsilon\downarrow 0} X(z \pm i\epsilon), z \in (-1,1)$ satisfy*

$$X_\pm(z) = X_-(z)\begin{bmatrix} 1 + phi(z) & -\phi(z)e^{2zt} \\ \phi(z)e^{-2zt} & 1-\phi(z)\end{bmatrix}.$$

1. *$X(z)$ is square-integrable on $[-1,1]$ and as $z \to \infty$, we require $X(z) = \mathbb{I} + X_1 z^{-1} + \mathcal{O}\left(z^{-2}\right), X_i = X_i(t)$.*

This problem is (uniquely) solvable for all $t > 0$ and have

$$\frac{d}{dt}\ln\det(1 - K_t \restriction_{L^2(-1,1)}) = -2X_1^{11}(t).$$

6.4.5 The analysis of orthogonal polynomials and random matrix models

By Fokas, Its, Kitaev, 1991.

Equip the vector space of $n\times n$ Hermitian matrices with the probability measure

$$P(M)\,dM = ce^{-N\mathrm{tr}V(M)}\,dM, c\int e^{-N\mathrm{tr}V(M)}\,dM = 1,$$

where $V : \mathbb{R} \to \mathbb{R}$ is real analytic and satisfies

$$\frac{V(x)}{\ln(x^2+1)} \to \infty \text{ as } |x| \to \infty.$$

Thanks to Dyson, Gaudin, Mehta, 1970, we know how to compute correlation and gap functions in terms of the Christoffel-Darboux kernel

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)}e^{-\frac{n}{2}V(y)}\sum_{j=0}^{n-1} p_{j,N}(x)p_{j,N}(y)$$

with

$$\int_{-\infty}^{+\infty} p_{j,N}(x)p_{k,N}(x)e^{-NV(x)}\,dx = \delta_{jk}.$$

However, now there is the issue that it's not immediately clear if we have good formulas for calculating some efficient asymptotic analysis of our orthogonal polymomials. It turns out we can characterize these through a Hilbert BVP:

Problem 6.10 (Fokas-Its-Kitaev Problem)**.** *For any $n \in \mathbb{Z}_{\geq 0}$, determine $X(z) = X(z;n) \in \mathbb{C}^{2\times 2}$ such that*

1. *$X(z)$ is analytic for $z \in \mathbb{C}\backslash\mathbb{R}$.*
2. *The limits $X_{\pm}(z) := \lim_{\epsilon\downarrow 0} X(z \pm i\epsilon), z \in \mathbb{R}$ satisfy*

$$X_+(z) = X_-(z)\begin{bmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{bmatrix}.$$

1. *As $z \to \infty$, we require $X(z)z^{-n\sigma_3} = \mathbb{I} + \mathcal{O}\left(z^{-1}\right)$.*

The above problem is uniquely solvable for a given $n \in \mathbb{Z}_{\geq 0}$ if and only if the nth monic orthogonal polynomial $\pi_n(x)$ for $e^{-NV(x)}\,dx$ exists. And

$$\pi_n(z) = X^{11}(z;n), z \in \mathbb{C}$$

6.5 The Swiss Army Knife

As we have seen, Hilbert factorization problems appears in a lot of scenarios, but we haven't seen what we have achieved by characterizing problems with the Hilbert BVP because there aren't able to write down explicit formula for many of these problems. Let's see what we have achieved.

why the Hilbert BVP characterization . But we haven't seen why the Hilbert BVP

We should think of the Hilbert factorization representation as a nonlinear contour integral representation which underlies a large class of integrable models. As such it allows us to

1. systematically derive dynamical systems for the quantities under consideration [Plemelj, 1908; Its, 1990].
2. analyze the models asymptotically in their thermodynamical limits [Its, Novokshenov, Kapaev, Kitaev, 1986; Deift, Zhou, 1993]

A few selected results that were derived from a Hilbert factorization problem:

6.5.1 Connection Formula for Painlevé Transcendents

Consider the Painleve-II with $(s_1, s_2, s_3) = (-i\sqrt{\gamma}, 0, i\sqrt{\gamma}), \gamma \geq 0$, the Ablowitz-Segur (1981) family of solutions. Then as $x \to +\infty$,

$$\mu(x) \sim \sqrt{\gamma}\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}.$$

Now we know how the solution behaves in one direction, the **connection problem** asks us to construct asymptotics in a different direction. It turns out that this connects to

Theorem 6.11 (Bounded Oscillatory, $0 \leq \gamma < 1$)**.** $u(x) \sim (-x)^{-\frac{1}{4}}\sqrt{-2\beta}\cos\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \beta\ln\left(8(-x)^{\frac{3}{2}}\right) + \phi\right), x \to -\infty$ *with* $\beta = \frac{1}{2\pi}\ln(1-\gamma)$ *and* $\phi = \frac{\pi}{4} - arg\Gamma(i\beta)$.

or

Theorem 6.12 (Unbounded algebra, $\gamma = 1$)**.**

$$u(x) \sim \sqrt{-\frac{x}{2}}, x \to -\infty$$

or

Theorem 6.13 (Singular oscillatory, $\gamma > 1$)**.**

$$u(x) \sim \frac{\sqrt{-x}}{\sin\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \hat{\beta}\ln(8(-x)^{\frac{3}{2}}) + \varphi\right)}, \quad x \to -\infty$$

with $\hat{\beta} = \frac{1}{2\pi}\ln(\gamma - 1)$ *and* $\varphi = \frac{\pi}{2} - \arg\Gamma\left(\frac{1}{2} + i\hat{\beta}\right)$.

The behavior at $x = -\infty$ can be uniformized with the help of Jacobi elliptic functions, which was done by the speaker in 2017.

6.5.2 Ulam's Problem

Choose a permutation $\pi \in S_n$. For $1 \leq i_1 < i_2 < ... < i_k \leq n$, we say that $\pi(i_1), \ldots, \pi(i_k)$ is an **increasing subsequence** of $\pi \in S_n$ of length k if

$$\pi(i_1) < \pi(i_2) < ... < \pi(i_k).$$

Denote with $\ell_n(\pi)$ the maximal length of all increasing subsequences of π.

Example 6.14. *Consider $\pi = (5, 1, 3, 2, 4) \in S_5$ which has increasing subsequences $(1, 3), (1, 2), (1, 4), (3, 4), (2, 4)$ and $(1, 3, 4), (1, 2, 4)$, so $\ell_5(\pi) = 3$.*

Problem 6.15 (Ulam's Problem)**.** *Suppose you pick $\pi \in S_n$ at random (uniformly distributed), how does ℓ_n behave statistically?*

Very many people contributed to this area, but we will only mention three results because they give the final answer.

Theorem 6.16 (The Baik-Deift-Johansson Theorem, 1999)**.**

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell_n - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq x\right) = \exp\left[-\int_x^\infty (y-x)u^2(y)\,dy\right], x \in \mathbb{R}$$

where $u(x) = u(x|(-i, 0, i))$ solves Painlevé-II.

This result follows from an intruiging combination of (de)-Poissonization arguments, Toeplitz determinants (Gessel formula), and OPUCs that were analyzed asymptotically based on a Hilbert factorization problem.

6.5.3 Universality For Invariant Random Matrix Models

Though the choice of the external field $V : \mathbb{R} \to \mathbb{R}$, the mean eigenvalue density $\frac{1}{n}K_{n,N}(x, x)$ has a limit

$$\lim_{\substack{n,N\to\infty \\ \frac{n}{N}\to 1}} \frac{1}{n} K_{n,N}(x, x) = \rho_V(x) \geq 0$$

where ρ_V is the density of the equilibrium measure μ_V in the presence of the external field V. The local eigenvalue statistics are in turn determined by its local characteristics:

Theorem 6.17 (Bulk Universality - Pastur, Shcherbina, 1997; Bleher, Its 1999; Deift et al, 1999)**.** *For $x^* \in supp(\mu_V)$ with $\rho_V(x^*) > 0$ we have:*

$$\lim_{n\to\infty} \frac{1}{n\rho_V(x^*)} K_{n,n}\left(x^* + \frac{x}{n\rho_V(x^*)}, x^* + \frac{y}{n\rho_V(x^*)}\right) = \frac{\sin(\pi(x-y))}{\pi(x-y)}$$

and

Theorem 6.18 (Edge Universality - Deift et al 1999; Deift and Gioev 2007)**.** *If ρ_V vanjishes as a square root at $x^* \in \partial(supp(\mu_V))$, then for a certain $c > 0$,*

$$\lim_{n\to\infty} \frac{1}{(cn)^{\frac{2}{3}}} K_{n,n}\left(x^* + \frac{x}{(cn)^{\frac{2}{3}}}, x^* + \frac{y}{(cn)^{\frac{2}{3}}}\right) = \int_0^\infty Ai(x+t)Ai(y+t)\,dt.$$

7 Anton Zabrodin: Integrability and Time Discretization of the Deformed Ruijsenaars-Schneider Model

Abstract

We will discuss the recently introduced deformed Ruijsenaars—Schneider (RS) many-body system. One the one hand, it is the dynamical system for poles of elliptic solutions to the Toda lattice with constraint of type B. On the other hand, equations of motion for this system coincide with those for pairs of RS particles which stick together preserving a special fixed distance between the particles. We prove integrability of the deformed RS system by finding the integrals of motion explicitly. We also obtain Backlund transformations and integrable time discretization of the deformed RS system.

Contents

7.1 The RS Model and Variants

7.1.1 The RS Model

Integrable many-body systems play a significant role in mathematical physics. The Calogero-Moser (CM) and Ruijsenaars-Schneider (RS) systems are the main examples.

These models exist in rational, trigonometric (or hyperbolic), and elliptic versions, in which the interaction between particles is described by rational, trigonometric (hyperbolic), and elliptic functions, respectively.

The elliptic models are the most general: the other ones can be obtained from them by appropriate degenerations, where one or two periods of the elliptic functions go to infinity.

Definition 7.1. *The **Weierstrass σ-function** is given by the infinite product*

$$\sigma(x) = \sigma(x|\omega_1, \omega_2) = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}$$

where $s = 2\omega_1 m_2 + 2\omega_2 m_2$ for integers m_1, m_2.

Here, ω_1 and ω_2 are periods but the Weierstrass σ-function is not periodic, but rather quasiperiodic because when we shift the argument by a period, some exponential factors appear in front of the Weierstrass σ-function. Nonetheless, this function is a very convenient building block for the construction of elliptic functions and doubly periodic functions as ratios of products of such σ-functions.

The Weierstrass ζ and $\wp$-functions are connected with the σ-function as follows:

$$\begin{aligned} \zeta(x) &= \sigma'(x)/\sigma(x) \\ \wp(x) &= -\zeta'(x) = -\partial_x^2 \log \sigma(x). \end{aligned}$$

The Weierstrass $\wp$-function is already doubly periodic - it has second order poles in all lattice points (including the origin). In the Laurent expansion, the first term is $\frac{1}{x^2}$.

Recall the CM model:

Definition 7.2. *The **CM model** is defined by the equations of motion*

$$\ddot{x}_i = 4 \sum_{j \neq i}^{N} \wp'(x_{ij}), \qquad x_{ij} = x_j$$

where dot means the time derivative.

The elliptic CM model is Hamiltonian and completely integrable, ie. it has N independent integrals of motion in involution. Integrability of the model was

proved by different methods by [Perelomov, 1977] and [Wojciechowski, 1977]. The proof by Perelomov was very direct - he proved that the eigenvalues of the lax matrix Poisson commute by direct computation.

Recall that RS model, a deformation of the CM model, which can be viewed as a "relativistic extension":

Definition 7.3. *The **RS model** is defined by the equations of motion*

$$\ddot{x}_i + \sum_{j \neq i}^{N} \dot{x_i}\dot{x_j} \left(\zeta(x_{ij} + \eta) + \zeta(x_{ij} - \eta) - 2\zeta(x_{ij})\right) = 0.$$

where ζ is the zeta function.

Two important propties of the zeta function is that it has first order pole in the origin and is an odd function.

Taking the limit $\eta \to 0$ leads to the equations of motions of the CM model. Additionally, the discrete second derivatives becomes a second continuous derivative, so we have Weierstrass $\wp$-function. Furthermore, $\dot{x}$ becomes $1+\dot{y}$ for some other $\dot{y}$, and this limit process becomes quite sophisticated.

Integrability of the RS system was proved by Ruijsenaars. The previous proof method does not work, and he proved this by considering the quantum version of this model and showing that all of the higher shift operators commute. When we have $\hbar \to 0$, we get Poisson brackets, and it appears that proving that quantum integrals of motion commute as operators is much easier than proving with Poisson brackets.

Now, we discuss the time discretization of the RS-model from [Nijhoff, Ragnisco, Kuznetsov, 1996].

Definition 7.4. *Let x_i^n be the coordinate of the ith particle at the nth step of discrete time. The equations of motion of the **time discretization of the RS-model** are:*

$$\begin{aligned}
&\prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1}) \\
+&\prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta) \\
=&0.
\end{aligned}$$

The properly taken continuous time limit yields the equations of motion of the RS model.

Remarkably, these equations are in the same form as the Bethe equations for the nested Bethe Ansatz for models with elliptic R-matrix. In applications to spin chains, n is the level of the Bethe Ansatz and it varies from 0 to $m = \text{rank } (G)$, a finite number of values. Here, it takes an infinite number of values, from $-\infty$ to ∞.

7.1.2 The Deformed RS Model

The RS model admits a deformation [Krichever, Zabrodin, 2022], the deformed RS model.

Definition 7.5. *The deformed RS model admits equations of motion:*

$$\ddot{x_i} + \sum_{j\neq i}^{N} \dot{x_i}\dot{x_j}(\zeta(x_{ij}+\eta) + \zeta(x_{ij}-\eta) - 2\zeta(x_{ij})) + g(U_i^- - U_i^+) = 0$$

where

$$U_i^{\pm} = \prod_{j\neq i}^{N} U^{\pm}(x_{ij}), \qquad U^{\pm}(x_{ij}) = \frac{\sigma(x_{ij}+2\eta)\sigma(x_{ij}\mp\eta)}{\sigma(x_{ij}\pm\eta)\sigma(x_{ij})}$$

and g is the deformation parameter.

At $g = 0$, we have the RS system. It is evident that $g \neq 0$ can be eliminated from the formulas by rescaling the time values $t \to g^{-\frac{1}{2}}t$. In what follows we fix g to be $g = \sigma(2\eta)$.

7.1.3 Integrable Time Discretization of The Deformed RS Model

Definition 7.6. *The equations of motion in the* ***time discretization of the deformed RS model*** *are:*

$$\begin{aligned}
&\mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1} - \eta) \\
+&\mu \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} + \eta)\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1}) \\
=&\mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1} - \eta)\sigma(x_i^n - x_k^n + \eta)\sigma(x_i^n - x_k^{n-1}) \\
+&\mu^{-1} \prod_{k=1}^{N} \sigma(x_i^n - x_k^{n+1})\sigma(x_i^n - x_k^n - \eta)\sigma(x_i^n - x_k^{n-1} + \eta),
\end{aligned}$$

where μ is a parameter related to the lattice spacing in the time lattice.

When $\mu \to 0$, only the right hand side remains. It was predicted by Krichever that there should exist an equation with four terms, with each term a product of cubic sigma functions. This result shows that he was correct.

Recall:

Definition 7.7. *The N-**particle elliptic RS model** is a completely integrable Hamiltonian system. The canonical Poisson brackets are given by $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution:*

$$I_n = \sum_{\mathcal{I}\subset\{1,\dots,N\},|\mathcal{I}|=n} \exp\left(\sum_{i\in\mathcal{I}} p_i\right) \prod_{i\in\mathcal{I},j\notin\mathcal{I}} \frac{\sigma(x_{ij}+\eta)}{\sigma(x_{ij})}.$$

We have two particularly important cases:

$$I_1 = \sum_{i=1}^{N} e^{p_i} \prod_{j\neq i} \frac{\sigma(x_{ij}+\eta)}{\sigma(x_{ij})}$$

which is the Hamiltonian H_1 of the chiral RS model and

$$I_N = \exp\left(\sum_{i=1}^{N} p_i\right).$$

One can also introduce integrals of motion I_{-n} (dependent on I_N but are still important) as

$$\begin{aligned} I_{-n} &= I_N^{-1} I_{N-n} \\ &= \sum_{\mathcal{I}\subset\{1,\dots,N\},|\mathcal{I}|=n} \exp\left(-\sum_{i\in\mathcal{I}} p_i\right) \prod_{i\in\mathcal{I},j\notin\mathcal{I}} \frac{\sigma(x_{ij}-\eta)}{\sigma(x_{ij})}. \end{aligned}$$

In particular,

$$I_{-1} = \sum_{i=1}^{N} e^{-p_i} \prod_{j\neq i} \frac{\sigma(x_{ij}-\eta)}{\sigma(x_{ij})}.$$

The Hamiltonian of the RS model is $I_1 + I_{-1}$, and the limit to the Calogero-Moser model is very straightforward.

It is also convenient to renormalize the integrals of motion:

$$J_n = \frac{\sigma(|n|\eta)}{\sigma^n(\eta)} I_n, \qquad n = \pm 1, \dots, \pm N.$$

The higher Hamiltonians of the RS model can be obtained from the equation of the spectral curve

$$z^N + \sum_{n=1}^{N} \theta_n(\lambda) J_n z^{N-n} = 0, \qquad \theta_n(\lambda) = \frac{\sigma(\lambda - n\eta)}{\sigma(\lambda)\sigma(n\eta)},$$

which is the determinant of the characteristic polynomial of the Lax matrix, as

$$H_n = \mathrm{Res}_{z=\infty}\left(z^{n-1}\lambda(z)\right).$$

For example,

$$\begin{aligned}
H_1 &= J_1\\
H_2 &= J_2 - \zeta(\eta)J_1^2\\
H_3 &= J_3 - (\zeta(\eta)+\zeta(2\eta))J_1J_2 + \left(\frac{3}{2}\zeta^2(\eta) - \frac{1}{2}\wp(n)\right)J_1^3
\end{aligned}$$

shown in [Prokofev, Zabrodin, 2021]. We also introduce the Hamiltonians

$$H_n^{\pm} = H_n \pm \overline{H}_n.$$

On the Toda lattice side, the RS dynamics corresponds to the dynamics of poles of elliptic solutions. In the Toda hierarchy, there are an infinite number of commuting flows, and each flow corresponds to a time variable. Here, the Hamiltonians $H_n^{\pm}$ generate the flows $\partial_{t_n} + \partial_{\bar{t}_n}$, where $t_n, \bar{t}_n$ are canonical higher times of the Toda lattice hierarchy.

The RS dynamics is the same as the dynamics of poles of elliptic solutions to the 2D Toda equation in the Toda times $t_1, \bar{t}_1$ [Krichever, Zabrodin, 1995]. Moreover, this correspondence extends to a complex isomorphism between the elliptic Ruijsenaars-Schneider model (with higher Hamiltonian flows) and elliptic solutions to the whole 2D Toda lattice hierarchy [Prokofev, Zabrodin, 2021].

Now, we can treat the deformed RS model as a dynamical system for pairs of RS particles. The restriction of the RS dynamics of $2N$ particles to the subspace $\mathcal{P}$ in which the particles stick together in N pairs such that

$$x_{2i} - x_{2i-1} = \eta, \qquad i = 1, \dots, N$$

leads to the equations of motion of the deformed RS system for coordinates of the pairs [Krichever, Zabrodin, 2022]. However, not all flows of the RS model preserve this configuration: when we have flows with respect to H_n^+, this configuration is completely destroyed. But H_n^- preserves this. It is natural to introduce the variables

$$X_i = x_{2i-1}, \qquad i = 1, \dots, N$$

which are coordinates of the pairs.

We pass from the initial $4N$-dimensional phase space $\mathcal{F}$ with coordinates $(\{x_i\}, \{p_i\})$ to the $2N$-dimensional subspace $\mathcal{P} \subset \mathcal{F}$ of pairs defined by the constraints

$$x_{2i} - x_{2i-1} = \eta, \qquad x_{2i-1} = X_i p_{2i-1} + p_{2i} = 2\log\sigma(\eta) + \sum_{j\neq i}\log\frac{\sigma(X_{ij}-2\eta)}{\sigma(X_{ij}+2\eta)}.$$

The coordinates in $\mathcal{P}$ are $(\{X_i\},\{P_i\})$. The subspace $\mathcal{P}$ is preserved by the H_1^--flow $\partial_t=\partial_{t_1}-\partial_{\bar t_1}$, but is destroyed by the H_1^+-flow $\partial_{t_1}+\partial_{\bar t_1}$. Therefore, to define the dynamical system, we should fix $T_1^+=\frac{1}{2}(t_1+\bar t_1)$ to be 0, ie. put $\overline{t_1}=-t_1$, and consider the evolution with respect to the time $t=T_1^-=\frac{1}{2}(t_1-\bar t_1)$.

Moreover, the subspace $\mathcal{P}$ is invariant not only with respect to the H_1^--flow but also with respect to all higher H_k^--flows. This gives the possibility to obtain integrals of motion J_n of the deformed RS model by restriction of the RS integrals of motion $\mathrm{J}_n,\mathrm{J}_{-n}$ to the subspace $\mathcal{P}$. We denote the restriction of J_k by J_k:

$$J_k((\{X_i\}_{N_0},\{P_i\}_{N_0}))=\mathrm{J}_k(\{x_\ell\}_N,\{p_\ell\}_N)|_{\mathcal{P}}$$

7.2 Main Results

The main result is as follows:

Proposition 7.8. *The explicit expressions for integrals of motion of the deforms RS system are given by*

$$J_n=\frac{1}{2}\sum_{m=0}^{\lfloor n/2\rfloor}\frac{\sigma(n\eta)\sigma^{2m-n}(\eta)}{m!(n-2m)!}\sum_{[i_1,\dots,i_{n-m}]}\dot x_{i_{m+1}}\cdots\dot x_{i_{n-m}}\prod_{\substack{\alpha,\beta=m+1\\ \alpha<\beta}}V(x_{i_\alpha i_\beta})$$
$$\times\left[\prod_{\gamma=1}^{m}\prod_{\ell\neq i_1,\dots,i_{n-m}}U^+(x_{i_\gamma\ell})+\prod_{\gamma=1}^{m}\prod_{\ell\neq i_1,\dots,i_{n-m}}U^-(x_{i_\gamma\ell})\right],$$

where

$$V(x_{ij})=\frac{\sigma^2(x_{ij})}{\sigma(x_{ij}+\eta)\sigma(x_{ij}-\eta)},$$
$$U^\pm(x_{ij})=\frac{\sigma(x_{ij}\pm 2\eta)\sigma(x_{ij}\mp\eta)}{\sigma(x_{ij}\pm\eta)\sigma(x_{ij})}.$$

Example 7.9. *Here are some examples:*

$$J_1 = \sum_{i=1}^{N} x_i,$$

$$J_2 = \frac{\sigma(2\eta)}{2\sigma^2(\eta)} \left[\sum_{i\neq j} x_i x_j V(x_{ij}) + \sigma^2(\eta) \sum_i \left(\prod_{\ell\neq i} U^+(x_{i\ell}) + \prod_{\ell\neq i} U^-(x_{i\ell}) \right) \right],$$

$$J_3 = \frac{\sigma(3\eta)}{6\sigma^3(\eta)} \left[\sum_{i\neq j,k;j\neq k} x_i x_j x_k V(x_{ij}) V(x_{ik}) V(x_{jk}) \right.$$
$$\left. +3\sigma^2(\eta) \sum_{i\neq j} x_j \left(\prod_{\ell\neq i,j} U^+(x_{i\ell}) + \prod_{\ell\neq i,j} U^-(x_{i\ell}) \right) \right].$$

The generating function of the integrals of motion is

$$R(z,\lambda) = \det_{1\leq i,j\leq N} \left(z\delta_{ij} - \dot{x}_i \phi(x_{ij}-\eta,\lambda) - \sigma(2\eta) z^{-1} U_i^- \phi(x_{ij}-2\eta,\lambda) \right)$$

where

$$\phi(x,\lambda) := \frac{\sigma(x+\lambda)}{\sigma(\lambda)\sigma(x)}.$$

The equation $R(z,\lambda) = 0$ defines the spectral curve which is an integral of motion.

It is quite hard to expand this determinant because for a RS model we only have to use the formula for determinant of a Cauchy elliptic matrix which is well known, but here we need to use a formula for a determinant of a sum of two matrices, which is possible but much harder: the generating function $R(z,u)$ is given by

$$R(z,u) = z^N + z^{-N} \frac{\sigma(u-2N\eta)}{\sigma(u)}$$
$$+ \sum_{k=1}^{N} z^{N-k} \frac{\sigma(u-k\eta)}{\sigma(u)\sigma(k\eta)} J_k$$
$$+ \sum_{k=1}^{N-1} z^{k-N} \frac{\sigma(u-2N\eta+k\eta)}{\sigma(u)\sigma(k\eta)} J_{-k}.$$

Note that this is not polynomial but Laurent polynomial, and it can be proved that $J_{-k} = J_k$.

Here are some properties of the spectral curve. The characteristic equation $R(z,u) = 0$ defines a Riemann surface $\tilde{\Gamma}$ which is a $2N$-sheet covering of the

λ-plane. Any point of it is $P = (z, u)$, where z, u are connected by equation $R(z, u) = 0$. There are $2N$ points above each point u. The Riemann surface $\tilde{\Gamma}$ is invariant under the simultaneous transformations

$$\begin{aligned} u &\mapsto u + 2\omega, \qquad & z &\mapsto e^{-2\zeta(\omega)\eta_z} \\ u &\mapsto u + 2\omega', \qquad & z &\mapsto e^{-2\zeta(w')\eta_z}. \end{aligned}$$

The factor of $\tilde{\Gamma}$ over these transformations is an algebraic curve Γ which covers the elliptic curve with periods $2\omega, 2\omega'$. It is the spectral curve of the deformed RS model.

The spectral curve Γ admits a holomorphic involution ι with two fixed points, which can be seen from the explicit form of the Laurent polynomial. Indeed, the equation $R(z, u) = 0$ is invariant under the involution

$$\iota : (z, u) \mapsto \left(z^{-1}, 2N\eta - u\right).$$

The fixed points lie above the points u_* such that $u_* = 2N\eta - u_*$ modulo the lattice with periods $2\omega, 2\omega'$, ie. $u_* = N\eta - \omega_\alpha$, where ω_α is either 0 or one of the three half periods $\omega_1 = \omega, \omega_2 = \omega', \omega_3 = \omega + \omega'$. Substituting this into the equation of the spectral curve and taking into account that $J_{-k} = J_k$, we conclude that the fixed points are $(\pm 1, N\eta)$ and there are no fixed points above $u_* = N\eta - \omega_\alpha$ with $\alpha \neq 0$.

What about the commutation representation ? There is no Lax representation for the deformed RS system. Instead, it admits the commutation representation in the form of the Manakov's triple.

Definition 7.10. *The* ***Lame-Hermite function*** *is given by*

$$\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(x)\sigma(\lambda)} e^{-\zeta(\lambda)x}.$$

Recall that

$$U_i^{\pm} = \prod_{j \neq i} \frac{\sigma(x_{ij} \pm 2\eta)\sigma(x_{jj} \mp \eta)}{\sigma(x_{ijh} \pm \eta)\sigma(x_{ij})}$$

and introduce $N \times N$ matrices L, M, R:

$$L_{ij}(z, \lambda) = \dot{x}_i \Phi(x_{ij} - \eta, \lambda) + z^{-1}\sigma(2\eta) U_i^- \Phi(x_{ij} - 2\eta, \lambda)$$

$$M_{ij}(z, \lambda) = \dot{x}_i(1 - \delta_{ij})\Phi(x_{ij}, \lambda) + z^{-1}\sigma(2\eta) U_i^+ \Phi(x_{ij} - \eta, \lambda) - \delta_{ij} \left(\sum_k \dot{x}_k \zeta(x_{ij} + \eta) - \sum_{k \neq i} \zeta(x_{ij}) \right)$$

$$R_{ij}(z, \lambda) = \sigma(2\eta) z^{-1} \Phi(x_{ij} - \eta, \lambda).$$

L is an analog of the Lax matrix, and removing the second term gives the Lax matrix for the RS model. M gives a Lax pair with L, it generates the flow, and removing the second term gives the M matrix from the RS model.

The equations of motion of the deformed RS model are equivalent to the Manakov's triple representation:

Definition 7.11. *The **Manakov's triple representation** is*

$$\dot{L} + [L, M] = R(L - zI).$$

From $\mathrm{Tr}R = 0$ (which can be proved from elliptic function identities) it follows that

$$\det(zI - L(z, \lambda))$$

is conserved in time. It is the generating function of integrals of motion.

Now, let's discuss the **Backlund transformation**. It is known that the integrable many-body systems of CM and RS type are dynamical systems for poles of singular solutions to nonlinear integrable differential and difference equations. The nonlinear integrable equations are known to serve as compatibility conditions for linear differential or difference equations for the "wave function" ψ. Poles of solutions to the nonlinear equations (zeroes of the tau-function) are simultaneously poles of the ψ-function, so the latter are subject to equations of motion of the CM or RS type. In fact zeroes of the ψ function are subject to the same equations, and this leads to the idea to obtain the Backlund transformation of the CM or RS system as passage from poles to zeroes. This idea works for many integrable many body systems, and we will apply it to the deformed RS model.

The first linear problem for the Toda lattice with constraint of type B was done by [Krichever, Zabrodin, 2022].

$$\partial_t \psi(x) = v(x) \left(\psi(x + \eta) - \psi(x - \eta)\right)$$

where $v(x)$ is expressed through the tau-function $\tau(x)$ as

$$v(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)}.$$

For elliptic solutions,

$$\tau(x) = \sum_{i=1}^{N} \sigma(x - x_i)$$

and the x_i's become the poles of the solutions and the ψ-function.

We can represent solutions for ψ in the form

$$\psi(x) = \mu^{\frac{x}{\eta}} e^{(\mu - \mu^{-1})t} \frac{\hat{\tau}(x)}{\tau(x)}$$

where μ is a parameter and

$$\hat{\tau}(x) = \prod_{i=1}^{N} \sigma(x - y_i)$$

with some y_i's, assuming there are no zero factors in the product.

The zeroes y_i and poles x_i of the ψ-function obey the system of equations

$$\begin{aligned}\dot{x}_i = \mu\sigma(-\eta) \prod_{j\neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)} \prod_k \frac{\sigma(x_i - y_k + \eta)}{\sigma(x_i - y_k)} \\ + \mu^{-1}\sigma(-\eta) \prod_{j\neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)} \prod_k \frac{\sigma(x_i - y_k - \eta)}{\sigma(x_i - y_k)},\end{aligned}$$

$$\begin{aligned}\dot{y}_i = \mu\sigma(-\eta) \prod_{j\neq i} \frac{\sigma(y_i - y_j + \eta)}{\sigma(y_i - y_j)} \prod_k \frac{\sigma(y_i - x_k - \eta)}{\sigma(y_i - x_k)} \\ + \mu^{-1}\sigma(-\eta) \prod_{j\neq i} \frac{\sigma(y_i - y_j - \eta)}{\sigma(y_i - y_j)} \prod_k \frac{\sigma(y_i - x_k + \eta)}{\sigma(y_i - x_k)}.\end{aligned}$$

where $x_i \to y_j$ is the Backlund transformation. This can be obtained by direct substitution of the ψ-function into the Toda lattice linear problem with constraint of type B. These equations are symmetric with respect to the permutation between x, y, $-\eta$.

Let's discuss discrete time dynamics. The Backlund transformation $x_j \to y_j$ can be regarded as a time evolution by one step of the discrete time. Denoting the discrete time variable by n, we then write

$$x_i = x_i^n, \qquad y_i = x_i^{n+1}.$$

Then the Backlund transformations can be read as the equations of motion in discrete time:

$$\mu\prod_{k=1}^{N}\sigma(x_i^n-x_k^{n+1})\sigma(x_i^n-x_k^n+\eta)\sigma(x_i^n-x_k^{n-1}-\eta)$$
$$+\mu\prod_{k=1}^{N}\sigma(x_i^n-x_k^{n+1}+\eta)\sigma(x_i^n-x_k^n-\eta)\sigma(x_i^n-x_k^{n-1})$$
$$=\mu^{-1}\prod_{k=1}^{N}\sigma(x_i^n-x_k^{n+1}-\eta)\sigma(x_i^n-x_k^n+\eta)\sigma(x_i^n-x_k^{n-1})$$
$$+\mu^{-1}\prod_{k=1}^{N}\sigma(x_i^n-x_k^{n+1})\sigma(x_i^n-x_k^n-\eta)\sigma(x_i^n-x_k^{n-1}+\eta)$$

Let's move on to continuous time limits. These equations admit different continuum limits. For one of them, we introduce the variables

$$X_j^n=x_j^n-n\eta$$

and assume that these variables behave smoothly when the time changes, ie. $X_j^{n+1}=X_j^n+O(\epsilon)$ as $\epsilon\to 0$, where we introduce the lattice spacing ϵ in the time lattice, so that the continuous time variable is $t=n\epsilon$. We should expand in powers of ϵ taking into account that

$$X_j^{n\pm1}=X_j+\epsilon\dot{X}_j+\frac{1}{2}\epsilon^2\ddot{X}_j+O(\epsilon^3)$$

as $\epsilon\to 0$. It is enough to expand up to the order ϵ. For consistency, one should require that μ^{-1} is of order ϵ. Putting $\mu^{-1}=\epsilon$, one obtains (in the leading order ϵ), the equations of motion of the deformed RS system.

Another possibility is to assume that the original variables x_j^n are smooth when the time changes, ie.

$$x_j^{n\pm1}=x_j\pm\epsilon\dot{x}_j+\frac{1}{2}\epsilon^2\ddot{x}_j+O(\epsilon^3).$$

In the general position, ie if $\mu^{-2}-1=O(1)$ as $\epsilon\to 0$, the leading order is ϵ adn the expansion gives the RS equations. However, if $\mu^{-2}=1+\alpha\epsilon$, then the first roder gives the identity $0=0$, and one should expand up to the second order in ϵ. In this case, one obtains the equations for dynamics of poles of elliptic solutions to the semi-discrete BKP equation, due to [Rudneva, Zabrodin, 2020]. These equations are very complicated so we do not show them - they require a lot of $\sum\dot{x}_i$ type of expressions.

7.3 Conclusion

We proved integrability of the deformed RS system by presenting all integrals of motion in explicit form. We also obtained the discrete time version of the deformed RS system by consider the Backlund transformations.

The connection between the standard RS system and the deformed one is not trivial. On one hand, the latter is an extension of the former and includes it as a particular case. However, on the other hand, the deformed RS system is contained in the RS system since it can be regarded as its reduction in the sense that its equations of motion are obtained by restriction of the RS dynamics to the subspace $\mathcal{P}$ of pairs.

8 Henry Liu: Invariance of Elliptic Genus Under Wall Crossing

Abstract

Elliptic genus, and its various generalizations, is one of the simplest numerical invariants of a scheme that one can consider in elliptic cohomology. I will present a topological condition which implies that elliptic genus is invariant under wall-crossing. It is related to Krichever—Höhn's elliptic rigidity. Many applications are possible; I will focus on elliptic Donaldson—Thomas theory for this talk.

Contents

8.1 Set Up

Let X be a smooth proper scheme over $\mathbb{C}$, with a torus $T = (\mathbb{C}^\times)^r$ acting on X. Associated to this data, we can think of three equivariant cohomology theories:

1. The T-equivariant cohomology: $\mathrm{Spec} H_T(X)$. These are modules for $\mathrm{Spec} H_T(\mathrm{pt}) \otimes \mathbb{C}$. This is identified with $\mathbb{C}^r = \mathrm{Lie} T$.
2. The T-equivariant K-theory $K_T(X)$. These are modules for $\mathrm{Spec} K_T(\mathrm{pt}) \otimes \mathbb{C}$. This is identified with $T = (\mathbb{C}^\times)^r$.
3. The T-equivariant elliptic cohomology $\mathrm{Ell}_T(X)$. These are modules for $\mathrm{Ell}_T(\mathrm{pt})$. This is already a proper scheme (as opposed to an affine scheme) so we do not need the Spec. This is identified with E^r.

To go between them, ie.

$$\mathrm{Lie}(T) = \mathbb{C}^r \xrightarrow{*} T = (\mathbb{C}^\times)^r \xrightarrow{**} E_r$$

where $*$ is the exp map and $**$ is via modding by $q^{\mathrm{cochar}\ T}$, where cochar T is the cocharacter lattice of T. So we can think of E as

$$E = \frac{\mathbb{C}^\times}{q^{\mathbb{Z}}}$$

where $q = e^{i\pi\tau}$ with $|q| << 1$ and τ is the elliptic modulus.

T-equivariant cohomology studies equivariant differential forms and their pullbacks/pushforwards, and T-equivariant K-theory is studying T-equivariant coherent sheaves and their pullbacks/pushforwards. In particular, in T-equivariant cohomology, push forward to a point is just integral, where in T-equivariant K-theory, push forward to a point is Euler characteristic of sheaves.

The elements of these cohomology theories are sections of line bundles on these schemes. In particular, we have some quantity which we should think of as some quantity in equivariant elliptical cohomology of a point - it will be a subsection of some line bundle on some number of copies of the elliptic curve. In particular, there is one very special subsection which we are interested in:

Definition 8.1. *The **odd Jacobi-Theta function** $\vartheta(x) \in \mathbb{C}^\times$ is given by*

$$\vartheta(x) = \left(1 - x^{-1}\right) \prod_{n>0} (1 - q^n x)(1 - q^n x^{-1}),$$

where

$$\phi := \prod_{n>0} (1 - q^n x)$$

is the q-Pochhammer symbol.

We can treat this formally as an element in $\mathbb{Z}[x^{\pm 1}][\![q]\!]$. We can check that ϑ is a subsection of a line bundle of degree 1; equivalently, we can check that it satisfies the difference equation

$$\vartheta(qx) = -(qx)^{-1} \vartheta(x).$$

Extend to vector bundles E. By the splitting principle, we can think $E = \bigoplus \mathcal{L}_i$. Define the functions

$$\Theta(E) := \prod_i \vartheta(\mathcal{L}_i)$$
$$\Phi(E) := \prod_i \phi(\mathcal{L}_i)$$

Again, these will be formal power series in q but now valued in objects such as the equivariant K-loop, so the coefficients of our series will be coherent sheaves or vector bundles.

Definition 8.2. *The **elliptic genus** of X, $E_{-y}(X) \in K_T(pt)[y^{-1}][\![q]\!]$ is*

$$E_{-y}(X) := \chi\left(X, \frac{\Theta(yT_X)}{\Phi(T_X)\Phi(T_X^\vee)}\right)$$

where $\chi = \sum(-1)^k H^k$ is the Euler characteristic.

The simplest situation is when the T action on X has isolated fixed points: if $X^T = \{p_1, \ldots, p_n\}$, by T-equivariant localization, we have

$$\sum_{i=1}^n \prod_{\omega \in T_{p_i}X} \frac{\vartheta(y\omega)}{\vartheta(\omega)}$$

which gives a meromorphic function on the torus T.

Is this really an elliptic quantity? Let T have coordinates t_i. Then we can check that

$$E_{-y}(X)|_{t \mapsto q^\sigma t} = \sum_{i=1}^n Y^{-\sum_{m \in T_{p_i}X} \langle \sigma, \omega_i \rangle} \prod \ldots$$

The simple case is when $\langle \sigma, \sum_{\omega \in T_{p_i}x} \omega \rangle \in N \cdot \mathbb{Z}$ and y is an N-th root of unity. This condition is equivalent to K_X admitting an Nth root. Later on, we'll see some specializations on y and the Nth roots of unity, as well as some conditions on whether or not roots of the canonical bundle exist. Morally, these conditions come from the requirement that something behaves elliptically.

8.2 The Wall Crossing Procedure

Start with M, a proper smooth scheme with the action $T \times \mathbb{C}^\times$, which we call the **master space**. The master space looks like $M^{\mathbb{C}^\times} = Z_- \sqcup Z_+ \sqcup Z_0$, where $Z_- = \{\zeta_- = 0\}$ (where ζ_- is a $\mathbb{C}^\times$ weight -1 action), $Z_+ = \{\zeta_+ = 0\}$, and Z_0, which is a slightly more complicated locus with many components, not be pure dimensional, etc.

We can obtain a relation between integrals on Z_- and Z_+, with correction terms coming from Z. This is a **Wall crossing formula** because in practice, Z_- and Z_+ are somehow stable loci for some variation of the stability condition, ie. we have some kind of locally constant behavior as we vary the stability condition, and at opposite ends the stable locus are at opposite signs.

Let's think about $\chi(M, \mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf. By $C^\times$ equivariant localization, this is equivalent to

$$\chi(M, \mathcal{F}) = \chi\left(Z_-, \frac{\mathcal{F}|_{Z_-}}{\text{euler}(N_-)}\right) + \chi\left(Z_+, \frac{\mathcal{F}|_{Z_+}}{\text{euler}(N_+)}\right) + \chi\left(Z_0, \frac{\mathcal{F}|_{Z_0}}{\text{euler}(N_0)}\right).$$

N_- and N_+ are line bundles because we assumed that they are divisors in the master space. Now, we want to extract a relation between the first and second terms on the right hand side, which we do by applying a "residue." We assume M is proper, and proper pushforwards preserve coherence so $\chi(M, \mathcal{F}) \in K_{T\times\mathbb{C}^\times}(\text{pt})$. But every term on the right hand side becomes a nontrivial rational functions of the coordinates on T and $C^\times$, so we can be very precise about where the poles of the rational functions will be. Call the coordinates on T, t, and on $\mathbb{C}^\times, S$. We get the third term on the right hand side: rational functions, with poles at $s^k t^\mu = 1$ for $k \in \mathbb{Z}, \mu$ T-weights. On the left hand size, we only have poles around 0 and ∞.

We apply $f \mapsto \frac{1}{2\pi i}\int_\gamma f\frac{ds}{s}$ where γ encloses every pole except the ones at 0 and at ∞. After applying this map, the equation simplifies drastically:

$$0 = \chi\left(Z_-, \mathcal{F}|_{Z_-}\Big|_{s=N_-}\right) + \chi\left(Z_+, \mathcal{F}|_{Z_+}\Big|_{s=N_+^\vee}\right) + \text{Res}\chi(Z_0, ...)$$

One key insight in wall-crossing is to try to control the last term as much as possible to get nice formulas.

For elliptic genus, we have

$$\mathcal{F} = \frac{\Theta(y \otimes T_M)}{\Phi(T_m)\Phi(T_M^\vee)}.$$

Executing the wall-crossing algorithm from earlier, we obtain an equation of the form

$$0 = (...)(E_{-y}(Z_-) - E_{-y}(Z_+)) + \chi\left(Z_0, ... \otimes \text{Res}\frac{yN_0}{N_0}\right).$$

We want to know: when does $\text{Res}\dfrac{yN_0}{N_0}$ vanish, ie. nothing happens when we cross the wall?

Theorem 8.3. *Write* $N_0 = \sum_{k\in\mathbb{Z}\setminus\{0\}} s^k(N_0)_k$. *If* $\sum_k k \cdot rank(N_0)_k \equiv 0 \pmod N$, *then* $E_{-\zeta_N}(Z_-) = E_{-\zeta_N}(Z_+)$, *where* $\zeta_N \neq 1$ *is an* N*-th root of unity.*

In literature, there are many different constructions of master spaces. Historically, the first is due to Thaddeus, who was studying a variation of GIT

quotients. A more general setting comes from thinking about how to remove the assumption that things are smooth, which is a very restrictive assumption in modern enumerative geometry where the moduli spaces we consider are often not smooth, ie. Hilbert scheme of points on $\mathbb{C}^3$, which contains basically every possible singularity known to man.

In modern enumerative geometry, we replace schemes with derived schemes, rings with dg-rings, and smooth with quasismooth. The derived version is roughly equivalent to a (very singular) scheme with perfect obstruction theory (POT). We want to study X, but it's singular, so we embed it $X = \{S = 0\} \subset M$ which is smooth, where s is a subsection of the vector bundle E. Let's call the inclusion i. POT is roughly a sufficiently functional way to record this data. In this situation, when the local picture is global, we can replace the fundamental cycle $[x]$ with the virtual fundamental cycle $i_*[x]^{\text{vir}} = [M] \cap \text{euler}(E)$, the structure sheaf $\mathcal{O}_X$ with the virtual structure sheaf $i_*\mathcal{O}_X^{\text{vir}} = \mathcal{O}_M \otimes \text{euler}^K(E)$, and the tangent bundle T_x is replaced by the virtual tangle bundle $T_X^{\text{vir}} = i^*(T_M - E)$.

Definition 8.4. *Suppose X is a scheme with POT such that*

$$T_X^{vir} = F - y^{-1} \otimes F^\vee$$

where y is a T-weight. Then we say the POT is ***equivariantly symmetric.***

In this setting, we can introduce the virtual chiral elliptic genus, which is a upgrade of the elliptic genus:

Definition 8.5. *The* ***virtual chiral elliptic genus*** *is*

$$E_{-y}^{vir/2}(X) := \chi\left(X, \frac{\mathcal{O}_X^{vir} \otimes \left(K_X^{vir}\right)^{\frac{1}{2}}}{\Phi(T_X^{vir})\Phi(T_X^{vir})^\vee}\right)$$

where $K_X^{vir} = \det(T_X^{vir})^\vee$

Let's compare our the virtual chiral elliptic genus to the original elliptic genus. Observe that:

$$\frac{1}{\Theta(T_X^{\text{vir}})} = \frac{\Theta(y^{-1}F^\vee)}{\Theta(F)}.$$

After twisting by $(K_X^{\text{vir}})^{\frac{1}{2}}$, this becomes

$$\text{factor}\frac{\Theta(yF)}{\Theta(F)}.$$

Theorem 8.6. *Write $N_0^{vir} = F_0 - y^{-1}F_0^\vee$. If*

$$\sum_k rank(F_0)_k \equiv 0 \pmod{N},$$

then

$$E_{-\zeta_N}^{vir/2}(Z_-) = E_{-\zeta_N}^{vir/2}(Z_+).$$

8.3 Elliptic Donaldson Theory

Let's explain the primary use case to care about, the elliptic Donaldson-Thomas theory. Let's start off with explaining what a Donaldson-Thomas theory is. Let Y be a smooth quasi-projective CY3 with a torus action T such that the CY3 form has T-weight y. Consider $\mathcal{M}_\alpha$, the moduli stack of coherent sheaves $\mathcal{E}$ on Y with Chern character α. For whatever notion of stability we have, we can think of the semi-stable locus

$$\mathcal{M}_\alpha^{\mathrm{sst}}(\sigma) = \{\sigma - \text{semistable } \mathcal{E}\}$$

where σ is the stability condition. A subset of this is the stable locus,

$$\mathcal{M}_\alpha^{\mathrm{st}}(\sigma) = \{\sigma - \text{stable } \mathcal{E}\}$$

Under the assumption that the semistable locus is the same as the stable locus, then $\mathcal{M}_\alpha^{\mathrm{sst}}(\sigma)$ has a POT with

$$T^{\mathrm{vir}}_{[\mathcal{E}]} = -\mathrm{Ext}_Y(\mathcal{E},\mathcal{E})$$

where $\mathrm{Ext} = \sum(-1)^k \mathrm{Ext}_Y^K$.

Definition 8.7. *The **elliptic DT invariant** is* $DT_{-Y}^{Ell/2}(\alpha;\sigma) = E_{-Y}^{vir/2}(\mathcal{M}_\alpha^{sst}(\sigma))$.

What is the wall crossing behavior for elliptic DT invariants as the stability condition is varied? This problem has been studied in ordinary cohomology and in K-theory, but not much with elliptic stuff. In particular, there are master spaces contributed by Mochizuki and Joyce, or in simple case, we can use the original methods.

Suppose the wall is "simple": we're thinking about a family of 1-parameter stability conditions with σ_- at one end and σ_+, with the only points where we have strictly semistable objects are between σ_- and σ_+, which we call σ_0. At σ_-, σ_+ we have sst=st, but everywhere else we have sst $\supsetneq$ *st*. The condition that the wall is simple if a-priori the sheaf could split into a sum of many semistable pieces, and it is semisimple if the wall can only be split into two pieces and not more, ie. $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Under this construction, we can construct a master space, and we can execute the procedure from earlier.

In particular, the locus which provides the wall crossing term is

$$Z_0 = \{[\mathcal{E}] \in \mathcal{M}_\alpha^{\mathrm{sst}}(\sigma_0) : \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2, \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_2)\}.$$

The virtual normal bundle looks like

$$N_0^{\mathrm{vir}} = -\mathrm{Ext}(\mathcal{E}_1,\mathcal{E}_2) + \mathrm{Ext}(\mathcal{E}_2,\mathcal{E}_1),$$

where $\mathcal{E}_1$ has $\mathbb{C}^\times$-weight 1. If we let $s \in \mathbb{C}^\times$, then we can rewrite it as

$$-s^{-1}\mathrm{Ext}(\mathcal{E}_1,\mathcal{E}_2) + s\mathrm{Ext}(\mathcal{E}_2,\mathcal{E}_1),$$

where by Serre duality the last term looks like $sy^{-1}\mathrm{Ext}(\mathcal{E}_1,\mathcal{E}_2)^\vee$. So we've rewritten the virtual normal bundle in the form that's required by the theorem.

Theorem 8.8. *At a simple wall, if* $\forall[\mathcal{E}] \in Z_0$*, then*

$$\dim Ext_Y(\mathcal{E}_1, \mathcal{E}_2) \equiv 0 \mod N$$

then

$$DT^{Ell/2}_{-\zeta_N}(\alpha, \sigma_-) = DT^{Ell/2}_{-\zeta_N}(\alpha, \sigma_+).$$

Usually, walls are not simple. When we have a non-simple wall, there's extra machinery that we can do that resolves the non-simple wall into a bunch of simple walls, and we cross those simple walls one by one. The cost of this is we replace the moduli stack/space of sheaves with moduli stack/space of sheaves with some extra data, and we have to play a combinatorial game with the extra data to make sure that it satisfies the topological criteria.

In some cases it's easy to do this, whereas in other cases it's not so easy. One easy case is when the CY3 is the total space of the canonical of some smooth projective surface, which is called a local surface. In this case, it's easy to control Ext. However, in general, it's not so easy, and more tools need to be introduced.

9 Alexei Borodin: Geometry of Dimer Models

Abstract

Random dimer coverings of large planar graphs are known to exhibit unusual and visually apparent asymptotic phenomena that include formation of frozen regions and various phases in the unfrozen ones. For a specific family of subgraphs of the (periodically weighted) square lattice known as the Aztec diamonds, the asymptotic behavior of dimers admits a precise description in terms of geometry of underlying Riemann surfaces. The goal of the talk is to explain how the surface structure manifests itself through the statistics of dimers. Based on joint works with T. Berggren and M. Duits.

Contents

9.1 The Simplest Example

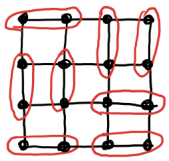
Dimer models study random **dimer coverings** (math terminology) or perfect matchings (physics terminology) on a given graph, which are a choice of subset of edges of the graph so that every vertex gets covered exactly once. In general, these graphs can be arbitrary, but all of the graphs in this talk are planar.

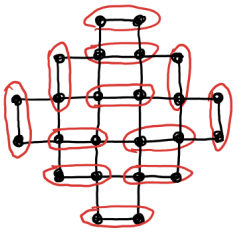
For some brief history:

- Introduced in 1937 by Fowler and Rushbrooke for chemistry in the context of "statistical theory of perfect solutions" in liquid mixtures.
- For planar graphs, the models were "solved" via Pfaffians/determinants in the early 1960s by Kasteleyn and Temperly-Fisher. This work connected these models to the Ising model of ferromagnetism, a significant breakthrough in statistical mechanics at the time.
- In the mid-1980s, Nienhuis, Hilhorst, and Blöte related these models to roughening transitions observed in equilibrium crystals. Their studies provided insights into how these models could describe physical phenomena such as the roughening of crystal surfaces at specific temperature thresholds.
- Since early 1990's, studied by mathematicians.

Now that we have explained what a dimer cover is, what is a dimer model? A **dimer model** is a way to choose a dimer cover with some probability. The simplest situation is when the graph is finite and all dimer covers are of equal probability. Now, the typical type of question a probabilist would ask is: "What does a random dimer look like?"

We will only one type of graph (subset of a square lattice) and only two possible subsets of the square: The image on the right is called an **Aztec diamond**,





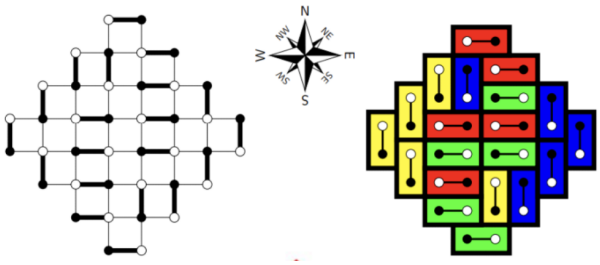
named by Jim Propp about 35 years ago.

It's very hard to see anything from these diagrams, so let's introduce a way to draw them in a more useful way.

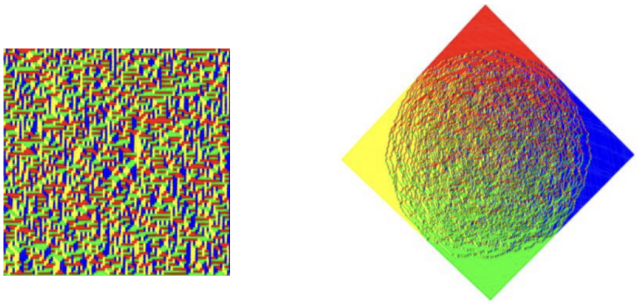
Notice that the graph is bipartite, which means that we can split the vertices into black and white so that the edges always go between a black and a white vertex. Furthermore, if a horizontal edges is chosen as a DI, they come in two

types: they can either go from white to black or from black to white as we move left to right. Similarly, vertical edges also come in two types: they can be either go from black to white or white to black as we move up to down.

Now, we can see that each edge of the graph can be covered by a 2×1 rectangle that looks like a domino:



Then we'll paint those dominoes in four different colors according to the four different types of of of edges. This gives the following picture:

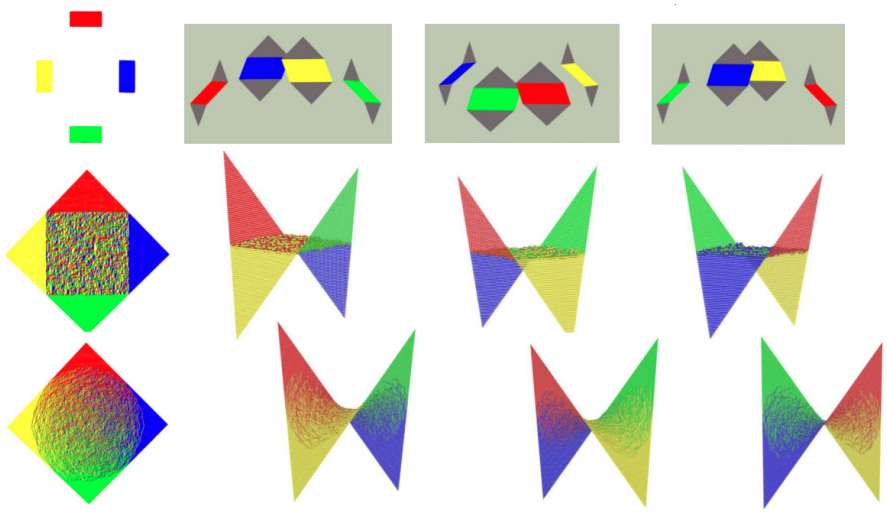


Now, it is clear that these two pictures are drastically different: there is chaos everywhere in the left image, but there is a very clear color separation in the corner in the right image.

The curve on the left that separates the mixed and unmixed colors is called a **arctic curve** (because everything outside the curve is frozen), which was introduced in [Elkies, Kuperberg, Larden, Propp, 1992]. In the right image, the arctic curve is a circle.

Now, we can see some difference between the two graphs. However, we can make things better by lifting the pictures up into 3d via the **height function**. We turn each domino into a 3d shape based on it's color, which gives us a graph of a continuous function.

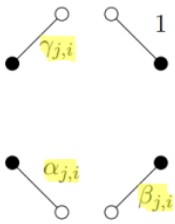
The point of this is that now we have a more suitable language to talk about dimers. It is now easier to guess that almost all dimer covers will life in the tiny neighborhood of a surface once we lift them to 3d - this is known as the **limit shape phenomenon**.



The Aztec model with uniform weights has been already been studied to death - mathematicians have studied it for 35 years and pretty much everything is known. So what is there left to do?

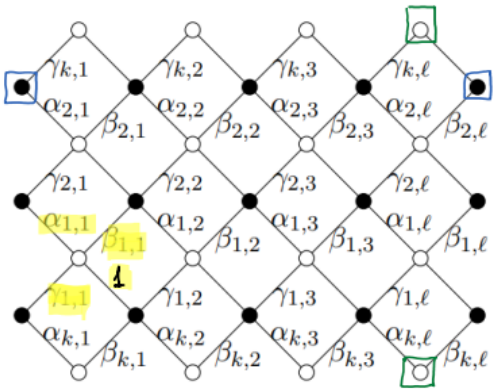
9.2 Dimers with Nonuniform Probability

Let's assign weights to edges of the lattice, and then when we have a particular domino cover, we will multiply the weights of the edges that are part of the domino. If we allow all of the edges to have different weights, but that's too much freedom and we need to restrict it for the model to be possible to analyze at a large scale, so we will only choose finitely many parameters. We will choose a piece of the square lattice of size $k \times \ell$, pick some positive numbers on the edges of that piece, and then duplicate them in all directions.



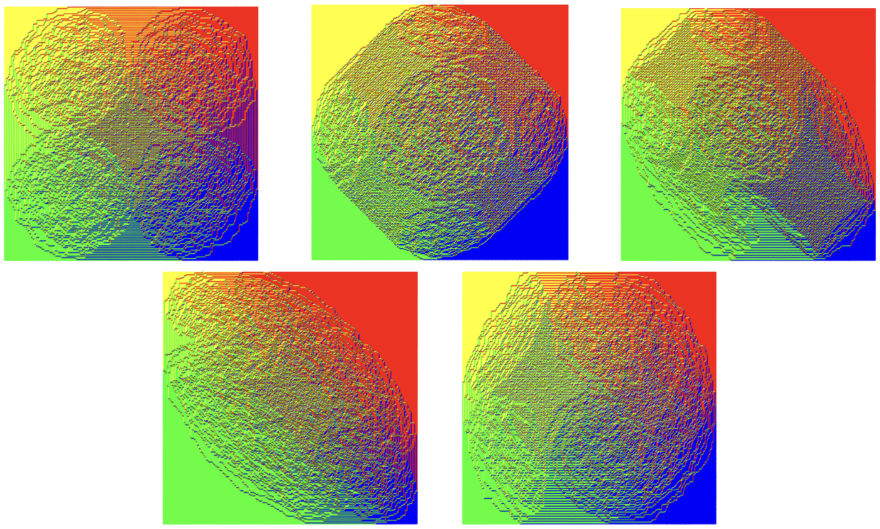
What we get is the following

So there is still the phenomenon of frozen corners, but there are also strange regions in the chaos. If we look at the height function near the star-like region (as indicated with arrows below), the height function is relatively flat, which is



the relation to the roughening phenomenon in chemistry - there is a appearance of a crystal phase in the flat region of the star-like subsection. It seems strange that we can achieve a new phase by just modifying weights, which is a key feature of the model that makes it interesting.

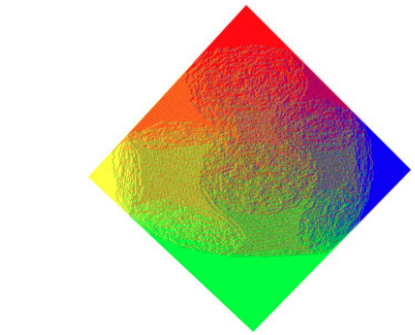
Here are more examples for different sizes of $k \times l$ with different weights.

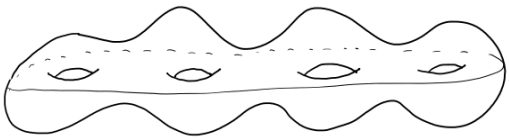


We can see that the images are very different. The main goal of the talk is to explain that we finally know why these pictures look like this.

9.3 The Main Result

Consider the following image:

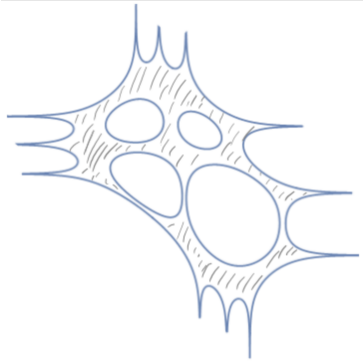




The main message of the talk is that this is a Riemann surface in disguise, particularly a compact Riemann surface of genus $(k-1)\cdot(\ell-1)$ (or a sphere with $(k-1)(\ell-1)$ handles).

The image is of 3×3 periodicity so it is an algebraic curve of genus 4. This lives in the 4d real space or the 2d complex space as zeroes of some polynomials.

To go between this image and the previous one, we need to go through an intermediary called an **amoeba**, which was introduced in [Gelfand, Kapranov, Zelevinsky, 1994].



In particular, we can go from the sphere with holes to the amoeba via the 2-to-1 map:

$$\{(z,w)\in\mathbb{C}^2 : P(z,w)=0\}\to\mathbb{R}^2(z,w)\mapsto(\log|z|,\log|w|)$$

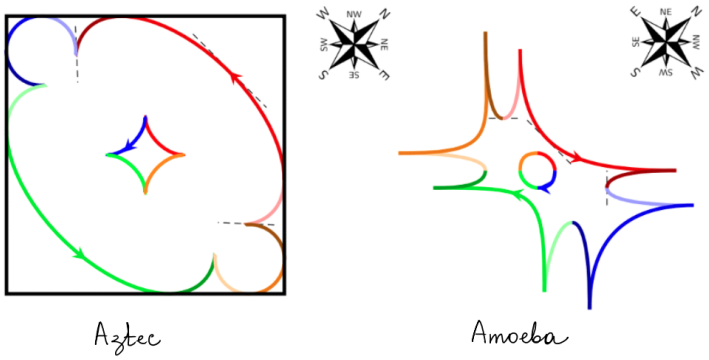
Then, these amoebas are in 1-to-1 correspondence the with the colorful pictures.

The four holes in the sphere with 4 holes corresponds to the four holes in the amoeba, and the tentacles of the amoeba go off to infinity and correspond to the tangency points of the Arctic curve.

The goal of the rest of the talk is to explain why this connection arises. This is not a very simple thing as it's a recent result but it's not too hard either and has to do with integrable systems.

Before we move on, one more cool fact: There is a diffeomorphism between these two images, and the restriction of the map to the boundaries, of the facets and of the amoeba, preserves the tangent lines, ie. the diffeomorphism will map points to points with the same slope of the tangent line.

The reason why the smooth ovals on the amoeba become star-like regions with four cusps is because of there is an orientation change that introduces $4 = (2\pi + 2\pi)/\pi$ cusps on inner components in the Aztec.



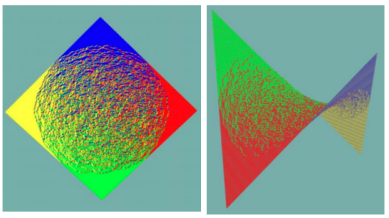
There is a diffeomorphism between these two images. Furthermore, the slope of the tangent line under this diffeomorphism for the boundaries is the same. However, the orientation changes.

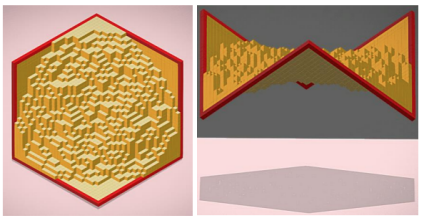
9.4 Dimers to Riemann Surfaces

How do we go from dimers to Riemann surfaces?

One well established path in statistical physics is through a variational problem. This path proves there must exist a limit shape: no matter what boundary conditions we choose there will always be this limiting surface. This surface is a solution to a variational problem.

We know that in the neighborhood of the limit shape there are lots of different possibilities for the height function, more than anywhere else. This is exactly the shape that maximizes the number of possibilities in a small neighborhood, so let's try to count how many possibilities we have in the neighborhood of some surface.





Let's take the surface and cut it into little pieces so that each piece is approximately flat. Let's assume that it's smooth and it's going to be smooth almost everywhere. Then, in each piece, we freeze the height function on the boundary of the piece and count how many possibilities I have inside that piece, fixing the boundary. Because each piece is approximately flat, we can solve this explicitly. It turns out there are

$$\prod_{k=1}^{m}\prod_{\ell=1}^{n}\left|2\cos\frac{\pi k}{m+1}+2i\cos\frac{\pi \ell}{m+1}\right|^{\frac{1}{2}}$$

ways to fill them with dominoes. It might seem that this is not an integer, but it is, which can be shown via a discrete Fourier analysis computation.

So the height function on the boundary of the square is 0, and we are computing the number of possibilities with the frozen flat boundary conditions. In general, we can freeze boundary conditions with different slopes and we can still do the Fourier analysis computation.

The product over flat pieces leads to the exponential of the integral of "surface tension" or "free energy" - the logarithm of the count of almost flat tilings with a given slope.

Let's explain how to go from the weights of a dimer model to an algebraic curve. Start with a fundamental domain and place it on a torus. Then, introduce additional variables w, z on the edges that intersect two particular cycles on the torus. Afterwards, compute all possible dimer covers of that graph on a torus and add them up, which gives a polynomial in w, z. Equating this to zero gives an algebraic curve that is responsible for the asymptotic behavior of the dimer model.

It turns out we can compute the functional/free energy as follows:

Theorem 9.1 (Kenyon, Okounkov, Sheffield, 2003)**.** *The free energy (aka surface tension) is the Legendre transform of the Ronkin function*

$$F(x,w)=\frac{1}{(2\pi i)^2}\int_{|z|=e^x}\int_{|z|=e^y}\log|P(z,w)|\,\frac{dz}{z}\frac{dw}{w}.$$

It is convex on the amoeba and linear on each connected component of the complement.

The high level overview is as follows: we compute how many things we have on each small piece, multiply them all together, and then take the limit, which gives a variational problem. Although we are multiplying things, when we take the log we add them up, which means we have a sum and eventually an integral. This creates a variational problem which is an integral of a certain functional on the surface, and that functional wants to be maximized in order to find the limit shape.

But it turns out that maximizing this functional is very nontrivial: we know the variational problem but we don't know how to find a solution nor whether an explicit solution exists. But once the solution is known, there is an upcoming work by verifying the solution is easy.

9.5 Relation to Integrable Systems

We are going to skip the relation of the dimer model to a matrix refactor problem, which was done by [Duits, Kuijlaars, 2017] and [Berggren, Duits, 2019]. There is a path that allows us to map random dimer problems, in particular domains, into the Riemann-Hilbert problem. We will skip straight to this part, where we want to solve the Riemann-Hilbert problem: we want to move poles of a matrix valued function, splitting a matrix depending on the complex parameter into two pieces, where one piece will have a zero of the determinant inside the circle and the other one will have it outside.

Let's briefly forget about all these words. Here is a question that one can ask an undergrad in their first algebra course: take two arbitrary matrices with z a variable and $\alpha, \beta, \gamma, \delta, a, b, c, d$ complex numbers. When we take these two matrices and multiply them. We want to write this matrix product as a matrix product of two other matrices so that the zero of the determinant goes from left to right okay, ie. swap the zero of the determinant. For the 2×2 case, we have an explicit equation:

$$\begin{pmatrix} \alpha & \gamma z \\ \beta & \sigma \end{pmatrix} \begin{pmatrix} a & c \\ \frac{b}{z} & d \end{pmatrix} = \begin{pmatrix} d & xc \\ \frac{b}{zx} & a \end{pmatrix} \begin{pmatrix} \delta x & \gamma z \\ \beta & \frac{\alpha}{x} \end{pmatrix}$$

Note that the first and fourth matrices have the same eigenvalues, and so do the second and third.

This is a nice birational map, but the issue is that it's not bilinear. In order to solve the problem, we need to iterate this map many, many times to get the

Wiener-Hopf factorization:

$$\begin{aligned}(P_{0,-}P_{0,+})^N &= P_{0,-}P_{0,+}...P_{0,-}P_{0,+}\\ &= P_{0,-}(P_{0,+}P_{0,-})^{N-1}P_{0,+}\\ &= P_{0,-}(P_{1,-}P_{1,+})^{N-1}P_{0,+}\\ &= P_{0,-}P_{1,-}(P_{2,-}P_{2,+})^{N-2}P_{1,+}P_{1,+}\\ &= ...\\ &= (P_{0,-}P_{1,-}...P_{N-1,-})(P_{N+1,+}...P_{1,+}P_{0,+})\end{aligned}$$

The beauty of it is that this problem has an explicit solution in terms of theta functions, done by [Moser, Veslov, 1991]. This is an offshoot of the finite gap integration method of the integrable partial differential equations, and the finite gap method has been developed in particular by Igor Krichever in the early part of his career.

So how does one linearize this integrable system? It is very useful that we can do this iteration over and over, and we can change the coordinates on the space in such a way that the map becomes linear: we have the flow

$$P(z) \mapsto \tilde{P}(z)$$

where

$$P_-(z)P_+(z) = P(z) = \begin{pmatrix} a_{11} & a_{12}+b_{12}z \\ a_{21}+b_{21}z^{-1} & a_{22}\end{pmatrix}$$

is the equation of our flow,

$$\tilde{P}(z) = P_+(z)P_-(z) = P_+(z)P(z)P_+^{-1}(z),$$

and + vs − refers to different zeroes of the determinant (inside vs outside the circle). Notice that this is a conjugation operation, and both the spectrum of $P(z)$ and $\det(P(z)-w)$ (called icospectrality**) are preserved. Hence, $\{(z,w) \in \mathbb{C}^2 : \det(P(z)-w) = 0\}$ is an invariant. It's natural compactification is the spectral curve; it has genus 1 (elliptic curve).

This is a very central idea in integrable systems: represent a nonlinear flow as a compatibility condition of linear problems via Lax pairs.

So we can write that the that characteristic polynomial and its coefficients are actually all integrals of motion of our system, and by equating the characteristic polynomial to zero we get the spectral curve. If we're on our curve, if the determinant of a matrix vanishes it means that it has a nullvector Ψ dependent on z, w. In the theory of integrable systems, this is the Baker-Akhiezer function in the more infinite-dimensional setup, but here it's just the eigenvector.

The Lax pair in our situation is given by

$$(P(z)-w)\Psi(z,w) = 0\tilde{\Psi}(z,w) = R(z)\Psi(z,w)$$

where the solution of $\det R(z) = 0$ is the $|z| < 1$ part of $\det P(z) = 0$. If Ψ satisfies $\left(\hat{P}(z) - w\right)\tilde{\Psi} = 0$ with similar $\hat{P}(z)$, then $R(z) = P_+(z)$ and $\hat{P} = \tilde{P}$ *up to conjugations by diagonal matrices.

The trick is that $\Psi(z, w)$ is a 2d meromorphic function on a spectral curve, with zeroes and poles satisfying some conditions dating back the early 19th century. Key transformations involve understanding how these zeros and poles change. Even though they reside on a Riemann surface where straight lines don't exist, mapping this surface onto the Jacobian allows for such straight-line representation. There is a standard operation that turns a Riemann surface into a torus of dimension equal to the genus of the surface, which aligns the movement of zeros into a straight line. We end up with a tricky change of variables, and to get the solution of the original problem we need to rewrite in terms of multidimensional Theta functions of Jacobians.

In slightly more detail, we normalize $\Psi(z, w) = \begin{bmatrix}\Psi_1 \\ \Psi_2\end{bmatrix}$ by $\Psi_1(z, w) + \Psi(z, w) \equiv 1$. Then, we show that $\Psi_1(z, w), \Psi(z, w)$ span the space of meromorphic functions on the compactification with two fixed simple poles. One zero of Ψ_1 is at 0, one zero of Ψ_2 is at ∞. The second zeroes of Ψ_1, Ψ_2 evolve by linear shifts on the Jacobian of the compactificaiton. The linearity follows from the singularity structure of P_+ and Abel's theorem.

This subsection could be three lectures and we do not do it justice, but let's show some other examples where the same technology of integrating matrix refactorization shows up in classical integrable systems because it's very common.

- The QR algorithm of numeric linear algebra is very popular, especially for calculating eigenvalues. Also closely related to the Toda lattices.
- Billiards inside an ellipsoid and geodesics on the ellipsoid. The 3d analog was one of the motivations for Theta functions.
- Discretization of the Euler equations of free rigid body motion.Dubrovin was one of the last people to care about this. It has not been studied for 30 years until it appeared now.

10 Alexander Bobenko: Dimers and M-curves

Abstract

We develop a general approach to dimer models analogous to Krichever's scheme in the theory of integrable systems. This leads to dimer models on doubly periodic bipartite graphs with quasiperiodic positive weights. Dimer models with periodic weights and Harnack curves are recovered as a special case. This generalization from Harnack curves to general M-curves, which are in the focus of our approach, leads to transparent algebro-geometric structures. In particular, the Ronkin function and surface tension are expressed as integrals of meromorphic differentials on M-curves. Based on Schottky uniformization of Riemann surfaces, we compute the weights and dimer configurations. The computational results are in complete agreement with the theoretical predictions. The talk is based on joint works with N. Bobenko and Yu. Suris.

Contents

10.1 Background

Borodin explained a lot of background in the previous talk, which we will use freely: we have dimer models, perfect matchings, and weights. We will go from Riemann surface theory to statistical mechanics.

Consider the following measure on dimer configurations (perfect matchings):

$$\mathbb{P}(D) = \frac{1}{Z} \prod_{e \in D} \nu(e),$$

with physical weights as face weights

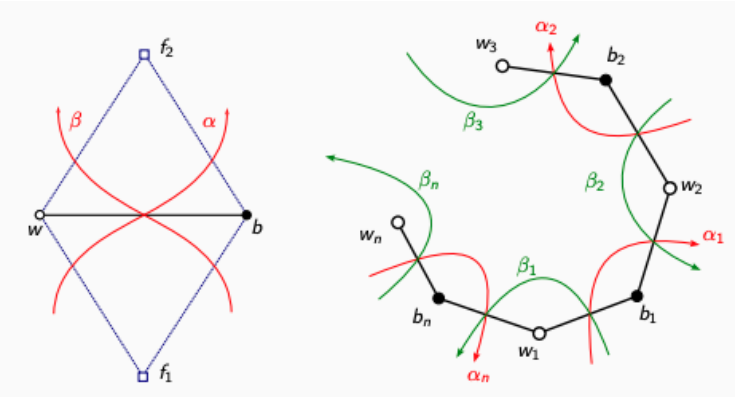
$$W_f = \frac{\nu(e_1)\nu(e_3)...\nu(e_{2n-1})}{\nu(e_2)\nu(e_4)...\nu(e_{2n})}$$

which we assume are real and positive. However, they can be complexified, and the sign is given via the Kasteleyn condition

$$\text{sign}(W_f) = (-1)^{(n+1)}$$

where n is the number of pairs of edges surrounding the face. If this condition is satisfied, then the model is physical.

We are considering bipartite graphs with white and black vertices with parameters associated to these weights, called **train tracks**:

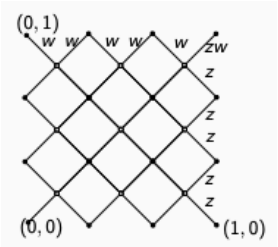


We are interested in integrable systems and these are the features of integrable systems. Here, we have one H with two parameters α and β, and a face of how these train tracks are arranged.

Let's discuss the problem we are going to solve. Similar to the previous talk, we will mostly consider the square lattice because that's the most interesting and also provides us with all essential properties of the models. More specifically, we will consider planar bipartite doubly periodic graph G. In our case, it's a square grid with white and black vertices with two sublattices. Then, we

consider a small piece of such a square and assumes that there are periodic weights associated to the edges of this graph. Placing these weights into the entries into a graph gives the **Kasteleyn matrix**.

Now, we define the spectral curve $K(z, w) = 0$. Taking a fundamental piece of the the weights, label the weights on one edge z and the other w, and then we multiply weights when the intersect. This gives the **spectral curve** $\det K(z, w) = 0$.



When we solve the direct problem for dimers, we consider the eigenfunctions of the modified Kasteleyn matrix:

$$K(z, w)\psi(P) = 0$$

where $P = (z, w)$.

Now, we can turn this spectral curve into a Riemann surface. This is how we generate Riemann surfaces via spectral curves. This is nice because the monodromies of $\psi(P)$ are z, w, and we can analyze the analytic properties of ψ on the spectral curve.

This is the direct problem for dimer models:

$$\text{Weights} \implies \text{Kasteleyn matrix} \implies \text{Spectral curve} \implies \text{Eigenfunction } \psi(P).$$

10.2 The Inverse Problem

Now, we consider the inverse problem: given a spectral curve/Riemann surface and analytic properties of $\psi(P)$, how do we get the explicit representation of $\psi(P)$, and then the corresponding weights?

In 1977, Krichever posed a slightly more general version of this inverse problem, where weights are not quasiperiodic weights (includes all periodic weights). This is now known as **Krichever's scheme**.

The original paper was about the KP equation

$$3u_{yy} = \partial_x(4u_t - 6uu_x - u_{xxx}).$$

Starting with a Riemann surface and some analytic data on it, we can construct the Baker-Akhiezer (BA) function:

$$\psi(x,y,t;P) = \frac{\theta(A(P)+U_1x+U_2y+U_3t+D)\theta(D)}{\theta(A(P)+D)\theta(U_1x+U_2y+U_3t+D)} \exp\left(\xi_1(P)x+\xi_2(P)y+\xi_3(P)t\right).$$

This function is meromorphic on $\mathcal{R}-P_0$ with essential singularity at P_0

$$\psi(P) = (1+o(1/k))\exp\left(kx+k^2y+k^3t\right), k\to\infty, P\to P_0$$

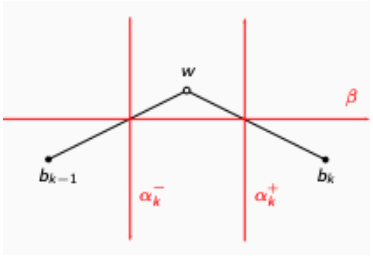
and pole divisor of degree genus of $\mathcal{R}$. It turns out that the BA function can be constructed explicitly, and once we have this, we have an explicit formula for the solution of the KP equation:

$$u(x,y,z) = 2\partial_x^2 \log\theta(U_1x+U_2y+U_3t+D).$$

This is the first case when this approach was applied to a general Riemann surface. There were several cases which were studied before, but this the paper where Krichever's scheme appeared became very influential because it was general.

In a way, we can apply Krichever's scheme to dimer models.

The data we are given is a compact Riemann surface $\mathcal{R}$ and train track parameters $\alpha_i \in \mathcal{R}$.



Then, we construct a meromorphic function ψ on $\mathcal{R}$ (BA-function), a function $\psi_b : \mathcal{R}\to\mathbb{C}$ on every black vertex b, with the property that ψ picks up a zero or a pole at $\alpha\in\mathcal{R}$ whenever crossing a train track.

As soon as we have these analytic properties, we can write a formula for this function:

$$\psi_b(P) = \frac{\theta(A(P)+\eta(b)+D)}{\theta(A(P)+\eta(b_0)+D)}\prod\frac{E(P,\alpha_k^-)}{E(P,\alpha_k^+)}\eta(b)-\eta(b_0) = \sum(A(\alpha_k^- - \alpha_k^+)).$$

This is quite similar to Krichever's formula in the KP case but now the function is meromorphic. These type of functions appear in the analysis of discrete integrable models.

It turns out that these functions are linearly dependent. If we consider a star associated to a white vertex, so that all neighboring vertices are black vertices, then there is a Baker-Akhiezer function sitting at all of these black vertices and they are linearly dependent. We can see this just by analyzing their analytic properties.

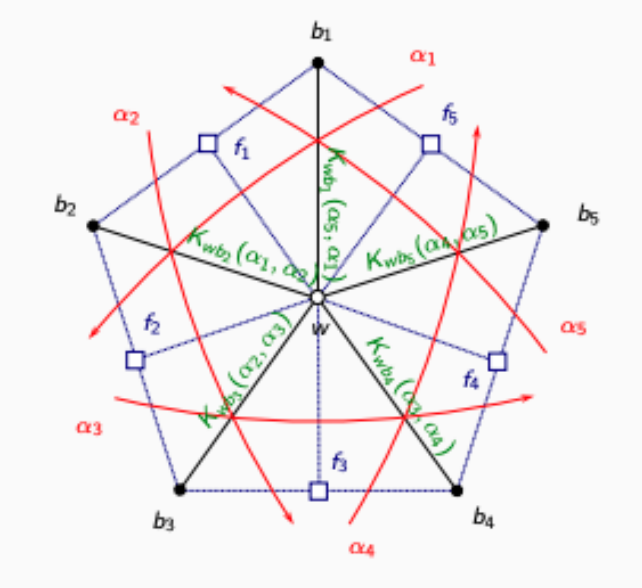
And exactly as in the KP case, from here, we can compute the corresponding coefficients, which are now independent of the point of the Riemann surface but depend on these parameters α. We have the **Dirac equation**

$$\sum_{k=1}^{n} K_{wb_k}(\alpha_{k-1}, \alpha_k)\psi_{b_k}(P) = 0,$$

and we can find the coefficients via the following equation:

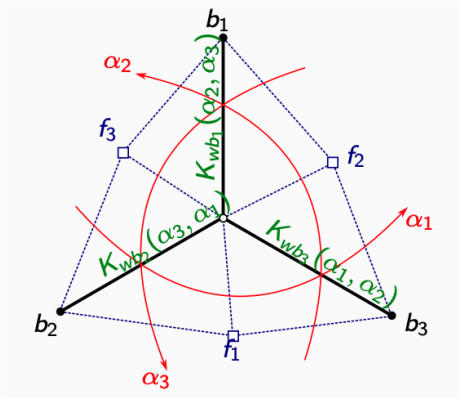
$$K_{wb}(\alpha, \beta) = \frac{E(\alpha, \beta)}{\theta(\eta(f_1) + D)\theta(\eta(f_2) + D)}.$$

These are called **Fock weights**.



Remark 10.1. *If we play this game for a triangle, we obtain a Dirac equation for 3 Baker-Akhiezer functions, and if we write down the equation explicitly, we get the famous* ***Fay identity****:*

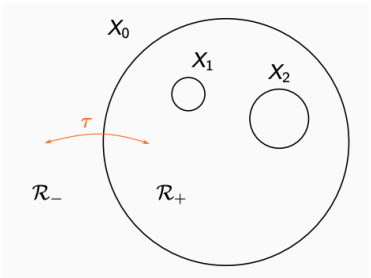
$$\begin{aligned}
&\theta(A(\alpha_2) + A(\alpha_3) + D)\theta(A(P) + A(\alpha_1) + D)E(\alpha_2, \alpha_3)E(P, \alpha_1) \\
+&\theta(A(\alpha_1) + A(\alpha_3) + D)\theta(A(P) + A(\alpha_2) + D)E(\alpha_3, \alpha_1)E(P, \alpha_2) \\
+&\theta(A(\alpha_1) + A(\alpha_2) + D)\theta(A(P) + A(\alpha_3) + D)E(\alpha_1, \alpha_2)E(P, \alpha_3) \\
=&0.
\end{aligned}$$



So after starting with a lot of complex analysis, we are now going in the direction of statistical mechanics.

It is important that the weights are real and positive, so we should take $\mathcal{R}$ to be an M-curve with antiholomorphic involutions τ and fixed ovals $X_0, \dots, X_g$.

It is important that we would like to have these weights real and positive. So we should specify that we have a real Riemann surface with an antiholomorphic involutions τ and fixed ovals $X_0, \dots, X_g$. In this case, we want $\mathcal{R}$ to be an M-curve: a Riemann surface of genus g with the maximal possible number of real ovals $g + 1$.



In this picture, we can see half of this Riemann surface. It's simply connected flat planar domain with two holes here, with all of the boundaries are round. We can take two copies and glue them together along the boundaries, which gives a Riemann surface that is an M-curve.

Or you just make an inversion, which is indicated here by τ, with respect to one of these circles x_0, and then we can generate the picture on the whole complex plane with the holes x_1 and x_2, and then we have $\mathcal{R}_+$ and $\mathcal{R}_-$ - two halves of the Riemann surface that are glued together into one compact Riemann surface, which is an M curve.

It is important that this is a Riemann surface that can be used for computations

and not just an illustration. In addition, all figures in this talk are not hand drawn but computed.

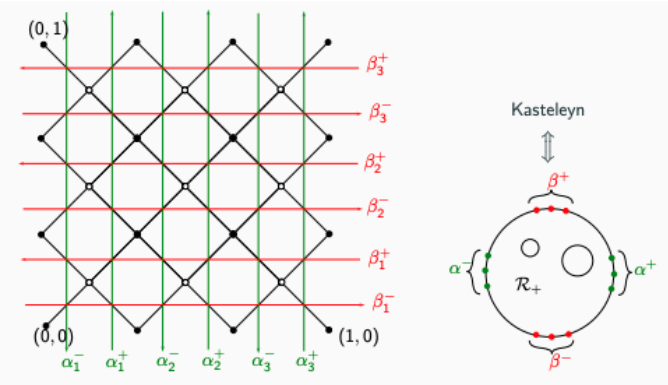
Now we've learned what is an M curve, how one can uniformize it, how one can represent it in a way that is numerically treatable, recall of the train track parameters, which were the points on the Riemann surface, they are now cons, they should sit just on one of these real ovals. If we choose these α's this way, then the weights become real. This is the **reality condition**.

We have one more condition: the sign of this real number should satisfy the Kasteleyn condition. We can do this as follows: the ordering condition on α_i, β_i gives

$$\operatorname{sign}(W_f) = (-1)^{(n+1)}.$$

In general, this condition was investigated on minimal graphs by [Buotillier, Cimasoni, de Tilière, '20, '21]. For $g = 0$, [Kenyon, 02] and [Kenyon, Okounkov, 2003] showed that these were isoradial weights.

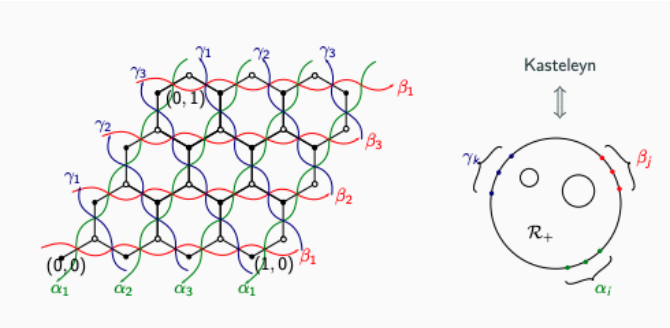
Now, let's come back to our square grid and simplify the picture. In the case of the square grid, we have train track parameters α_+, α_-, and train track parameters β_+, β_-. If we examine the Kasteleyn condition, we obtain the result on the right-hand side.



All points on the Riemann surface should lie on one real oval, and moreover, the parameters should be clustered so that they belong together and are not separated by any other points. The weights are positive if and only if the corresponding Riemann surface is an M-curve.

Similarly, this can be applied to the hexagonal grid, where there are three different types of train tracks: α, β, γ. They should also be clustered similarly as shown.

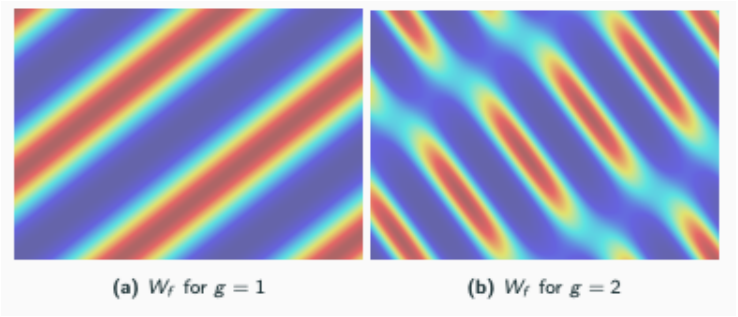
What we are studying here is not just complex analysis; this is statistical mechanics because all the weights satisfy this positivity property.



The formula for the weight is

$$W_f = \prod_{i=1}^{n} \frac{\theta[\Delta]\left(\int_{\alpha_i}^{\beta_i} \omega\right) \theta(\eta(f_{2i}) + D)}{\theta[\Delta]\left(\int_{\beta_i}^{\alpha_{i+1}} \omega\right) \theta(\eta(f_{2i-1}) + D)}$$

Performing some computations gives pictures for W_f (for $g = 1, 2$):



(a) W_f for $g = 1$ **(b)** W_f for $g = 2$

In the case of genus 0, we end up with isoradial weights, so all functions become rational functions.

If we revert to the doubly periodic case, we recover the spectral curve which was done by [Kenyon, Okounkov, Sheffield, 2007], and the corresponding Riemann surface is a Harnack curve [Mikhalkin, 2000].

The weights we are considering here, which are solutions for the KP equations, are not necessarily periodic; they are quasi-periodic. They arise from linearization on the Jacobian variety, and this linearization hyperplane/2 plane can cross the Jacobi variety in a way that avoids rational points and becomes irrational. Then, we can see that the weights are periodic if and only if $\sum_i \left(\alpha_i^- - \alpha_i^+\right), \sum_i \left(\beta_i^- - \beta_i^+\right)$ are principal divisors.

In the case when $g > 0$, the train track parameters repeating periodically does not necessarily imply that the weights K_{wb}, W_f are periodic.

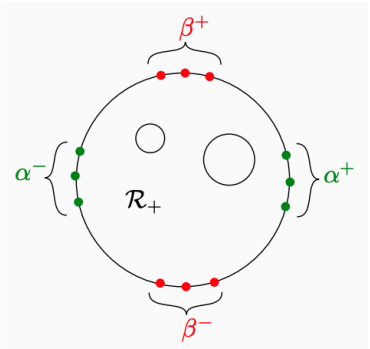
10.3 Algebra-Geometric Description

Now, let's delve deeper into the algebraic-geometric description that arises here.

The goal is to describe limiting objects in treatable form so that we can compute their characteristics.

The data we are given is an M-curve $\mathcal{R}$ with antiholomorphic involution τ and parameters $\{\alpha_i^{\pm}, \beta_j^{\pm}\} \in X_0$ with clustering condition.

Now, we introduce $d\zeta_1, (\, d\zeta_2)$ normalized meromorphic differentials with residues ∓ 1 at $\alpha_i^{\pm}, (\beta_i^{\pm})$ and zero a-periods (the a periods are the small circles in the image)



Now, we can take

$$\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$$

which is an abelian integral that is well define on $\mathcal{R}_+$ since the a-periods vanish.

Proposition 10.2 (Krichever, 2014)**.** *$(x_1, x_2), (y_1, y_2)$ are coordinates on $\mathcal{R}_+^\circ$.*

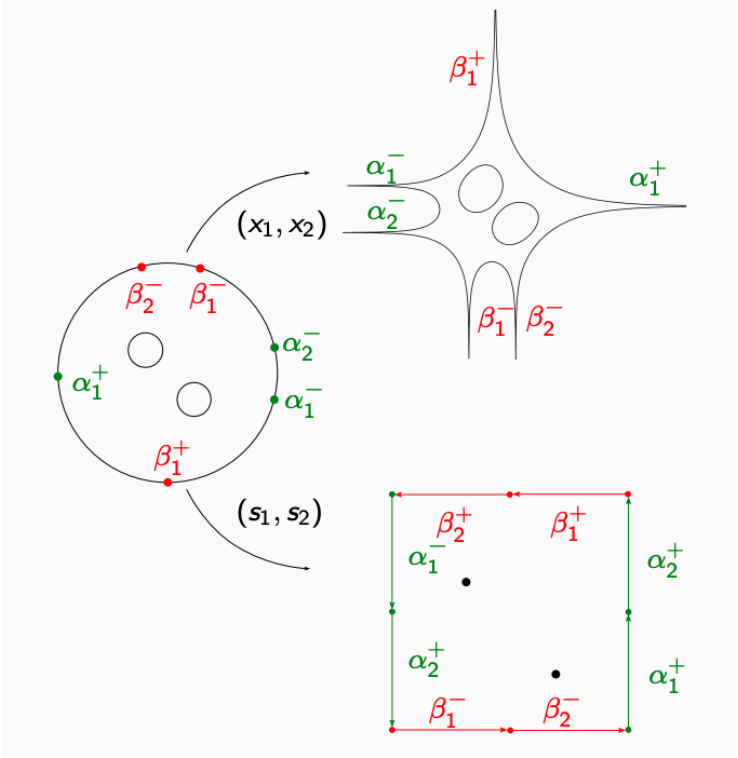
For historical reasons, $(s_1, s_2) = \frac{1}{\pi}(y_2, y_1)$ was used, which is is basically the same thing. We have the following image

There is a diffeomorphism from the left to the top right image. On the other hand, the map from the left to the bottom right image gives a polygon, namely the Newton polygon in the periodic case.

How can we use this? Using this method, we can introduce new definitions for the Ronkin function and the surface tension:

Definition 10.3. *The **Ronkin function** is given by*

$$\rho(P) = \rho(x_1, x_2) = -\frac{1}{\pi} Im \int_\ell \zeta_2 \, d\zeta_1 + x_2 s_2.$$


β_1^+
α_1^-
α_1^+
α_2^-
(x_1, x_2)
β_2^-
β_1^-
β_1^-
β_2^-
α_2^-
α_1^+
α_1^-
β_1^+
(s_1, s_2)
β_2^+
β_1^+
α_1^-
α_2^+
α_2^+
α_1^+
β_1^-
β_2^-

Definition 10.4. *The* ***surface tension*** *is given by*

$$\sigma(P) = \sigma(s_1, s_2) = \frac{1}{\pi} Im \int_\ell \zeta_2 \, d\zeta_1 - x_1 s_1$$

Proposition 10.5. *They satisfy the* ***Legendre dual*** *property:*

$$\nabla\sigma(s_1, s_2) = (x_1, x_2) \nabla\rho(x_1, x_2) = (s_1, s_2).$$

These definitions agree with the algebraic ones in the doubly periodic case:

$$F(x_1, x_2) := \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |\mathcal{P}(e^{x_1} z, e^{x_2} w)| \, \frac{dz}{z} \frac{dw}{w}.$$

This case is more general: it is not doubly periodic and there is no spectral curve, only the Riemann surface. However, if we consider the doubly periodic case, we can show that it coincides the the previous equation. They coincide up to an affine function because their Hessians coincide, but fortunately for us this function is not important for variational principles because if we change ρ or Σ by an affine function, it doesn't change my variational principle. Therefore, we can use both Ronkin functions consider variational problems. For the purposes of this talk, the first one is much more useful because we can analyze the convergence and many other things.

Now, we have extended this picture with two more maps.

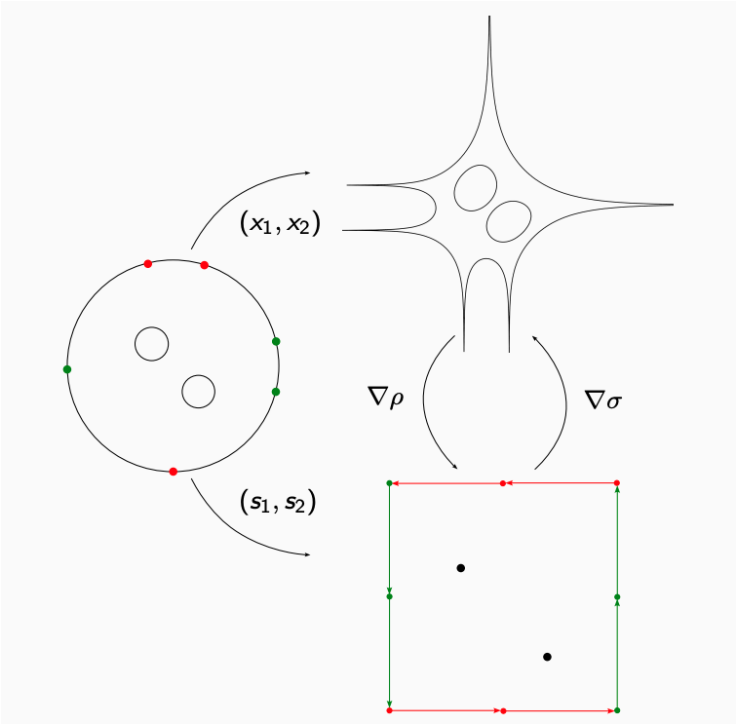
So there's a dictionary

Object	Doubly Periodic Notation	Our Setup Notation
Spectral Curve	$\det(P(z, w)) = 0$	$\mathcal{R}$
Monodromies	(z, w)	$\left(\frac{\psi_{1,0}}{\psi_{0,0}}, \frac{\psi_{0,1}}{\psi_{0,0}}\right)$
Main Differentials	$(dz/z, dw/w)$	$(d\zeta_1, d\zeta_2)$
Amoeba Map	$(\log \|z\|, \log \|w\|)$	$(\text{Re } \zeta_1, \text{Re } \zeta_2)$

Now, let's discuss the height function. The discrete height function $h : G^\times \to \mathbb{Z}$ is in bijection to the dimer configuration. So we can analyze dimer configurations by analyzing the corresponding height function. Additionally, the boundary conditions are purely combinatorial and deterministic. Shown in [Cohn, Kenyon, Propp, 2001], the limiting height function converges to the minimizer of $\int_\Omega \sigma(\nabla h)$. In our more general situation with quasiperiodic weights, σ is slightly different, but this is still true, shown in [Bobenko, Bobenko, Suris].

What does the minimizer look like? [Berggren, Borodin, 2023] gave the answer for periodic weights for Aztec diamond via a purely variational proof and more general boundary conditions.

Now, we will extend this general picture to include the Aztec Diamond. The first definition of complex structure on Diamond was obtained in [Kenyon, Okounkov,



2005], but the very crucial result is [Berggren, Borodin, 2023] where we define $d\zeta = d\zeta_{(u,v)} := -u\,d\zeta_2 + v\,d\zeta_1 + d\zeta_3$ for the Aztec diamond. The analytic properties are chosen as follows:

Res	α_i^-	β_i^-	α_i^+	β_i^+
$d\zeta_1$	1	0	-1	0
$d\zeta_2$	0	1	0	-1
$d\zeta_3$	1	-1	1	-1

Now, we can compute the number of zeroes and poles:

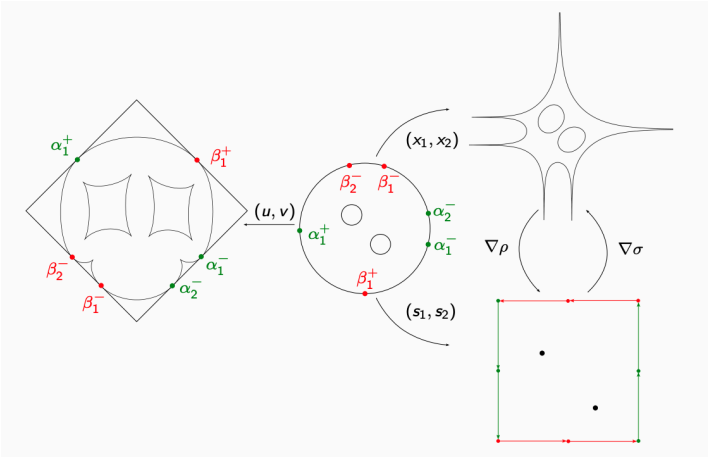
$$\#\text{zeroes} = \#\text{poles} + 2g - 2$$

For any $(u, v) \in (-1, 1)^2$, we have: 2 zeroes on any inner oval, 1 zero between $\alpha_i^\pm, \alpha_{i+1}^\pm$, 1 zero between $\beta_i^\pm, \beta_{i+1}^\pm$, and 2 free zeroes.

Definition 10.6. *If we have conjugated free zeroes $(P, \tau P), P \in \mathcal{R}_+^\circ$, then $(u, v) \in \mathcal{F}_S$ is a **liquid region**.*

Proposition 10.7 (Berggren, Borodin, 2023)**.** *$\mathcal{F} : P \in \mathcal{R}_+^\circ \mapsto (u, v) \in \mathcal{F}_S$ is a diffeomorphism.*

So now the picture looks like: This looks very complicated, so let's see how it works for the Arctic curve in the isoradial case ($g = 0$). Let $d\zeta_i(z) = f_i(z)\,dz$.



Then

$$f_1(z) = \sum_i \frac{1}{z - \alpha_i^-} - \frac{1}{z - \alpha_i^+}$$
$$f_2(z) = \sum_i \frac{1}{z - \beta_i^-} - \frac{1}{z - \beta_i^+}$$
$$f_3(z) = \sum_i \frac{1}{z - \alpha_i^-} + \frac{1}{z - \alpha_i^+} - \frac{1}{z - \beta_i^-} - \frac{1}{z - \beta_i^+}$$

imply

$$u = \frac{W(f_1, f_3)}{W(f_1, f_2)}, v = \frac{W(f_1, f_3)}{W(f_1, f_2)}$$

where W is the Wronksian

$$W(f_i, f_j) = \begin{vmatrix} f_i & f_j \\ f_i' & f_j' \end{vmatrix}$$

Let's generalize a bit.

Definition 10.8. *Define the* ***height function***

$$H(u, v) = \int_\Omega d\zeta_{(u,v)}$$

where $H = \frac{1}{\pi} g + ih$.

We can now show that the integrals of the divergence free fields are

$$J\nabla g = (x_1, x_2), \nabla h = (s_1, s_2),$$

and h can be extended affinely to holes and frozen regions.

Theorem 10.9. *h is a surface tension minimizer*

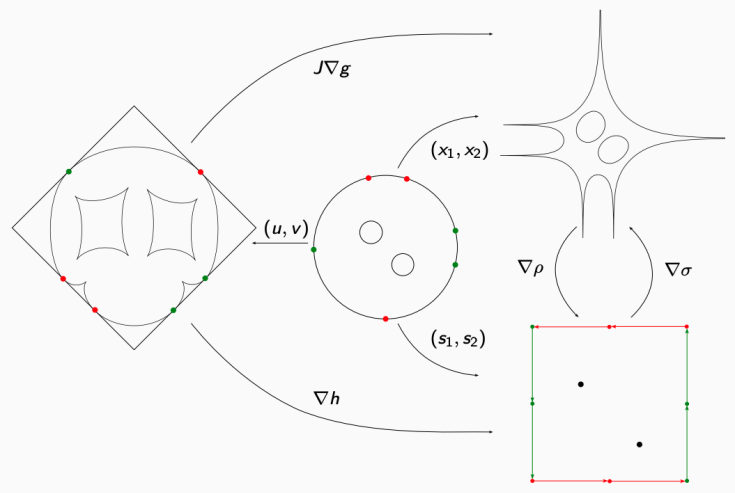
$$h = arg\ min_f \int_{[-1,1]^2} \sigma(\nabla f).$$

There is also a dual theorem:

Theorem 10.10. *g is a magnetic tension minimizer*

$$g = arg\ min_f \int_{[-1,1]^2} \rho(\nabla(f)).$$

This gives two more arrows in the diagram which makes it complete:



In a more general setup, we can still compute all of the formulas efficiently via Schottky uniformization.

Definition 10.11. *The **Schottky group** is a free group G generated by inversions in circles X_i, with differentials given by Poincaré theta series:*

$$d\zeta_1(z) = \sum_{g \in G} \sum_i \left(\frac{1}{z - g(\alpha_i^-)} - \frac{1}{z - g(\alpha_i^+)} \right) dz$$

$$d\zeta_1(z) = \sum_{g \in G} \sum_i \left(\frac{1}{z - g(\beta_i^-)} - \frac{1}{z - g(\beta_i^+)} \right) dz.$$

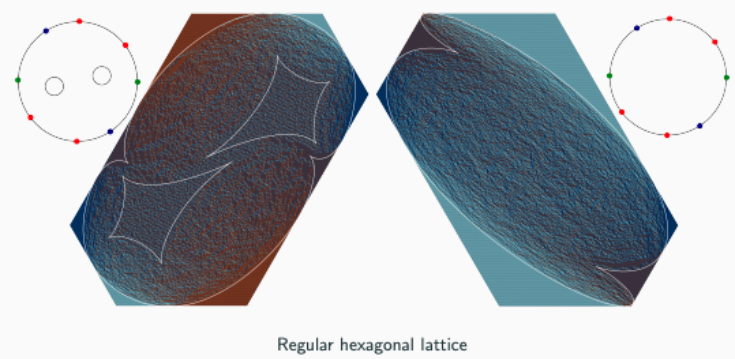
10.4 Proof Idea

The proof idea is to minimize $\int_{[-1,1]^2} \sigma(\nabla h)$, consider the Euler-Lagrange equation $\text{div}(\nabla\sigma(\nabla h)) = 0$. In the liquid region, this is just $\text{div}(x_1, x_2)$. But that's

not the end of the story: the surface tension is non-differentiable because it has conical singularities, which makes it challenging to prove that what we obtained is indeed a minimizer.

We should allow non-differentiabilities in σ, which gives a generalized Euler-Lagrange in terms of subgradients. The case $g = 0$ has down by [Astala, Duse, Prause, Zhong, 2020] and the case $g > 0$ was done by [Bobenko, Bobenko, Suris], which follows from the existence of the extension of g to gas bubbles and frozen regions.

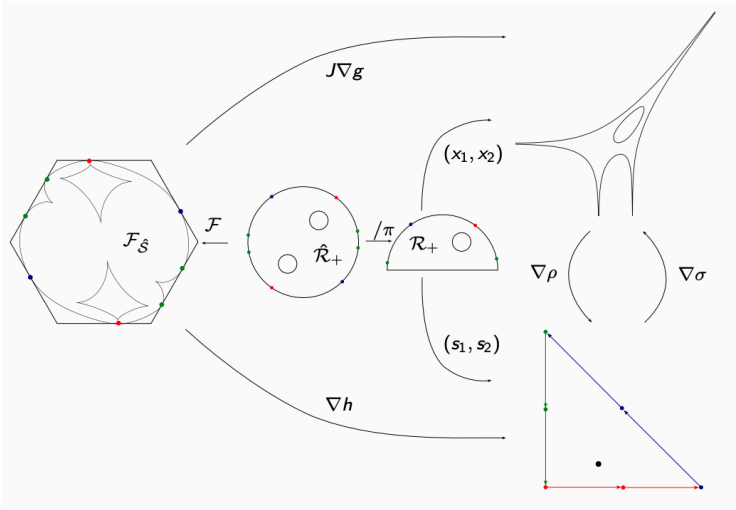
Other problems can also be solved in a similar idea:



This is a picture of a hexagonal Riemann surface, which has a holomorphic symmetry. This is a genus zero example, and this is an example of genus two. Here, the Riemann surface we are considering is a ramified covering of a Riemann surface, and the holes are mapped to the gas regions of our Dimer model. Furthermore, there is a one-to-one map to the liquid domain. The complete maps looks like this:

So, in the last picture for the hexagonal case, our Riemann surface is a double cover. The weights live on the factor of this double cover. We have a defined morphism to the Diamond liquid domain from the double cover and a defined morphism to the Newton polygon from this factor where the weights live.

We can also investigate how these gas regions behave, how they can come together, and become hexagonal, which will be presented by Nikolai Bobenko at RTISART.



11 Youjin Zhang: Bihamiltonian Integrable Systems and their Classification

Abstract

Bihamiltonian structure plays an important role in the theory of integrable systems. For a system of evolutionary PDEs with one spatial variable which possesses a bihamiltonian structure, one is able to find, under a certain appropriate condition, infinitely many conservation laws of the system from the bihamiltonian recursion relation and to arrive at its integrability. In the case when the bihamiltonian structure of the system of evolutionary PDEs possesses a hydrodynamic limit, one can further obtain from it a flat pencil of metrics, and relate it to Frobenius manifold structures or their generalizations under a certain condition, such a relationship may help one to find applications of the integrable system in different research areas of mathematical physics. In this talk, we will recall the notion of bihamiltonian integrable systems, explain their relationship with Frobenius manifold structures or their generalizations, and review the results on the classification of bihamiltonian integrable hierarchies which possess semisimple hydrodynamic limits.

Contents

11.1 Introduction

Let's first explain what kind of bihamiltonian integrable systems we will discuss. The most prototypical version is the integrable nonlinear evolutionary PDE known as the **Korteweg-de Vries (KdV) equation**:

Definition 11.1. *The **KdV equation** is*

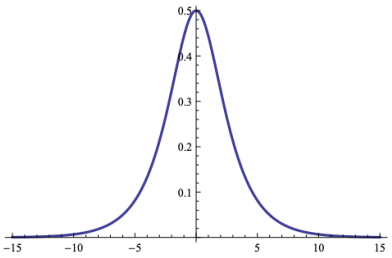
$$u_t + 6uu_x + u_{xxx} = 0, u = u(x,t).$$

It was introduced in [Korteweg, de Vries 1895] to model shallow water solitary waves which were observed by [Russell 1834] in the Union Canal near Edinburgh.

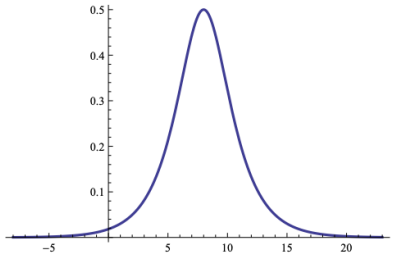
There exists a solitary wave solution of the KdV equation, given by

$$u(x,t) = 2\partial_x^2 \log \tau = 2k^2 \mathrm{sech}^1(kx - 4k^3 t), \tau = \cosh(kx - 4k^3 t).$$

Two graphs of this solution are graphed below:



(a) with $k = \frac{1}{2}, t = 0$



(b) with $k = \frac{1}{2}, t = 8$

Around 70 years later, it was discovered that there is a 2-soliton solution of KdV given by

$$u(x,t) = 2\partial_x^2 \log \tau,$$

where

$$\tau = \det \begin{pmatrix} -k_1(c_1 e^{\phi_1} - e^{-\phi_1}) & c_1 e^{\phi_1} + e^{-\phi_1} \\ -k_2(c_2 e^{\phi_2} - e^{-\phi_2}) & c_2 e^{\phi_2} + e^{-\phi_2} \end{pmatrix}$$

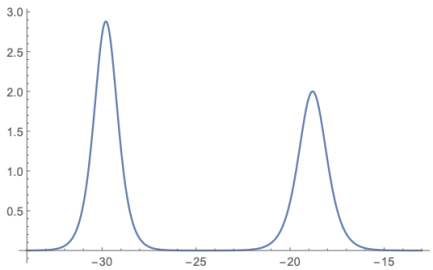
and

$$\phi_i = k_i x - 4k_i^3 t, 0 < k_1 < k_2.$$

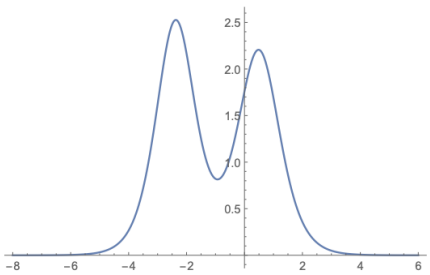
With the parameters $k_1 = 1, k_2 = 1.2, c_1 = 1, c_2 = 1$, we have the following diagrams:

Furthermore, there is a phase shift after the interaction: For the 2-soliton solution, when one moves at the speed $v = 4k_1$, then

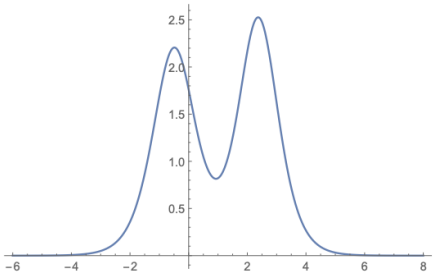
$$u(x,t) \sim 2k_1^2 \mathrm{sech}^2(k_1 x - 4k_1^3 t + \theta), t \to -\infty, u(x,t) \sim 2k_1^2 \mathrm{sech}^2(k_1 x - 4k_1^3 + \tilde{\theta}), t \to +\infty,$$



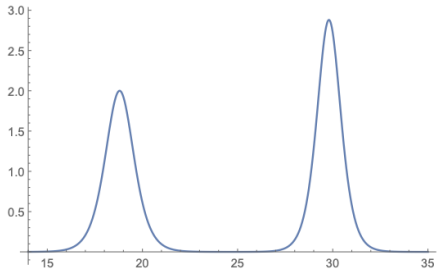
(a) with $t = -5$



(b) with $t = -0.2$



(c) with $t = 0.2$



(d) with $t = 5$

where

$$\theta = \frac{1}{2}\log\frac{c_1(k_2-k_1)}{k_2+k_1}, \tilde{\theta} = \frac{1}{2}\log\frac{c_1(k_2+k_1)}{k_2-k_1},$$

and we have the phase shift

$$\tilde{\theta} - \theta = \log\frac{k_2-k_1}{k_2+k_1}.$$

This observation means that there are some integrability (possesses enough symmetry or has many conservation laws) behind these soliton equations. Indeed:

Definition 11.2 (Miura, 1967)**.** *The* ***Miura transformation*** *is defined as*

$$u = -v^2 - v_x$$

The Miura transformation relates the KdV equation to the modified KdV equation as follows:

$$u_t + 6uu_x + u_{xxx} = (-2v - \partial_x)(v_t - 6v^2v_x + v_{xxx}).$$

It's generalization [Miura, Garner, Kruskal, 1967]

$$u = w - \epsilon w_x - \epsilon^2 w^2, u_t + 6uu_x + u_{xxx} = (1 - 2\epsilon^2 w - \epsilon\partial_x)(w_t + t(w - \epsilon^2 w^2)w_x + w_{xxx})$$

leads to the fact that the KdV equation has infinitely many conservation laws. This was a very important step toward the discovery of the integrability of the KdV equation.

Then, from the generalized Miura transformation one obtains, after the transformation

$$w = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\partial_x \log\psi,$$

the Schrödinger equation

$$\psi_{xx} + u\psi = \lambda\psi, \lambda = \frac{1}{4\epsilon^2}.$$

leads to the inverse scattering method for solving the initial value problem of the KdV equation [Gardner, Greene, Kruskal, Miura 1967]. This is arguably the most influential result in the early days of studying solitons.

Another important development was the KdV hierarchy and their Lax pair formalism: Together with the KdV equation, there are higher order KdV equations which are also related to the Schrödinger operator

$$L = \partial_x^2 + u,$$

they form the KdV hierarchy and can be represented by Lax equations [Lax 1968]

$$\frac{\partial L}{\partial t_k} = [B_k, L], k = 0, 1, 2, ...$$

Here, $\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t}, B_0 = \partial_x, B_1 = -4\partial_x^3 + 12u\partial_x + 6u_x, ...$

Now, we introduce the dispersion parameter ϵ by a rescaling, and consider the KdV equation of the form

$$u_i = uu_x + \frac{\varepsilon^2}{12}u_{xx}.$$

An important property of the KdV equation is the bihamiltonian structure:

Proposition 11.3 (Faddeev, Zakharov, 1971; Gardner 1971; Magri 1978)**.** *The KdV equation has bihamiltonian structure*

$$\frac{\partial u}{\partial t} = \mathcal{P}_1 \frac{\delta H_1}{\delta u} = \frac{3}{2}\mathcal{P}_2 \frac{\delta H_0}{\delta u}.$$

Here, the Hamiltonian operators are given by

$$P_1 = \partial_x, \quad \mathcal{P}_2 = u(x)\partial_x + \frac{1}{2}u_x + \frac{\varepsilon^2}{8}\partial_x^3.$$

The Hamiltonians $H_k = \int h_k dx, k = 0, 1$ of the KdV equation can be obtained can be obtained by using the bihamiltonian recursion relation

$$P_1 \frac{\delta H_k}{\delta u} = \left(k + \frac{1}{2}\right) P_2 \frac{\delta H_{k-1}}{\delta u}, \quad k \geq 0$$

starting from the Casimir $H_{-1} = \int u\, dx$ of the first Hamiltonian structure $\mathcal{P}_1$. We have

$$h_0 = \frac{1}{2}u^2 + \frac{\varepsilon^2}{12}u_{xx}, \quad h_1 = \frac{1}{6}u^3 + \frac{\varepsilon^2}{24}\left(u_x^2 + 2uu_{xx}\right) + \frac{\varepsilon^4}{240}u^{(4)}, \cdots$$

Thus from the bihamiltonian recursion relation one arrives at infinitely many conserved quantities of the KdV equation, and one also obtains the KdV hierarchy:

Proposition 11.4. *The KdV equation satisfies the* ***KdV hierarchy****:*

$$\frac{\partial u}{\partial t_k} = p_1 \frac{\delta H_k}{\delta u}, \quad k = 0, 1, 2, .. \ .$$

The densities of the Hamiltonians $H_k = \int h_k(u, u_x, \cdots) \, \mathrm{d}x$ can be chosen to satisfy

$$\frac{\partial h_{p-1}}{\partial t_q} = \frac{\partial h_{q-1}}{\partial t_p} = \partial_x \Omega_{p,q}(u, u_x, \cdots),$$

This leads to the definition of tau function for the KdV hierarchy.

Definition 11.5. *The* ***tau function*** *for the KdV hierarchy satisfies*

$$\Omega_{p,q}(u, u_x, \ldots)|_{u \to u(x,t_0,t_1,\ldots)} = \varepsilon^2 \frac{\partial^2 \log \tau}{\partial t_p \partial t_q}, \quad p, q \geq 0.$$

In particular, we have

$$u = \varepsilon^2 \frac{\partial^2 \log \tau}{\partial t_0 \partial t_0} = \varepsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial x}.$$

Here is a timeline with some results related to tau functions:

1. Bilinear equations for soliton equations [Hirota, 1970's].
2. Solution of KP equation and Grassmannians [Sato, 1980's].
3. The notion of tau functions for monodromy preserving deformation equations of linear ODEs, and for finite dimensional Hamiltonian systems [Jimbo, Miwa, Ueno, 1980's].
4. Relation of integrable hierarchies with infinite dimensional Lie algebras [Date, Kashiwara, Jimbo, Miwa; Kac, Wakimoto, 1980's].
5. Tau structures for infinite dimensional Hamiltonian systems.[Dubrovin, Zhang, 2001]

One important application of the KdV hierarchy is 2d gravity: The partition function of the 2d topological gravity is a tau function of the KdV hierarchy. This was conjectured by Witten in 1991 and proved by Kontsevich 1992. It can be written as follows:

$$u = \varepsilon^2 \frac{\partial^2 \log \tau(t_0, t_1, \dots)}{\partial x^2} = \sum_{q \geq 0} \varepsilon^{2g} \frac{\partial^2 \mathcal{F}_g(t_0, t_1, \dots)}{\partial x^2}, \quad x = t_0,$$

where $\tau = e^{\sum_{g \geq 0} \varepsilon^{2g-2} \mathcal{F}_g}$ is the partition function of 2d topological gravity with

$$\mathcal{F}_g = \sum \frac{1}{k!} t_{p_1} \cdots t_{p_k} \int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{p_1} \wedge \cdots \wedge \psi_k^{p_k}.$$

Apart from the KdV hierarchy, there are many bihamiltonian integrable systems with tau functions. We present a few below:

Example 11.6. *All these the bihamiltonian structures and the associated integrable hiearrchies possess hydrodynamic limits:*

- *Drinfeld-Sokolov hierarchies associated to untwisted affine Kac-Moody algebras [Drinfeld-Sokolov 1981].*
- *The q-deformed N-th KdV hierarchy [Frenkel-Reshetikhin 1996; Frenkel 1996]*
- *The Toda lattice hierarchy and its generalizations—the bigraded Toda lattice hierarchies [Carlet 2006].*
- *A class of integrable evolutionary PDEs of hydrdynamic type that arise and play important roles in the study of 2d topological field theory, and more generally, the theory of Frobenius manifolds [Dubrovin 1992].*

Now, we present an example of an integrable hierarchy that shares properties of the KdV hierarchy:

Example 11.7. *Consider*

$$\frac{\partial v}{\partial t_k} = \frac{1}{k!} v^p v_x, \quad k \geq 0,$$

and its bihamiltonian structure

$$\frac{\partial v}{\partial t_k} = \mathcal{P}_1^{[0]} \frac{\delta H_k^{[0]}}{\delta v} = \left(k + \frac{1}{2}\right) \mathcal{P}_2^{[0]} \frac{\delta H_{k-1}^{[0]}}{\delta v},$$

where

$$P_1^{[0]} = \partial_x, \quad P_2^{[0]} = v(x)\partial_x + \frac{1}{2} v_x; \quad H_k^{[0]} = \frac{1}{(k+1)!} \int v^{k+1} \, \mathrm{d}x.$$

The tau function of the dispersionless KdV hierarchy is

$$\frac{v^{p+q+1}}{p!q!(p+q+1)} = \frac{\partial^2 \log \tau^{[0]}}{\partial t_p \partial t_q}, \quad p, q \geq 0.$$

11.2 Infinite Jet Space and Hamiltonian Structures

Let's describe the class of bihamiltonian structures and the associated integrable hierarchies which possess hydrodynamic limit and tau functions. We will start from the study of a class of so called **flat exact bihamiltonian structures of hydrodynamic type**, show the existence of an integrable hierarchy of systems of hydrodynamic type, and the tau functions. Then we study the classification of deformations of these bihamiltonian integrable hierarchies.

First, we consider the finite dimensional case of Hamiltonian structures.

11.2.1 The Finite Case

Definition 11.8. *Let M^n be a smooth manifold. A **Poisson structure** (or Hamiltonian structure) on M is defined by a Poisson bracket*

$$\{\ ,\ \}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M).$$

Equivalently, it is defined by a Poisson bivector P satisfying the condition $[P, P] = 0$, where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket defined on the space of multivector fields $\Lambda^* = \Gamma(\Lambda^*(TM))$.

Proposition 11.9. *The Poisson bivector defines a complex ($\Lambda^\star = \bigoplus_{k\geq 0} \Lambda^k$, d) with the coboundary operator*

$$d:\ \mathrm{A}^k \to \mathrm{A}^{k+1}, \quad a \mapsto [P, a].$$

Definition 11.10 (Lichnerowicz, 1977)**.** *The Poisson cohomology is defined by*

$$\mathcal{H}^k(M,P) := \frac{\text{Ker } d|_{\Lambda^{k-1}}}{\text{Im } d|_{\Lambda^{k-1}}}, \quad k \geq 0.$$

Now, we present the Bihamiltonian cohomology (in the finite dimensional case).Let M be endowed with a bihamiltonian structure, i.e., a pair of compatible Poisson structures (P_1, P_2) satisfying

$$[P_1, P_1] = [P_1, P_2] = [P_2, P_2] = 0.$$

Then we have a bicomplexes (Λ^*, d_1, d_2) such that

$$d_1 d_2 + d_2 d_1 = 0.$$

They induce another complex (Λ^*, d_1, d_2), and its cohomology is called the bihamiltonian cohomology

$$BH^k(M, P_1, P_2) = \frac{\text{Ker } d_2|_{\tilde{\Lambda}^k}}{\text{Im } d_2|_{\tilde{\Lambda}^{k-1}}} = \frac{\text{Ker } d_1|_{\Lambda^k} \cap \text{Ker } d_2|_{\Lambda^{k-1}}}{\text{Im } d_2|_{\tilde{\Lambda}^{k-1}}}$$

(following the definition given in [Dubrovin,Zhang, 2001] for infinite dimensional bihamiltonian structures).

Furthermore, there exists a super manifold formalism: A convenient way to present the definition of the Schouten-Nijenhuis bracket is to use the super manifold $\hat{M} = \prod(T^*M)$ of dimension $(n|n)$ obtained by reversing the parity of fibers of the cotangent bundle T^*M.

We have $C^\infty(\hat{M}) = \Gamma(\Lambda^*(TM))$, and the symplectic structure on $\hat{M}$ yields the the definition of Schouten-Nijenhuis bracket

$$[P, Q] = \frac{\partial P}{\partial \theta_\alpha}\frac{\partial Q}{\partial u^\alpha} + (-1)^p \frac{\partial P}{\partial u^\alpha}\frac{\partial Q}{\partial \theta_\alpha}, \quad P \in \Lambda^p,\ Q \in \Lambda^q$$

on a local trivialization $\hat{U} = U \times \mathbb{R}^{0|n}$ of $\hat{M}$ with coordinates $u^1, \ldots, u^n$ of U and the dual coordinates $\theta_1, \ldots, \theta_n$ on the fiber $\mathbb{R}^{0|n}$ which satisfy the relations

$$\theta_\alpha \theta_\beta + \theta_\beta \theta_\alpha = 0, \quad \alpha, \beta = 1, \ldots, n.$$

11.2.2 The Infinite Case

Now, consider the infinite jet space

$$J^\infty(\hat{M}) = \varprojlim_k$$

where $\hat{M} = \prod(T^*M)$. We have:

1. Local coordinate system on $\hat{M}$:

$$\left(\hat{U};\, u^\alpha, \theta_\alpha, \alpha = 1, \ldots, n\right).$$

2. Local coordinate system on $J^\infty(\hat{M})$:

$$(\hat{U} \times \mathbb{R}^\infty;\, u^{\alpha,p}, \theta^p_\alpha,\ 1 \le \alpha \le\ n,\ p \ge 0).$$

3. The ring of differential polynomials

$$\hat{A} = C^\infty(\,U)[[u^{\alpha,s+1}, \theta^s_\alpha \mid \alpha = 1, \cdots, n; s \ge 0]].$$

Proposition 11.11. *The ring of* ***differential polynomials*** $\hat{A}$ *has the differential and the super gradations*

$$\deg_x u^{\alpha,s} = \deg_x \theta^s_\alpha = s; \quad \deg_\theta u^{\alpha,s} = 0, \quad \deg_\theta \theta^s_\alpha = 1.$$

We denote the space of homogeneous elements by

$$\begin{aligned}
\hat{\mathcal{A}}_d &= \{f \in \hat{\mathcal{A}} \mid \deg_x f = d\},\\
\hat{\mathcal{A}}^p &= \{f \in \hat{\mathcal{A}} \mid \deg_\theta f = p\},\\
\hat{\mathcal{A}}_d &= \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}_d.
\end{aligned}$$

$\lim_{\leftarrow_k} J^k(\hat{M})$

The global vector field

$$\partial_x = \sum_{s \ge 0} u^{\alpha,s+1} \frac{\partial}{\partial u^{\alpha,s}} + \theta^{s+1}_\alpha \frac{\partial}{\partial \theta^s_\alpha}$$

on $J^\infty(\hat{M})$ induces a derivation on $\hat{A}$. The space of local functionals $\hat{\mathcal{F}}$ is defined as follows:

Definition 11.12. *The* ***space of local functionals on*** $\hat{M}$ *is*

$$\hat{\mathcal{T}} := \hat{\mathcal{A}}/\partial_x \hat{A}.$$

Additionally, we denote the image of an element $f \in \hat{\mathcal{A}}$ in $\hat{\mathcal{F}}$ by $\int f$, so we have

$$0 \to \hat{\mathcal{A}}/\mathbb{R} \xrightarrow{\partial_x} \hat{\mathcal{A}} \xrightarrow{\int} \hat{\mathcal{F}} \to 0.$$

$\hat{\mathcal{F}}$ also admits differential and super gradations induced from that of $\hat{\mathcal{A}}$, and we denote the spaces of homogeneous elements of differential degree d and super degree p by $\hat{\mathcal{F}}_d$ and $\hat{\mathcal{F}}^p$ respectively.

Definition 11.13. *A graded Lie algebra structure on $\hat{\mathfrak{F}}$ is given by the* ***Schouten-Nijenhuis bracket***

$$[P,Q]=\int\left(\frac{\delta P}{\delta\theta_\alpha}\frac{\delta Q}{\delta u^\alpha}+(-1)^p\frac{\delta P}{\delta u^\alpha}\frac{\delta Q}{\delta\theta_\alpha}\right),\quad \forall P\in\hat{\mathcal{F}}^p,\forall Q\in\hat{\mathcal{F}}^q.$$

It is preserved under the change of coordinates of the form

$$\begin{aligned}u^{\alpha,s}\mapsto w^{\alpha,s}&=\textstyle\sum_{k\geq 0}\partial_x^s f_k^\alpha(u),\qquad f_k^\alpha(u)\in\hat{A}_k^0,\qquad s\geq 0\\ \theta_\alpha\mapsto\sigma_\alpha^s&=\partial_x^s=\partial_x^s\textstyle\sum_{k\geq 0}(-\partial_x)^k\left(\frac{\partial u^\beta}{\partial w^{\alpha,k}}\theta_\beta\right),\qquad s\geq 0,\end{aligned}$$

which is induced from a change of coordinates (called Miura-type transformation) on $J^\infty(M)$:

$$u^\alpha\mapsto w^\alpha=f^\alpha(u)=\sum_{k\geq 0}f_k^\alpha(u),\quad \det\left(\frac{\partial f_0^\alpha(u)}{\partial u^\beta}\right)\neq 0.$$

For any local functional $X=\int X^\alpha\theta_\alpha\in\hat{\mathcal{F}}^1$, we can associate with it an evolutionary PDEs of the form

$$\frac{\partial u^\alpha}{\partial t}=X^\alpha,\qquad \alpha=1,\dots,n,$$

here we need to make the replacement $u^{\alpha,s}\mapsto\partial_x^s u^\alpha$.

Definition 11.14. *We call $X\in\hat{\mathcal{F}}^1$ a* ***Hamiltonian evolutionary PDE*** *if there exist $P\in\hat{\mathcal{F}}^2$ and $H\in\hat{\mathcal{F}}^0$ such that*

$$X=[H,P],[P,P]=0.$$

Here P and H are called the Hamiltonian structure and the Hamiltonian of X respectively.

We can represent P and H in the form

$$\frac{\partial u^\alpha}{\partial t}=\mathcal{P}^{\alpha\beta}\frac{\delta H}{\delta u^\beta},\quad\text{with}\quad\mathcal{P}^{\alpha\beta}=\sum_{s\geq 0}\mathcal{P}_s^{\alpha\beta}\partial_x^s.$$

Definition 11.15. *The evolutionary PDE X is called a* ***bihamiltonian system*** *if there exist $P_1,P_2\in\hat{\mathcal{F}}^2$ and $H,G\in\hat{\mathcal{F}}^0$ such that*

$$X=[H,P_1]=[G,P_2],\qquad [P_1,P_1]=[P_2,P_2]=[P_1,P_2]=0.$$

Example 11.16 (KdV Hierarchy)**.** *For example, let M be a smooth manifold of dimension one with local coordinate u. We consider the local functionals*

$$X_0=\int u_x\theta,\quad X_n=\frac{2^n}{(2n+1)!!}\int(R^n u_x)\,\theta,\quad n\geq 2,$$

where

$$\mathcal{R} = \frac{\varepsilon^2}{8}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1}.$$

These local functionals correspond to the KdV hierarchy

$$\frac{\partial u}{\partial t_0} = u_x, \quad \frac{\partial u}{\partial t_1} = uu_x + \frac{\varepsilon^2}{12}u_{xxx}, ...$$

Example 11.17 (Bihamiltonian structure of the KdV hierarchy)**.** *The KdV hierarchy has a bihamiltonian structure given by the following local functionals*

$$P_1 = \frac{1}{2}\int \theta\theta^1, \quad P_2 = \frac{1}{2}\int \left(u\theta\theta^1 + \frac{\varepsilon^2}{8}\theta\theta^3\right).$$

It can be represented as

$$\frac{\partial u}{\partial t_p} = \mathcal{P}_1\frac{\delta H_p}{\delta u} = \left(p + \frac{1}{2}\right)^{-1}\mathcal{P}_2\frac{\delta H_{p-1}}{\delta u}, \quad p \geq 0,$$

where the Hamiltonian operators and the Hamiltonians are given by

$$P_1 = \partial_x, \quad \mathcal{P}_2 = u\partial_x + \frac{1}{2}u_x + \frac{\varepsilon^2}{8}\partial_x^3, H_{-1} = \int u, \quad H_0 = \int \left(\frac{1}{2}u^2 + \frac{\varepsilon^2}{12}u_{xx}\right), ...$$

11.3 Flat exact bihamiltonian structures of hydrodynamic type

Let (P_1, P_2) be a bihamiltonian structure of hydrodynamic type on the jet space $J^\infty(M^n)$. In the local coordinates $v_1, ..., v_n$ the compatible Hamiltonian operators have the expressions

$$\mathcal{P}_a^{\alpha\beta} \equiv g_a^{\alpha\beta}(v)\partial_x + \Gamma\,{}_{a,\gamma}^{\alpha\beta}(v)v_x^\gamma, \quad a = 1, 2.$$

Here $(g_1^{\alpha\beta})$, $(g_2^{\alpha\beta})$ are symmetric and nondegenerate, and $(g_1^{\alpha\beta})^{-1}$, $(g_2^{\alpha\beta})^{-1}$ are flat metrics on M and

$$\Gamma_{a,\gamma}^{\alpha\beta} = -g_a^{\alpha\xi}\Gamma_{\xi\gamma}^\beta, \quad a = 1, 2$$

are the contravariant components of the Levi-Civita connections of these metrics respectively [Dubrovin, Novikov, 1983], and $(g_1^{\alpha\beta})$, $(g_2^{\alpha\beta})$ form a flat pencil of metrics [Dubrovin, 1998].

Definition 11.18. *The bihamiltonian structure (P_1, P_2) of hydrodynamic type is called* ***semisimple*** *if the roots $u^1(v), ..., u^n(v)$ of the characteristic equation*

$$\det(g_2^{\alpha\beta}(v) - u\, g_1^{\alpha\beta}(v)) = 0$$

are pairwise distinct for generic point of v and are non-constant. They can be used as local coordinates, called ***canonical coordinates*** *of the semisimple bihamiltonian structure.*

In the canonical coordinates the two flat metrics are diagonal

$$g_1^{ij} = f^i(u)\delta_{ij}, \qquad g_2^{ij} = u^i f^i(u)\delta_{ij}.$$

The Bihamiltonian structure (P_1, P_2) is called exact if there exists a vector field Z on M such that

$$[Z, P_1] = 0, [Z, P_2] = P_1.$$

Here $[\,,]$ is the Schouten-Nijenhuis bracket defined on the space of local functionals, and

$$Z \in \hat{\mathcal{F}}^1, P_1, P_2 \in \hat{\mathcal{F}}^2.$$

Definition 11.19. *We call Z the **unity vector field** of the exact bihamiltonian structure.*

The exact bihamiltonian structure $(P_1, P_2; Z)$ is called flat exact if the unity vector field Z is flat with respect to the first flat metric g_1, i.e.

$$\nabla Z = 0$$

where ∇ is the Levi-Civita connection of g_1.

Let (P_1, P_2) be semisimple, then Z must take the form

$$Z = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}$$

in canonical coordinates [Falqui, Lorenzoni, 2012].

In the semisimple case, Z is flat if and only if the first diagonal metric $ds_1^2 = \sum_{i=1}^n f_i(u)(du^i)^2$ with $f_i := \left(f^{\,i}\right)^{-1}$ $(i = 1, ..., n)$ satisfy the Egoroff condition

$$\frac{\partial f_i}{\partial u^j} = \frac{\partial f_j}{\partial u^i}, \quad \forall 1 \leq i, j \leq m.$$

ie. the flat diagonal metric g_1 is a Egoroff metric.

11.4 Flat exact bihamiltonian structures and Frobenius manifolds

Given a semisimple flat exact bihamiltonian structures of hydrodynamic type $(P_1, P_2; Z)$ with canonical coordinates $u^1, \ldots, u^n$. Let γ_{ij} be the rotation coefficients of the first flat metric g_1:

$$\gamma_{ij}(u) = \frac{1}{2\sqrt{f_i f_j}} \frac{\partial f_i}{\partial u^j}, \quad i \neq j, \quad \gamma_{ii} = 0,$$

then $\gamma_{ij} = \gamma_{ji}$ and the condition that (P_1, P_2) is a bihamiltonian structure is equivalent to the following equations for γ_{ij}

$$\frac{\partial \gamma_{ij}}{\partial u^k} = \gamma_{ik}\gamma_{jk}, \quad \text{for distinct } i, j, k, \sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u^k} = 0, \qquad \sum_{k=1}^{n} u^k \frac{\partial \gamma_{ij}}{\partial u^k} = -\gamma_{ij}.$$

Definition 11.20. *The semisimple bihamiltonian structure (P_1, P_2) is called* ***reducible*** *at $u \in M$ if there exists a partition of the set $\{1, 2, ..., n\}$ into the union of two nonempty nonintersecting sets I and J such that*

$$\gamma_{ij}(u) = 0, \quad \forall i \in I, \ \forall j \in J.$$

(P_1, P_2) is called ***irreducible*** *on a certain domain $D \subset M$, if it is not reducible at any point $u \in D$.*

We will impose this irreducibility condition on the class of semisimple flat exact bihamiltonian structures.

We consider the linear system

$$\frac{\partial \psi_j}{\partial u^i} = \gamma_{ji}\psi_i, \quad i \neq j, \frac{\partial \psi_j}{\partial u^i} = -\sum_{k \neq i} \gamma_{ki}\psi_k,$$

The above conditions for γ_{ij} ensure the compatibility of this linear system, so its solution space has dimension n, and we can find a fundamental system of solutions

$$\Psi_\alpha = (\psi_{1\alpha}(u), \dots, \psi_{n\alpha}(u))^T, \quad \alpha = 1, \dots, n,$$

defined on a domain D.

Here are some properties of a Frobenius manifold structure:

- The symmetric non-degenerate constant matrix

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}(u)\psi_{i\beta}(u), \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}.$$

- One can define local coordinates $v^1, \dots, v^n$ by

$$dv^\alpha = \sum_{i=1}^{n} \eta^{\alpha\gamma}\psi_{i1}\,\psi_{i1}\,\psi_{i\gamma}du^i, \quad i = 1, \dots, n,$$

  then the flat metric of the Frobenius manifold is given by $ds^2 = \eta_{\alpha\beta}dv^\alpha dv^\beta$.

- The structure constants of the Frobenius algebra are given by

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}, \quad \frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = \eta^{\gamma\xi}c_{\alpha\beta\xi}\frac{\partial}{\partial v^\gamma}.$$

Proposition 11.21 (Dubrovin-Liu 2018)**.** *There exists a smooth function $F(v)$ defined on D such that*

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma}, \quad \eta_{\alpha\beta} = \frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^1},$$

and it gives the potential of a Frobenius manifold structure (without, in general, the quasi-homogeneity condition) on D with the unit vector field

$$e = \frac{\partial}{\partial v^1},$$

i.e. it satisfies the WDVV equations of associativity

$$\frac{\partial^3 \mathcal{F}}{\partial v^\alpha \partial v^\xi \partial v^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}}{\partial v^\nu \partial v^\zeta \partial v^\beta} = \frac{\partial^3 \mathcal{F}}{\partial v^\beta \partial v^\xi \partial v^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}}{\partial v^\nu \partial v^\zeta \partial v^\alpha}.$$

Different choices of the fundamental system of solutions $\Psi_1, \dots, \Psi_n$ of the above-mentioned linear system lead to different Frobenius manifold structures which are related by Legendre transformations.

Since the Euler vector field

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}$$

acts on the space of solution of the linear system as a linear transformation, so we can fix an eigenvector with eigenvalue μ_1 and take it to be Ψ_1. We choose other basis $\Psi_2, \dots, \Psi_n$ such that the matrix of E takes the Jordan normal form. Then the Frobenius manifold structure corresponding to the above Ψ_1 is quasi-homogeneous with the Euler vector field E and the charge $d = -2\mu_1$ [Dubrovin, Liu, Zhang, 2018].

On the infinite jet space of the Frobenius manifold there is defined a semsimple bihamiltonian structure of hydrodynamic type, which is given by the flat metrics

$\langle du^i, du^j \rangle_1 = \delta_{ij} \psi_{i1}^{-2}$, the flat metric of the Frobenius manifold
$\langle du^i, du^j \rangle_2 = \delta_{ij} u^i \psi_{i1}^{-2}$, the intersection form of the Frobenius manifold

When we take

$$\psi_{i1} = f_i^{\frac{1}{2}} = (f^i)^{-\frac{1}{2}}, \qquad i = 1, \dots, n,$$

then this bihamiltonian structure coincides with the original one $(P_1, P_2; Z)$. We call this Frobenius manifold structure the canonical one. In what follows we will use this canonical Frobenius manifold to construct the bihamiltonian integrable hierarchy.

11.5 The Principal Hierarchy and its tau structure

Let us given a flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$, and let $v^1, ..., v^n$ be a system of flat coordinates of the first metric g_1 such that

$$Z = \frac{\partial}{\partial v^1}.$$

In the flat coordinates, the first Hamiltonian structure P1 is given by the Hamiltonian operator

$$n^{\alpha\beta}\frac{\partial}{\partial x}.$$

We have the following definitions:

Definition 11.22. *A collection of smooth functions*

$$\{h_{\alpha,p}|\alpha=1,\dots,n;\ p=-1,0,1,...\}$$

is called a ***calibration*** *of* $(P_1,P_2;Z)$ *if*

1. $H_{\alpha,p}=\int(h_{\alpha,p})$ *are bihamiltonian conserved quantities, ie.*

$$[P_2,[P_1,H_{\alpha,p}]]=0.$$

2. *The recursion* $Z(h_{\alpha,p})=h_{\alpha,p-1}$ *for* $p\geq 0$ *holds true.*
3. *Normalization:* $h_{\alpha,-1}=v_\alpha=\eta_{\alpha\gamma}v^\gamma$, *and* $\frac{\partial}{\partial t^{1,0}}=\frac{\partial}{\partial x}$.

Definition 11.23. *The hierarchy of quasi-linear PDEs*

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}}=\eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial h_{\beta,q}(v)}{\partial v^\gamma}\right),\quad 1\leq\alpha,\beta\leq n,\ q\geq 0$$

is called the ***Principal Hierarchy*** *of the flat exact bihamiltonian structure* $(P_1,P_2;Z)$ *with the calibration* $\{h_{\alpha,p}\}$.

Theorem 11.24 (Dubrovin-Liu 2018)**.** *Let* $\{h_{\alpha,p}\}$ *be a calibration of* $(P_1,P_2;Z)$, *then the associated Principal Hierarchy is a bihamiltonian integrable hierarchy of hydrodynamic type with*

$$\frac{\partial v^\alpha}{\partial t^{1,0}}=v^\alpha_x.$$

Moreover, for any $(\alpha,p),(\beta,p)$, *we have*

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}}=\frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}},$$

and there exists differential polynomials $\Omega_{\alpha,p;\beta,q}$ *such that*

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}}=\frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}=\partial_x\Omega_{\alpha,p;\beta,q}.$$

Definition 11.25. *A collection of smooth functions*

$$\{\Omega_{\alpha,p;\beta,q}|\alpha,\beta=1,\dots,n;\ p,q=0,1,2,...\}$$

is called a tau structure of the flat exact bihamiltonian structure $(P_1,P_2;Z)$ *and the Principal Hierarchy associated to a fixed calibration* $\{h_{\alpha,p}\}$ *if the following conditions are satisfied:*

1. $\partial\Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p-1}}{\partial u^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}$.
2. $\Omega_{\alpha,p,\beta,q} = \Omega_{\beta,q;\alpha,p}$.
3. $\Omega_{\alpha,p+1,1,0} = h_{\alpha,p}$.

Now let us proceed to construct a calibration for the canonical Frobenius manifold structure $F(v)$ of $(P_1, P_2; Z)$. We first define the functions

$$\theta_{\alpha,0}(v) = v_\alpha = \eta_{\alpha\gamma}v^\gamma, \quad \theta_{\alpha,1}(v) = \frac{\partial F(v)}{\partial v^\alpha}, \quad \alpha = 1, \ldots, n.$$

By adding to the function $F(v)$ a certain quadratic term in $v^1, \ldots, v^n$, if needed, we can assume that

$$\frac{\partial^2 F(v)}{\partial v^1 \partial v^\alpha} = \eta_{\alpha\gamma}v^\gamma = v_\alpha.$$

We then define the functions $\theta_{\alpha,p}(v)$ for $p \geq 2$ recursively by using the following relations:

$$\frac{\partial^2 \theta_{\gamma,p+1}(v)}{\partial v^\alpha \partial v^\beta} = c_{\alpha\beta\xi}\eta^{\xi\zeta}\frac{\partial \theta_{\gamma,p}(v)}{\partial v^\xi}, \quad \alpha, \beta, \gamma = 1, \ldots, n$$

We can require that the functions $\theta_{\alpha,p}$ also satisfy the following normalization conditions

$$\frac{\partial \theta_\alpha(v;z)}{\partial v^\xi}\eta^\xi\frac{\partial \theta_\beta(v;-z)}{\partial v^\xi} = \eta_{\alpha\beta}, \quad \alpha, \beta = 1, \ldots, n.$$

Here

$$\beta_\alpha(v;z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^p.$$

Now we define the functions $h_{\alpha,p}(v)$ so that their generating functions

$$h_\alpha(v;z) = \sum_{p\geq -1} h_{\alpha,p}(w)z^{p+1}$$

satisfy the following defining relations

$$h_\alpha(v;z) = \frac{1}{z}\frac{\partial \theta_\alpha(v;z)}{\partial v^1} - \frac{1}{z}\eta_{\alpha 1}.$$

We proceed by construcing the τ structure: The tau structure

$$\{\Omega_{\alpha,p;\beta,q}(v) \mid \alpha, \beta = 1, ...; p,\ q = 0, 1, 2, ...\}$$

of the calibration $\{h_{\alpha,p}\}$ is given by

$$\frac{\partial h_\alpha(v;z_1)}{\partial v^\xi}\eta^{\xi\zeta}\frac{\partial h_\beta(v;z_2)}{\partial v^\zeta} - \eta_{\alpha\beta} = (z_1+z_2)\sum_{p,q\geq 0}\Omega_{\alpha,p,\beta,q}(v)z_1^p z_2^q.$$

The tau function of the Principal Hierarchy is defined by

$$\frac{\partial^2 \log\tau}{\partial t^{\alpha,p}\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}(v(t)).$$

Theorem 11.26 (Dubrovin-Liu 2018). *1. The set of functions* $\{h_{\alpha,p}\}$ *and* $\{\Omega_{\alpha.p;\beta,q}\}$ *give a calibration and a tau structure of the flat exact semisimple bihamiltonian structure of hydrodynamic type* $(P_1, P_2; Z)$.

2. The Principal Hierarchy also possesses the Galilean symmetry

$$\frac{\partial v^\alpha}{\partial s} = \delta_1^\alpha + \sum_{\beta,q} t^{\beta,q+1} \frac{\partial v^\alpha}{\partial t^{\beta,q}}.$$

11.6 Deformations of semisimple bihamiltonian structure of hydrodynamic type

1. Uniquely parameterized by central invariants $c_1(u^1), ..., c_n(u^n)$. [Dubrovin, Liu, Zhang, 2006]
2. Existence of deformation with a given set of central invariants. [Liu, Zhang, 2012] for KdV case, [Carlet, Posthuma, Shadrin, 2015] for general case.
3. The space of bihamiltonian conserved quantities and bihamiltonian vector fields of the deformed bihamiltonian structure are isomorphic to those of the original bihamiltinian structure. [Dubrovin, Liu, Zhang, 2018]

Theorem 11.27 (Falqui, Lorenzoni 2012). *A deformation* $(\tilde{P}_1, \tilde{P}_2)$ *of the exact semisimple bihamiltonian structure of hydrodynamic type* $(P_1, P_2; Z)$ *is exact if and only if its central invariants* $c_1, ..., c_n$ *are constant functions. Moreover, there exists a Miura type transformation* g *such that*

$$g(\bar{P}_1) = P_1, \quad g(\bar{P}_2) = P_2 + \varepsilon^2 Q_1 + \varepsilon^4 Q_2 + \dots,$$

and the unit vector field $\tilde{Z}$ *is transformed to the undeformed one*

$$g(\vec{C}) = Z = \sum_{i=1}^n \frac{\partial}{\partial u^i}.$$

Let $(P_1, P_2; Z)$ be a flat exact semisimple bihamiltonian structure of hydrodynamic type, with a fixed calibration and a tau structure $\{h_{\alpha,p}\}, \{\Omega_{\alpha,p;\beta,q}\}$. Given a set of constants $c_1, \dots, c_n$ we have a unique, up to Miura type transformations, deformation $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ with

$$\bar{P}_1 = P_1, \quad \bar{P}_2 = P_2 + \varepsilon^2 Q_1 + \varepsilon^4 Q_2 + \dots, \tilde{Z} = \sum_{i=1}^n \frac{\partial}{\partial u^i}.$$

Then we have a unique deformation

$$\tilde{H}_{\alpha,p} = H_{\alpha,p} + \varepsilon^2 H^{[1]}_{\alpha,p} + \varepsilon^4 H^{[2]}_{\alpha,p} + \dots$$

of the bihamiltonian conserved quantities $H_{\alpha,p} = \int (h_{\alpha,p})$.

Theorem 11.28 (Dubrovin-Liu 2018)**.** *The deformation* $(\tilde{P}_1, \tilde{P}_2, Z)$ *and the associated deformed Principal Hierarchy possess a tau structure.*

The densities of the Hamiltonians are given by

$$\bar{h}_{\alpha,p} = \delta_Z \bar{H}_{\alpha,p+1} = \sum_{i=1}^{n} \frac{\delta \bar{H}_{\alpha,p+1}}{\delta u^i}, \quad \alpha = 1, \dots, n, \ p = -1, 0, 1, 2, \dots.$$

The differential polynomials $\tilde{\Omega}_{\alpha_r,p;\beta,q}$ are given by

$$\frac{\partial \tilde{h}_{\alpha,p-1}}{\partial t^{\beta,q}} = \partial_x \tilde{\Omega}_{\alpha,p;\beta,q}.$$

There is still plenty of work to be done in this area. One important goal is to provide a constructive approach to fix a representative of the deformations of a flat exact semisimple bihamiltonian structure of hydrodynamic type with constant central invariants, and to fix the tau structure, so that the deformation of the Principal Hierarchy can be used as candidate of integrable hierarchies that control 2D TFT.

When the central invariants $c_1, ..., c_n$ are chosen to be $\frac{1}{24}$, a representative of the deformed bihamiltonian integrable hierarchy can be constructed by using the algorithm of [Dubrovin, Zhang, 2001] via the quasi-Miura transformation given by the solution of the loop equation of the associated semisimple Frobenius manifold.

Part III

Week 2 Talks

There were eleven talks in week 2. There were no talks on Thursday, July 4th, and Friday, July 5th.

Week 2

Sunday, June 30

- Sergei Lando: Weight Systems Associated to Lie Algebras

  V. A. Vassiliev's theory of finite type knot invariants allows one to associate to such an invariant a function on chord diagrams, which are simple combinatorial objects, consisting of an oriented circle and a tuple of chords with pairwise distinct ends in it. Such functions are called "weight systems". According to a Kontsevich theorem, such a correspondence is essentially one-to-one: each weight system determines a certain knot invariant.

  In particular, a weight system can be associated to any semi-simple Lie algebra. However, already in the simplest nontrivial case, the one for the Lie algebra $\mathfrak{sl}(2)$, computation of the values of the corresponding weight system is a computationally complicated task. This weight system is of great importance, however, since it corresponds to a famous knot invariant known as the colored Jones polynomial.

  Last few years was a period of significant progress in understanding and computing Lie algebra weight systems, both for $\mathfrak{sl}(2)$- and $\mathfrak{gl}(N)$-weight system, for arbitrary N. These methods are based on an idea, due to M. Kazarian, which suggests a recurrence for $\mathfrak{gl}(N)$-weight system extended to permutations. The recurrence immediately leads to a construction of a universal $\mathfrak{gl}$-weight system taking values in the ring of polynomials $C[N, C_1, C_2, C_3, \dots]$ in infinitely many variables and allowing for a specialization to $\mathfrak{gl}(N)$ and $\mathfrak{sl}(N)$-weight systems for any given value of N. A lot of new explicit formulas were obtained.

  Simultaneously, Zhuoke Yang extended the construction to the Lie superalgebras $\mathfrak{gl}(N|M)$ and, together with M. Kazarian, to other classical series of Lie algebras. It happened that certain specializations of the universal $\mathfrak{gl}(N)$-weight system lead to well-known combinatorial invariants of graphs, allowing thus to extend these invariants to permutations.

  Certain integrability properties of the Lie algebra weight systems will be discussed. The talk is based on a joint work of M. Kazarian, the speaker,

and N. Kodaneva, P. Zakorko, and others.

- Senya Shlosman: Pedestals Matrices: Polynomial Matrices with Polynomial Eigenvalues

  I will explain a construction which for every finite poset X (such as a Young diagram) produces a square matrix M^X. Its matrix elements are indexed by pairs P, Q of linear orders on X (pairs of standard tableaux in the case of Young diagrams). The entries of M^X are monomials in variables x_i. Our main result is that the eigenvalues of M^X are polynomials in x_i with integer coefficients. Joint work with Richard Kenyon, Maxim Kontsevich, Oleg Ogievetsky, Cosmin Pohoata, and Will Sawin.

Monday, July 1

- Ivan Cherednik: Q-zeta Revisited

  The fundamental feature of practically all zeta-functions and L-functions is that their meromorphic continuations to complex s provide a lot of information about the corresponding objects. However, complex values of s have generally no direct arithmetic/geometric meaning, and occur as a powerful technical tool. We will discuss the refined theory, which is basically the replacement of the terms $\frac{1}{n^s}$ by the invariants of lens space $L(n,1)$, certain q,t,a-series. One of their key properties is the superduality $q \leftrightarrow t^{-1}$, which is related to the functional equation of the Hasse-Weil zetas for curves, the symmetry $\epsilon_1 \leftrightarrow \epsilon_2$ of Nekrasov's instantons, and to other refined theories in mathematics and physics. These invariants have various specializations, including Rogers-Ramanujan identities and the topological vertex. We will begin the talk with the Riemann q-zeta-hypothesis in type A_1, in full detail.

- Alexander Braverman: Introduction to Symplectic Duality and Coulomb Branches of 3D Quantum Field Theories

  I will give a survey of the series of my joint works with Finkelberg and Nakajima giving a mathematical construction of the so called Coulomb branches of 3D $\mathcal{N} = 4$ super-symmetric gauge theories (no knowledge of any of these words will be needed). I will also explain its connection with the (purely mathematical subject) of symplectic duality.

Tuesday, July 2

- Paul Wiegmann: Peierls phenomenon via Bethe Ansatz: reflection of Krichever's works on Peierls model

  In the 1930s Rudolf Peierls argued that the one-dimensional electrons interacting with phonons undergo an instability, leading to the formation of a periodic structure known as an electronic crystal. Peierls's instability stands in a short list of major phenomena of condensed matter physics.

  From a mathematical perspective, a comprehensive solution to the Peierls

problem was given in papers by Igor Krichever and co-authored by Natasha Kirova, Sergei Brazovski, and Igor Dzyaloshinsky In the early 80's. It was found that electronic crystals are periodic solutions of soliton equations, falling within the framework of Krichever-Novikov's theory of finite-gap potentials.

The Peierls phenomenon also emerges as a limiting case of models of interacting fermions, such as Gross-Neveu models with a large rank symmetry group when the rank of the group tends to infinity. These models are solvable by the Bethe Ansatz for finite rank groups. The talk presents the result of a recent paper co-authored by Konstantin Zarembo, Valdemar Melin, and Yoko Sekiguchi, where Krichever's finite-gaps solutions of soliton equations were obtained as a singular large rank limit of the Bethe Ansatz solution of models with Lie group symmetry.

- Andrei Marshakov: Krichever tau-function: basics and perspectives

  I plan to start with the definition of quasiclassical tau-function, introduced by Igor Krichever in 1992, formulate its main properties with some simple proofs, and discuss certain particular cases, which include the Seiberg-Witten prepotentials, matrix models etc. Then I am going to turn to certain modern developments, related with this object, which include the relation with instanton partition functions, isomonodromic tau-dunctions and even some unexpected relations with other famous relations in mathematical physics.

- Alexander Veselov: Harmonic locus and Calogero-Moser spaces

  The harmonic locus consists of the monodromy-free Schroedinger operators with rational potential quadratically growing at infinity. It is known after Duistermaat and Grunbaum that in the multiplicity-free case the poles $z_1, ..., z_N$ of such potentials satisfy the following algebraic system

$$\sum_{j \neq i}^{N} \frac{2}{(z_i - z_j)^3} - z_i = 0, \quad i = 1, ..., N,$$

  describing the complex equilibriums of the corresponding Calogero-Moser system. Oblomkov proved that the harmonic locus can be identified with the set of all partitions via Wronskian map for Hermite polynomials. We show that the harmonic locus can also be identified with the subset of the Calogero-Moser spaces introduced by Wilson, which is invariant under a natural symplectic action of $\mathbb{C}^\times$ As a corollary, for the multiplicity-free part of the locus we effectively solve the inverse problem for the Wronskian map by proving that the spectrum of Moser's matrix coincides with the set of contents of the corresponding Young diagram. We also compute the characters of the $\mathbb{C}^\times$-action at the fixed points, proving a conjecture of Conti and Masoero. The talk is based on a joint work with Giovanni Felder.

- Stanislav Smirnov: Coulomb gas and lattice models

  Even before the introduction of Conformal Field Theory by Belavin, Polyakov and Zamolodchikov, it appeared indirectly in the work of den Nijs and Nienhuis using Coulomb gas techniques. The latter postulate (unrigorously) that height functions of lattice models converge to the Gaussian Free Field, allowing to derive many exponents and dimensions of 2D lattice models.

  This convergence is in many ways mysterious, in particular it was never formulated in the presence of a boundary, but rather pn a torus or a cylinder. We will discuss possible formulations on general domains or Riemann surfaces and their relations to CFT, SLE and conformal invariance of critical lattice models. Interestingly, new objects in complex geometry and potential theory seem to arise.

- Grigori Olshanski: Macdonald-level extension of beta ensembles and multivariate hypergeometric polynomials

  A beta ensemble (or log-gas system) on the real line is a random collection of N point particles $x_1, ..., x_N$ whose joint probability distribution has a special form containing the Vandermonde raised to the power $\beta > 0$. I will survey results related to some discrete analogs of beta ensembles, which live on q-lattices, and large-N limit transitions.

Wednesday, July 3

- Da-jun Zhang: Elliptic solitons related to the Lamé functions

  In this talk I will report recent progress on the elliptic solitons related to the Lamé functions. Apart from the classical solitons that are composed by usual exponential type plane wave factors, there exist "elliptic solitons" which are composed by the Lamé-type plane wave factors and expressed using Weierstrass functions. Recently, we found vertex operators to generate tau functions for such type of solitons. We also established an elliptic scheme of direct linearization approach.

- Anton Dzhamay: Geometry and Symmetry of Painlevé Equations

  We begin by an overview of how geometric ideas entered the theory of differential Painlevé equations in the work of of K.Okamoto, which led to the better understanding of their symmetries (Backlünd transformations) in terms of affine Weyl groups. These ideas were then extended by H.Sakai to the discrete (elliptic, multiplicative, and additive) Painlevé equations and resulted in the beautiful Sakai classification scheme for both differential and discrete Painlevé equations. In the latter case, it is the symmetry group that is the source of a discrete dynamics. In the second part of the talk we discuss the notion of an abstract discrete Painlevé equation and its various concrete realizations. This leads to the study of a refined identification problem, which is a classification of different orbits for the

same abstract discrete Painlevé dynamic, and results in the appearance of special symmetry groups that are not a part of the general (i.e., generic) Sakai classification scheme. We illustrate this by an example of a discrete Painlevé-II equation and its symmetry group. This is based on a joint work with Yang Shi, Alex Stokes, and Ralph Willox.

12 Sergei Lando: Weight Systems Associated to Lie Algebras

Abstract

V. A. Vassiliev's theory of finite type knot invariants allows one to associate to such an invariant a function on chord diagrams, which are simple combinatorial objects, consisting of an oriented circle and a tuple of chords with pairwise distinct ends in it. Such functions are called "weight systems". According to a Kontsevich theorem, such a correspondence is essentially one-to-one: each weight system determines a certain knot invariant.

In particular, a weight system can be associated to any semi-simple Lie algebra. However, already in the simplest nontrivial case, the one for the Lie algebra $\mathfrak{sl}(2)$, computation of the values of the corresponding weight system is a computationally complicated task. This weight system is of great importance, however, since it corresponds to a famous knot invariant known as the colored Jones polynomial.

Last few years was a period of significant progress in understanding and computing Lie algebra weight systems, both for $\mathfrak{sl}(2)$- and $\mathfrak{gl}(N)$-weight system, for arbitrary N. These methods are based on an idea, due to M. Kazarian, which suggests a recurrence for $\mathfrak{gl}(N)$-weight system extended to permutations. The recurrence immediately leads to a construction of a universal $\mathfrak{gl}$-weight system taking values in the ring of polynomials $C[N, C_1, C_2, C_3, \dots]$ in infinitely many variables and allowing for a specialization to $\mathfrak{gl}(N)$ and $\mathfrak{sl}(N)$-weight systems for any given value of N. A lot of new explicit formulas were obtained.

Simultaneously, Zhuoke Yang extended the construction to the Lie superalgebras $\mathfrak{gl}(N|M)$ and, together with M. Kazarian, to other classical series of Lie algebras. It happened that certain specializations of the universal $\mathfrak{gl}(N)$-weight system lead to well-known combinatorial invariants of graphs, allowing thus to extend these invariants to permutations.

Certain integrability properties of the Lie algebra weight systems will be discussed. The talk is based on a joint work of M. Kazarian, the speaker, and N. Kodaneva, P. Zakorko, and others.

Contents

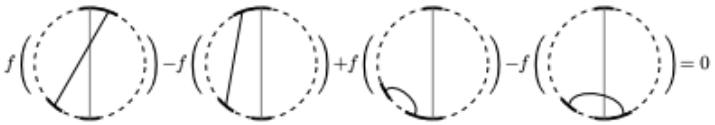
12.1 Chord Diagrams and Weight Systems

Any knot invariant v with values in a commutative ring admits an extension to singular knots according to the following Vassiliev skein relation:

A knot invariant is of **order at most** n if its extension to singular knots with more than n double points vanishes.

Each knot invariant of order at most n determines a function on chord diagrams with n chords; this function satisfies Vassiliev's 4-term relations:

Definition 12.1. *The **Vassiliev's 4-term relations** is:*



What about the converse? It is also true: According to Kontsevich's theorem, each weight system with values in an algebra over a field of characteristic 0 arises from a finite type knot invariant. So there is more or less a 1-to-1 correspondence

$$\text{weight systems} \leftrightarrow \text{finite type knot invariant.}$$

It is interesting to construct weight systems in order to construct some functions on diagrams that satisfy the 4-term relations. There are two main sources of such constructions: graph invariants and Lie algebras.

- Graph invariants: easy to construct, easy to compute, but not powerful;
- Lie algebras: easy to construct, hard to compute, but very powerful.

We will see how the graph invariants source works but we will mostly focus on the Lie algebras source.

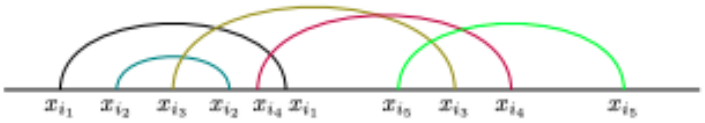
12.2 Constructing Weight Systems from Lie Algebras

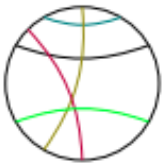
We are given the following initial data: a finite dimensional Lie algebra g with a nondegenerate invariant scalar product,

$$(\mathfrak{g}, (\cdot,\cdot)); ([x,y],z) = (x,[y,z]) \forall x,y,z;\ d = \dim\ \mathfrak{g}.$$

Then, we proceed as follows:

- Pick an orthonormal basis $x_1, \dots, x_d$ in $\mathfrak{g}$, $(x_i, x_j) = \delta_{ij}$.
- Cut the circle of a chord diagram D at some point and make it into an arc diagram A. Pick a numbering $\nu : V(A) \to \{1, \dots, d\}$ of the arcs of A.
- Put letters $\chi_{\nu(a)}$ at the ends of each arc a; the result is a word in $U\mathfrak{g}$. Sum over all the numberings $\nu : V(A) \to \{1, \dots, d\}$.



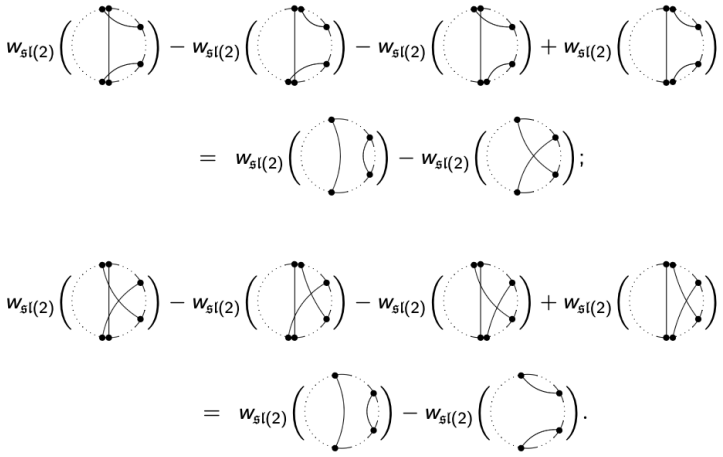


$$D \mapsto \sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x_{i_1}x_{i_2}x_{i_3}x_{i_2}x_{i_4}x_{i_1}x_{i_5}x_{i_3}x_{i_4}x_{i_5}$$

Theorem 12.2 (Bar-Natan, Kontsevich)**.** *The result is independent of the choice of the orthonormal basis $\{x_i\}$ and the cut point; it belongs to the center of $U\mathfrak{g}$ and satisfies 4-term relations.*

Unfortunately, it is a very difficult task to compute this result because we are obliged to make computations in a non-commutative algebra (the universal enveloping algebra of the Lie algebra). Despite the fact that the result belongs to the center of this universal enveloping algebra, which is a commutative algebra, all the intermediate computations are to be made in a noncommutative situation which makes them extremely complicated, with one exception: For $\mathfrak{g} = \mathfrak{sl}_2$, there is a recurrence relation due to Chmutov and Varchenko (1997) so we can calculate the values of the $\mathfrak{sl}_2$ weight system directly, and not through intermediate computations.

We can see that after the initial chord diagram, the next three have fewer intersections while the last two have fewer chords, so it is indeed a recursive relation. We can now proceed recursively to compute the value of this $\mathfrak{sl}_2$ weight system. However, even this algorithm is rather complicated because we replaced one chord diagram with five. So we are left with an exponential complicated algorithm which does not allow us to proceed much further. Until recently, the explicit values of the $\mathfrak{sl}_2$ weight system have been known only for fairly simple chord diagrams and families of chord diagrams.



12.3 Constructing Weight Systems from Graphs

[Chmutov, Lando, 2007] showed that the value of the $\mathfrak{sl}(2)$-weight system on a chord diagram depends on the intersection graph of the chord diagram rather than on the diagram itself. The definition of intersection graphs is as follows:

Definition 12.3. *The **intersection graph** of a chord diagram is the graph whose vertices are the chords of the diagram, and two vertices are connected by an edge if and only if the corresponding chords intersect one another.*

To continue, we need to explain another invariant of graphs which is in fact a weight system. This is invariant is very well known: the chromatic polynomial, which counts the number of proper colorings of the vertices of a graph into a given number of colors.

The chromatic polynomial for complete graphs on n variables looks very simple:

$$\chi_{K_n}(c) = c(c-1)\cdots(c-n+1) = (c)_n.$$

There is a very nice formula for the generating function, which has the continued fraction form

$$\sum_{n=0}^{\infty} \chi_{K_n}(c)t^n = \cfrac{1}{1 - ct + \cfrac{ct^2}{1 - (c-2)t + \cfrac{(2c-2)t^2}{1 - (c-4)t + \cfrac{(3c-6)^2}{1 - (c-6)t + \ldots}}}},$$

where the kth row is

$$1 - (c - 2(k-1))\,t + \left(kc - \frac{k(k-1)}{2}\right)t^2.$$

It turns out that for the values of the $\mathfrak{sl}_2$ weight system on complete graphs, there is a similar continued fraction for the generating function:

Theorem 12.4 (Lando's Conjecture, 2014; Proved by Zakorko, 2021)**.** *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_n)t^n &= 1 + ct + c(c-1)t^2 + c(c-1)\big(c-2\big)t^3 \\ &\quad + +c(c^3 - 6c^2 + 13c - 7)t^4 + ... \\ &= \cfrac{1}{1 - ct + \cfrac{ct^2}{1-(c-2)t+\cfrac{(4c-3)t^2}{1-(c-6)t+\cfrac{(9c-18)^2}{1-(c-12)t+\dots}}}}\end{aligned}$$

where the kth row is

$$1 - (c - k(k-1))\,t + \left(k^2 c - \frac{k^2(k^2-1)}{4}\right)t^2.$$

If we compare with the chromatic continued fraction: the kth row is

$$1 - (c - 2(k-1))t + \left(kc - \frac{k(k-1)}{2}\right)t^2.$$

In addition to proving this theorem, Zakorko also developed several instruments for treating $\mathfrak{sl}_2$ weight systems on objects called shares which led to the following:

Theorem 12.5 (Kazarian, Zinova)**.** *For the generating functions*

$$G_m(t) = \sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}(K_{m,n})t^n,$$

we have

$$G_m(t) = \frac{c^m + t \cdot \sum_{i=0}^{m-1} s_{i,m} G_i(t)}{1 - \left(c - \frac{m(m+1)}{2}\right)t}$$

with the initial condition

$$G_0(t) = \frac{1}{1-ct}.$$

There is an explicit formula for the coefficients $s_{i,m}$. So in the last couple of years, we have gotten numerous highly nontrivial families of chord diagrams for which we know the values of the $\mathfrak{sl}_2$ weight system explicitly.

The last direction we mention is the $w_{\mathfrak{sl}_2}$-duality, which generalizes the results for the complete bipartite graphs in the following form: If one replaces complete bipartite graphs sequences $K_{m,n}, n = 0, 1, 2, ...$, with the sequences of **joins** (G, n) of a given graph G with discrete graphs on $n = 0, 1, 2, ...$ vertices, the

form of the previous formula remains the same: the generating function for the values of the $\mathfrak{sl}_2$ weight system is

$$\sum_{n=0}^{\infty} w_{\mathfrak{sl}(2)}((G,n))t^n = \sum_{m=0}^{|V(G)|} \frac{P_m^G(c)}{1-\left(c-\frac{m(m+1)}{2}\right)t},$$

for some sequence of polynomials $P_0^G, P_1^G, ...$

Theorem 12.6 (Zakorko, Zinova)**.** *If we replace a graph G with its complement $\overline{G}$, then the polynomials P_k^G remain the same up to a sign: $P_k^{\overline{G}} = (-1)^{V(G)-k}P+k^G$.*

Here the complement graph $\overline{G}$ has the same set of vertices as G, and the complementary set of edges.

But what about extending $\mathfrak{gl}(N)$-weight system to permutations? There is nothing similar to Chmutov-Varchenko recurrence for other Lie algebras! $\mathfrak{sl}_2$ is just a single example which corresponds to a very powerful influential knot polynomial, the colored Jones polynomial, but there are much more powerful Lie algebras, and for $\mathfrak{gl}(N)$, Kazarian came up with an extremely fruitful idea: For the Lie algebra $\mathfrak{gl}(N)$, a recurrence arises if we extend the weight system from chord diagrams to arbitrary permutations.

In order to explain this construction, we will modify the construction of the weight system from the Lie algebra. Pick an arbitrary basis $\{x_1, ..., x_d\}$, not necessarily orthonormal, and write $\chi_{\nu(a)}$ on the left end of an arc a and the $(\cdot,\cdot)$-dual element $x^*_{\nu(a)}$ on its right end. In the previous example,

$$\sum_{i_1,i_2,i_3,i_4,i_5=1}^{d} x^*_{i_1}x^*_{i_2}x^*_{i_3}x^*_{i_2}x^*_{i_4}x^*_{i_1}x^*_{i_5}x^*_{i_3}x^*_{i_4}x^*_{i_5}$$

The resulting element of the center of the universal enveloping algebra of $\mathfrak{g}$ coincides with the one above.

For $\mathfrak{g} = \mathfrak{gl}(N)$, with the scalar product $(A,B) := \mathrm{Tr}AB$, choose the basis consisting of matrix units $E_{ij}, i,j = 1, ..., N$, with the duality $E^*_{ij} = E_{ji}$.

Definition 12.7. *For $\sigma \in S_m$, a permutation of m elements, define*

$$W_{\mathfrak{gl}(N)} : \sigma \mapsto \sum_{i_1,i_2,...,i_m=1}^{N} E_{i_1,i_{\sigma(1)}}E_{i_2,i_{\sigma(2)}}\cdots E_{i_m,i_{\sigma(m)}} \in U\mathfrak{gl}(N).$$

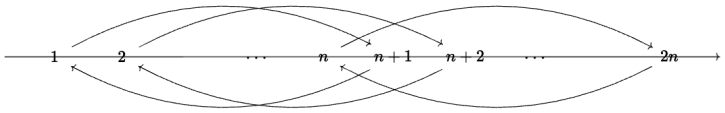
Theorem 12.8. *For any permutation $\sigma, w_{\mathfrak{gl}(N)}(\sigma)$ lies in the center $ZU\mathfrak{gl}(N)$ of $U\mathfrak{gl}(N)$.*

In order to explain the recursion, we need to introduce the notion of a digraph of the permutation:

Definition 12.9. *A permutation can be represented as an oriented graph. The m vertices of the graph correspond to the permuted elements. They are placed on the horizontal line, and numbered from left to right in the increasing order. The arc arrows show the action of the permutation (so that each vertex is incident with exactly one incoming and one outgoing arc edge). The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m oriented edges.*

Example 12.10.

$$G((1\ n+1)(2\ n+2)\cdot\cdots\cdot(n\ 2n)) =$$



Now, we can see that chord diagrams are permutations of special kind: involutions without fixed points. For them, the initial definition coincides with the one above.

12.4 More on Lie Algebras

Let's briefly discuss the center $ZU\mathfrak{gl}(N)$.

Definition 12.11. *The **Casimir elements** $C_m \in U\mathfrak{gl}(N), m = 1, 2...$:*

$$C_m = w_{\mathfrak{gl}(N)}((1, 2, \ldots, m)) = \sum_{i_1, i_2, \ldots i_m = 1}^{N} E_{i_1, i_2} E_{i_2, i_3} \cdots E_{i_m, i_1};$$

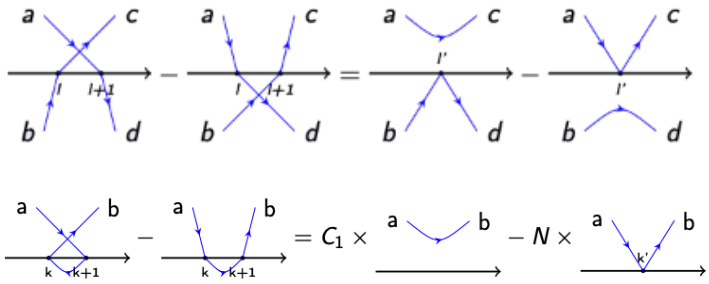
associated to the standard cycles $1 \mapsto 2 \mapsto 3 \mapsto ... \mapsto m \mapsto 1$.

Theorem 12.12. *The center $ZU\mathfrak{gl}(N)$ of the universal enveloping algebra $U\mathfrak{gl}(N)$ of $\mathfrak{gl}(N)$ is identified with the polynomial ring $\mathbb{C}[C_1, \ldots, C_n]$.*

Now, we have arrived to the recurrence relation:

Theorem 12.13 (Yang)**.** *The $w_{\mathfrak{gl}(N)}$ invariant of permutations possesses the following properties:*

- *For the empty permutation, the value of $w_{\mathfrak{gl}(N)}$ is equal to 1;*
- *$w_{\mathfrak{gl}(N)}$ is multiplicate with respect to concatenation of permutations;*
- ***Recurrence rule**: For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $l, l+1$, of the permuted set $\{1, 2, ..., m\}$, we have for the values of the $w_{\mathfrak{gl}(N)}$ weight system.*



For the special case $\sigma(k+1) = k$, the recurrence looks like follows:

Example 12.14.

$$\begin{aligned} w_{\mathfrak{gl}(N)}((1\,3\,2)) &= w_{\mathfrak{gl}(N)}((1\,2\,3)) + C_1 \cdot w_{\mathfrak{gl}(N)}((1)) - N \cdot w_{\mathfrak{gl}(N)}((1\,2)) \\ &= C_3 + C_1^2 - NC_2 \end{aligned}$$

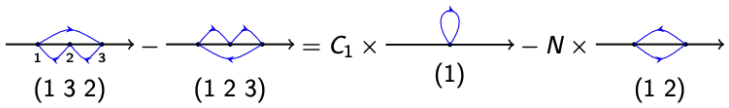
We state an immediate and important corollary of this recurrence relation:

Corollary 12.15. *The $\mathfrak{gl}(N)$-weight systems, for $N = 1, 2, \ldots$ are combined into a universal $\mathfrak{gl}$-weight system $w_{mathfrakgl}$ taking values in the ring of polynomials in infinitely many variables $\mathbb{C}[N; C_1, C_2, \ldots]$.*

After substituting a given value of N and an expression of high Casimirs $C_{N+1}, C_{N+2}, \ldots$ in terms of lower ones $C_1, C_2, \ldots, C_N$, this weight system specifies into the $\mathfrak{gl}(N)$-weight system.

For the chord diagram of order 5 such that any two chords intersect one another, we have

$$\begin{aligned} w_{gl}(K_5) = {} & 24C_2N^4 + (24C_3 - 50C_2^2 - 24C_2^1)N^3 \\ & - (24C_4 + 10C_2C_3 - 35C_2^3 - 70C_2^1C_2 + 72C_1C_2 - 32C_2)N^2 \\ & + (10C_2C_4 + 96C_1C_3 - 10C_4^2 - 50C_2^1C_2^2 + 30C_1C_2^2 - 82C_2^2 \\ & - 20C_4^1 + 48C_3^1 - 32C_1^2)N \\ & - 40C_1C_2C_3 + C_5^2 + 10C_2^1C_3^2 + 30C_3^1C_2 + 15C_4^1C_2 \\ & - 20C_3^1C_2 + 10C_2^1C_2, \end{aligned}$$



What kind of information does this polynomial contain? One way to make this question more precise is to ask which graph invariants can be extracted from these polynomials.

It is easy to show that no substitution for $N, C_1, C_2, ...$ makes $w_{\mathfrak{gl}}$ into the chromatic polynomial of the intersection graph of a chord diagram: the corresponding system of equations for K_1, K_2, K_3, K_4, K_5 has no solutions.

Theorem 12.16. *Under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, \ldots$, the value of $w_{\mathfrak{gl}}$ on a chord diagram becomes a polynomial in N whose leading term is the chromatic polynomial of the intersection graph of the chord diagram.*

Theorem 12.17. *The assertion remains true if one replaces chord diagram with an arbitrary positive permutation.*

Definition 12.18. *A permutation is **positive** if each of its disjoint cycles is strictly increasing, with the exception of the last element.*

Theorem 12.19. *There is a substitution for N and C_k , $k = 1, 2, 3, ...$, which makes the value of $w_{\mathfrak{gl}}$ on a chord diagram into the interlace polynomial of its intersection graph.*

We don't state the definition of an interlace polynomial, but it's very interesting and well investigated.

There is a similar construction of weight systems from Lie superalgebras endowed with nondegenerate invariant scalar product [Vaintrob, 1994].

Theorem 12.20 (Yang)**.** *There is an extension of the Lie superalgebra $\mathfrak{gl}(m|n)$ weight system to permutations similar to that for the Lie algebra $\mathfrak{gl}(N)$. The corresponding universal weight system, which works for all values of m and n together, coincides with the result of substitution $N = m - n$ into the universal weight system $w_{\mathfrak{gl}}$.*

For the other classical series of Lie algebras and Lie superalgebras, the corresponding construction is elaborated by [Kazarian, Yang].

12.5 Krichever's Works

Krichever, in cooperation with Grushevsky, applied effectively real-normalized differentials to the study of geometry of moduli spaces of complex curves.

Definition 12.21. *A meromorphic differential ω on a complex curve X is said to be **real-normalized** if all its periods are real, that is $\int_\gamma \omega$ is real, for an arbitrary closed curve $\gamma : S^1 \to X$ not passing through the poles of ω*

Any meromorphic differential ω on X determines a line field V_ω on $X \backslash \{$ poles of$\omega\}$: at each point q the line $V_\omega(x)$ looks the direction where the imaginary part of $\int_q \omega$ increases, the real part being constant.

Definition 12.22. ***Separatirces** of the line field V_ω are its integral trajectories passing through the zeroes of ω.*

For a given point $A \in X$ not belonging to the separatrices, the function $q \mapsto \int_A^q \omega$ determines a mapping from X cut along the separatrices to $\mathbb{C}$. If ω is real normalized, with a single pole of order 2, then the image is $\mathbb{C}$ cut along several vertical half-lines.

For $(X, \omega), \omega$ real normalized, with a single pole of order 2, and ω in general position, the vertical cut half-lines split into pairs starting at the same height, and determine thus a chord (or arc) diagram. Such a cut diagram determines the pair (X, ω) uniquely: X is reconstructed by gluing the opposite sides of the cuts belonging to the same pair, and ω is the image of dz. Under isoperiodic deformations, the diagram is subject to second Vassiliev moves. This construction has been applied to the study of the isoperiodic foliation in the space of real normalized differentials by [Krichever, Lando; Skripchenko, 2021].

For a more general real normalized differential, the corresponding cut diagram determines a chord diagram no longer. Instead, it determines a diagram of a permutation.

12.6 Open Problems

- The $\mathfrak{sl}(2)$-weight system depends on the intersection graph of a chord diagram rather than on the diagram itself. Whether the $\mathfrak{sl}(2)$-weight system can be induced from a polynomial graph invariant satisfying 4-term relations for graphs?

  A partial answer [Fomichev, Karev, 2024]. The value of the $\mathfrak{sl}(2)$-weight system at c = 3/4 admits a natural extension to graphs.

- The chromatic polynomial of the intersection graph of a chord diagram is the leading term in N of the universal gl-weight system under the substitution $C_k = xN^{k-1}$, $k = 1, 2, 3, ...$ What is the combinatorial meaning of the coefficient of the next term in N? of the other terms?

- What is the combinatorial meaning of the chromatic substitution for permutations? Same questions about interlace polynomial.

- Chord diagrams are orientable maps with a single vertex. Permutations are orientable hypermaps with a single vertex. How can one extend the construction of $\mathfrak{gl}$-weight system to arbitrary hypermaps?

- Stratification of the moduli spaces of meromorphic differentials by strata corresponding to permutations suggests that we must consider permutations (and chord diagrams as a special case) as metrized rather than just combinatorial objects. What is the correct way to impose Vassiliev's 4-term relations and construct corresponding invariants in continuous case?

13 Senya Shlosman: Pedestals Matrices: Polynomial Matrices with Polynomial Eigenvalues

Abstract

I will explain a construction which for every finite poset X (such as a Young diagram) produces a square matrix M^X. Its matrix elements are indexed by pairs P, Q of linear orders on X (pairs of standard tableaux in the case of Young diagrams). The entries of M^X are monomials in variables x_i. Our main result is that the eigenvalues of M^X are polynomials in x_i with integer coefficients. Joint work with Richard Kenyon, Maxim Kontsevich, Oleg Ogievetsky, Cosmin Pohoata, and Will Sawin.

Contents

13.1 Introduction

We begin with a brief reminder of Young diagrams:

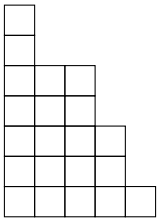
Definition 13.1. *Let us express the integer k as follows:*

$$k = \pi(1) + \pi(2) + \cdots + \pi(n),$$

*where $\pi(i) \geq 0$ and $\pi(i) \geq \pi(i+1)$. We call π a **partition of** k **into (at most)** n **parts**. Let $\mathcal{Y}_n$ denote the set of all partitions π of arbitrary integers. These are referred to as **Young diagrams with at most** n **columns**. The integer k is called the **volume** of the diagram π.*

Example 13.2. *Consider the following partition:*

$$21 = 7 + 5 + 5 + 3 + 1$$



Let g_k denote the number of partitions π of k. The generating function for the sequence g_k is given by:

$$G_n(t) = \prod_{i=1}^{n} \frac{1}{1-t^i}.$$

In fact, we have:

$$\prod_{l=1}^{n} \frac{1}{1-t^l} = (1+t+t^2+t^3+\cdots)(1+t^2+t^4+\cdots)(1+t^3+t^6+\cdots)\cdots(1+t^n+t^{2n}+t^{3n}+\cdots)$$

The 1-to-1 correspondence between the terms in the product and the diagrams holds because, in the first factor, there are two rows of length 1, and the second term is t^2, and so on.

Next, we turn our attention to plane partitions, which are positioned over a rectangle of dimensions $n \times m$.

Let g_k be the number of plane partitions of volume k placed over a rectangle of size $n \times m$. The generating function for the sequence g_k is given by the MacMahon formula:

Proposition 13.3 (MacMahon formula)**.** *The generating function for plane partitions is:*

$$G_{n\times m}(t) = \prod_{l=1}^{n}\prod_{s=1}^{m} \frac{1}{1-t^{l+s-1}},$$

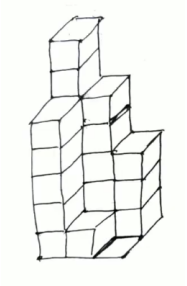


Figure 1: Plane partitions over a rectangle

where $l + s - 1$ is the hook length of the cell (l, s).

The hook length is represented by the shaded region in the following diagram:

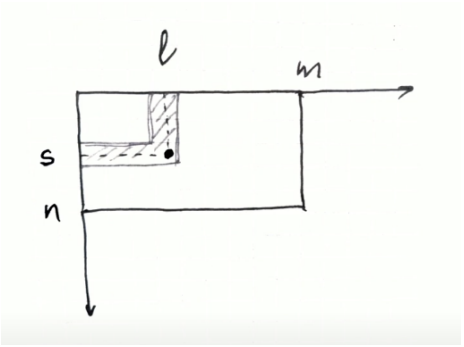


Figure 2: Hook length visualization

There is a natural map from the set $\sqcap_{n,m}$ of plane partitions sitting over the rectangle $n \times m$ onto the Young diagrams $\mathcal{Y}_{nm}$. However, the inverse map is not straightforward, as the preimages of various diagrams do not have the same number of preimages.

The rectangle $n \times m$ has a natural partial order. Let us fix some linear order P on it, which extends the partial order. This linear order is simply a map from the rectangle $n \times m$ onto the segment $[1, nm]$. Let Q be any other linear order on the rectangle.

Definition 13.4. *We call the node $Q^{-1}(k)$ a (P, Q)-**disagreement node** (or descent) if and only if $P(Q^{-1}(k-1)) > P(Q^{-1}(k))$.*

In the diagram, there is only one place of conflict, marked by a circle.

Given the pair of orders P, Q, we define the function q_{PQ} on the rectangle $n \times m$

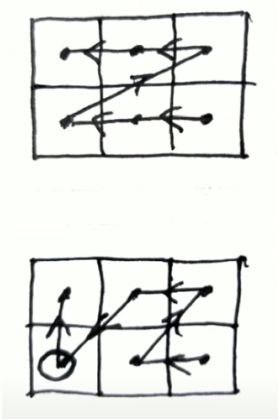
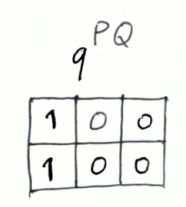


Figure 3: Disagreement node example

by:

$$q_{PQ}(Q^{-1}(k)) = \#\{I : I \le k, Q^{-1}(I) \text{ is a } (P,Q)\text{-disagreement node}\}.$$

Clearly, the function q_{PQ} is non-decreasing on the rectangle. It is called the pedestal of Q with respect to P. Let $\mathcal{E}_P$ denote the set of all pedestals q_{PQ}. For example, for the previous diagram, we obtain the following pedestal diagram:



Why is this useful?

Theorem 13.5. *There exists a bijection between the set $\mathcal{P}_{n,m}$ of non-decreasing functions (i.e., 3D diagrams) and the direct product $\mathcal{E}_P \times \mathcal{Y}_{nm}$, preserving the volumes.*

This bijection is constructed as follows: to each pedestal q_{PQ} and each partition π (i.e., 2D diagram), we associate the function p on the rectangle $n \times m$:

$$p\left(Q^{-1}(k)\right) = q_{PQ}\left(Q^{-1}(k)\right) + \pi(k), \quad k = 1, \ldots, nm.$$

Clearly, the function thus defined is non-decreasing on the rectangle $n \times m$.

Therefore, we need to fix some ordering P on the rectangle $n \times m$, consider all the pedestals q_{PQ}, and take the generating function:

$$\sqcap_P(t) = \sum_Q t^{\nu(q_{PQ})}$$

(which is actually a generating polynomial) of the sequence of the number of pedestals with a given volume. Then, we have the identity:

$$G_{n\times m}(t) = \sqcap_P(t) G_{nm}(t) \equiv \sqcap_P(t) \prod_{l=1}^{nm} \frac{1}{1-t^l}.$$

This gives another generating function for plane partitions, which is simply the generating function of Young diagrams times the generating function of all pedestals. In particular, the polynomial $\sqcap_P(t)$ does not depend on P, and can thus be denoted by $\sqcap_{n\times m}(t)$. This is particularly useful because we can apply this construction not just to plane partitions, but to any partially ordered set.This bijection is constructed as follows: to each pedestal q_{PQ} and each partition π (i.e., 2D diagram), we associate the function p on the rectangle $n \times m$:

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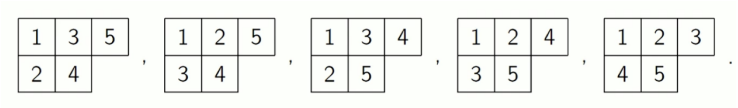
In particular, we can compare two formulas:

$$\sqcup_{n\times m}(t) = \frac{\prod_{l=1}^{nm}(1-t^l)}{\prod_{l=1}^{n}\prod_{s=1}^{m}(1-t^{l+s-1})},$$

which reveals fine cancellations. This ratio is actually a polynomial, a fact that is not immediately obvious.

13.2 Example 1

The standard tableaux for the 2×3 tableaux are shown below:



Definition 13.6. *The **Pedestal matrix** is given by*

$$\begin{pmatrix} 1 & q^3 & q & q^4 & q^2 \\ q^3 & 1 & q^4 & q & q^2 \\ q & q^4 & 1 & q^3 & q^2 \\ q^4 & q & q^3 & 1 & q^2 \\ q^4 & q & q^3 & q^2 & 1 \end{pmatrix}$$

Now, we attempt to calculate the eigenvalues. The miracle is that these eigenvalues are exactly the polynomials we encountered earlier:

$$-(-1+q)(1+q)$$
$$(-1+q)^2(1+q+q^2)$$
$$-(-1+q)(1+q)(1+q+q^2)$$
$$-(-1+q)(1+q)(1-q+q^2)$$

The fact that the function $\sqcap_P(t)$ does not depend on the order P in our rectangle has the following generalization. Instead of characterizing the pedestal q_{PQ} solely by its volume, we associate with it the monomial

$$m_{PQ}(x_1, x_2, x_3, \dots) = x_1^{l_1-1} x_2^{l_2-l_1} \dots x_r^{l_r-l_{r-1}} x_{r+1}^{n-l_r+1},$$

where r is the number of (P,Q)-disagreement nodes, and $l_1, \dots, l_r$ are their locations. Note that for $m_{PQ}(1, t, t^2, \dots)$, we have $t^{v(q_{PQ})}$.

We have shown with Oleg Ogievetsky that the polynomial

$$\mathfrak{h}\left(x_1, x_2, x_3, \dots\right) = \sum_Q m_{PQ}\left(x_1, x_2, x_3, \dots\right)$$

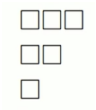
is also independent of P, so it can be denoted as $\mathfrak{h}_{n\times m}(x_1, x_2, x_3, ...)$ and in a sense we have a stochastic matrix.

We now extend to the general case. Instead of considering the partially ordered set - the rectangle $n \times m$ - we take any finite poset X. We denote by Tot_X the set of all possible linear orders on X. One way to express the property that $\mathfrak{h}_P, P \in \mathrm{Tot}_X$ depends only on X is by stating that the matrix M_X, of size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$ with entries $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \dots)$, is stochastic. Specifically, the vector $(1, 1, \dots, 1)$ is the right eigenvector, with eigenvalue $\mathfrak{h}_X(x_1, x_2, x_3, \dots)$. The matrix M_X is the **pedestal matrix**.

Theorem 13.7 (Kenyon, Kontsevich, Ogievtsky, Pohoata, Sawin, Shlosman)**.** *For every poset X, all the eigenvalues of the $|Tot_X| \times |Tot_X|$ matrix M_X with entries $(M_X)_{PQ} = m_{PQ}(x_1, x_2, x_3, \dots)$ are polynomials in $x_1, x_2, x_3, \dots$ with integer coefficients.*

13.3 Example 2

Consider the partition $(3, 2, 1)$:



The 16 standard tableaux (i.e., the orders P, Q on our Young tableau) are given by:

$$\begin{array}{cccc}
\{1,4,6,2,5,3\} & \{1,3,6,2,5,4\} & \{1,2,6,3,5,4\} & \{1,3,6,2,4,5\} \\
\{1,2,6,3,4,5\} & \{1,4,5,2,6,3\} & \{1,3,5,2,6,4\} & \{1,2,5,3,6,4\} \\
\{1,3,4,2,6,5\} & \{1,2,4,3,6,5\} & \{1,2,3,4,6,5\} & \{1,3,5,2,4,6\} \\
\{1,2,5,3,4,6\} & \{1,3,4,2,5,6\} & \{1,2,4,3,5,6\} & \{1,2,3,4,5,6\}
\end{array}$$

To save space, we express the pedestal matrix, where the replacement

$$(x_6^1, x_5^1x_2, x_4^1x_2^2, x_4^1x_2x_3, x_3^1x_3^2, x_3^1x_2^2x_3, x_2^1x_4^2, x_2^1x_3^2x_3, x_2^1x_2^2x_2^3, x_2^1x_2^2x_3x_4) \to (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$$

is made.

We obtain the matrix:

$$\begin{pmatrix}
a_1 & a_5 & a_7 & a_3 & a_9 & a_2 & a_6 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_1 & a_7 & a_3 & a_9 & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_7 & a_1 & a_9 & a_3 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_8 & a_2 & a_{10} & a_4 & a_6 \\
a_5 & a_3 & a_9 & a_1 & a_7 & a_6 & a_2 & a_8 & a_4 & a_{10} & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_7 & a_1 & a_6 & a_8 & a_2 & a_{10} & a_4 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 \\
a_2 & a_6 & a_8 & a_3 & a_9 & a_1 & a_5 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_1 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_7 & a_1 & a_9 & a_3 & a_5 & a_{10} & a_4 & a_8 & a_2 & a_6 \\
a_6 & a_2 & a_8 & a_4 & a_{10} & a_5 & a_3 & a_9 & a_1 & a_7 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_1 & a_5 & a_9 & a_3 & a_8 & a_2 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_5 & a_1 & a_9 & a_3 & a_8 & a_6 & a_2 \\
a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_4 & a_{10} & a_2 & a_8 & a_6 & a_1 & a_7 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_8 & a_2 & a_6 & a_{10} & a_4 & a_8 & a_2 & a_6 & a_7 & a_1 & a_9 & a_3 & a_5 \\
a_6 & a_4 & a_{10} & a_2 & a_8 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_3 & a_9 & a_1 & a_7 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_9 & a_2 & a_6 & a_9 & a_3 & a_7 & a_1 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_8 & a_6 & a_2 & a_9 & a_3 & a_7 & a_5 & a_1
\end{pmatrix}$$

This matrix has eigenvalues:

$$\begin{aligned}
(a_1 - a_4 - a_7 + a_{10})_3,\\
a_1 - a_4 + a_7 - a_{10},\\
(a_1 + a_2 - a_5 - a_6)_2,\\
(a_1 - a_2 - a_5 + a_6),\\
(a_1 - a_2 - a_3 + a_4 + a_7 - a_8 - a_9 + a_{10})_2,\\
(a_1 - a_2 - a_3 + a_4 - a_7 + a_8 + a_9 - a_{10})_2,\\
(a_1 - a_4 + a_5 - a_6 + a_7 - a_{10})_2,\\
a_1 + 2a_2 + 2a_3 + a_4 - a_7 - 2a_8 - 2a_9 - a_{10},\\
a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + a_7 + 2a_8 + 2a_9 + a_{10}
\end{aligned}$$

This is another instance of the previously mentioned phenomenon.

13.4 The Proof

We present a rough plan of the proof:

1. We introduce the class of matrices M_F, such that the matrix M_X can be expressed as a linear combination of M_F-s with integer coefficients.
2. We show that all M_F-s can be made upper-triangular by conjugating them with the same matrix. The resulting upper-triangular matrices will have integer entries on their diagonal.

To clarify these steps, we first introduce some definitions:

Definition 13.8. *A **filter** F is a surjective map $F : X \to [1, 2, \ldots, k]$ with $k \leq n$, such that if $\alpha_i \leq \alpha_j$, then $F(\alpha_i) \leq F(\alpha_j)$. For integers $b_1, \ldots, b_r$ summing to n, we define $\mathcal{F}_{b_1,\ldots,b_r}$ as the set of all filters $F : X \to [1, 2, \ldots, r]$ such that $|F^{-1}(i)| = b_i$ for all $i = 1, \ldots, r$.*

Let P be a linear order on X, and let F be a filter on X. We define a new linear order $Q(P, F)$ by the following rules:

1. For α_i, α_j in the same stratum, i.e., $F(\alpha_i) = F(\alpha_j)$, we have $Q(\alpha_i) < Q(\alpha_j)$ if and only if $P(\alpha_i) < P(\alpha_j)$.
2. For α_i, α_j in different strata, we have $Q(\alpha_i) < Q(\alpha_j)$ if and only if $F(\alpha_i) < F(\alpha_j)$.

We define the matrix M_F by:

$$(M_F)_{PQ} = \begin{cases} 1 & \text{if } Q = Q(P, F) \\ 0 & \text{otherwise} \end{cases}$$

In particular, the matrix M_F has exactly one non-zero entry in each row.

It is easier to explain the rest geometrically, for which we introduce the following definitions:

Definition 13.9. *The **central real hyperplane arrangement** A_n (braid arrangement) consists of hyperplanes $\{H_{ij} : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$, where each hyperplane is defined by $H_{ij} = \{(x_1, \ldots, x_n) : x_i = x_j\}$.*

Definition 13.10. *A **chamber** is an open connected component of*

$$\mathbb{R}^n \setminus \left\{ \bigcup H_{ij} \right\}.$$

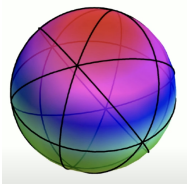
Definition 13.11. *A **cone** is a convex union of the closures of chambers.*

Let $\mathfrak{D}(n)$ denote the set of all distinct cones.

Given a poset X of n elements with a binary relation $\preccurlyeq$, we associate with each pair $i, j \in X$ such that $i \leq j$ a corresponding half-space $K_{ij} = \{x_i \leq x_j\} \subset \mathbb{R}^n$. The cone

$$A(X, \preccurlyeq) = \left\{ \bigcap_{i,j; i \preccurlyeq j} K_{ij} \right\} \in \mathfrak{D}(n)$$

is formed by the intersection over all pairs i, j such that $i \preccurlyeq j$. The correspondence $(X, \preccurlyeq) \to A(X, \preccurlyeq)$ defines a one-to-one map between the set of all partial orders on $\{1, 2, \ldots, n\}$ and the set of all (convex) cones in $\mathfrak{D}(n)$.



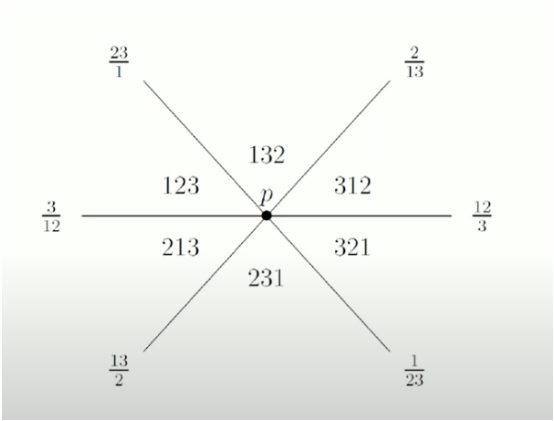
In the image above, the central real hyperplane arrangement A_4 in $\mathbb{R}^4$ is projected to $\mathbb{R}^3$ along the line $x = y = z = t$, and intersected with the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

Example 13.12 ($n = 4$)**.** *We want to know how many partial orders we can put on this. Each triangle is almost like a platonic solid, except the triangles have two angles that are $60°$ and one angle that is $90°$. We can partition $\mathbb{S}^2$ into 24 equal triangles with angles $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right)$, which leads to convex unions: the sphere, the hemisphere, the moon, an elementary triangle (e-triangle), a pair of e-triangles with a common side, a triangle formed by three e-triangles, a square formed by four e-triangles, a triangle made from a square with an adjacent e-triangle, and a triangle formed by six e-triangles sharing a common $\frac{\pi}{3}$-vertex. Their counts are $1, 12, 60, 24, 36, 48, 6, 24, 8$, totaling 219. This matches the number of partial orders on a set of four distinct elements.*

Next, consider the semigroup structure on the hyperplanes and all faces of lower order. Let f', f'' be two faces in $A(X) = A(X, \preccurlyeq)$. Define the face

$f = f''(f') \in A(X)$, called the face-product of f'' and f'. Choose points $x' \in f'$, $x'' \in f''$ in general position. Let $s_{x'x''} : [0,1] \to \mathbb{R}^n$ be the line segment joining these points, where $s_{x'x''}(0) = x'$ and $s_{x'x''}(1) = x''$. Consider the face $f \in A(X)$ containing all points $s_{x'x''}(1-\epsilon)$ for small $\epsilon > 0$. By definition, $f''(f') = f$.

The following figure illustrates the projection of the braid arrangement in $\mathbb{R}^e$ to the plane orthogonal to the main diagonal:



This face-product has the following properties:

- If f'' is a chamber, then $f''f' = f''$.
- If f'' is a chamber, then $f'f''$ is also a chamber. Therefore, faces act on chambers.
- The face-product is associative. For all faces $f, g, h \in A(X, \preccurlyeq)$, we have:

$$f(gh) = (fg)h.$$

Definition 13.13. *The semigroup $A(X, \preccurlyeq)$ is a **left-regular band**, meaning:*

$$ff = f, fgf = fg.$$

The semigroup $A(X, \preccurlyeq)$ recovers the poset X.

Filters correspond to the same structure. Let F be a filter on X of rank k, i.e., a surjective map $F : X \to \{1, \ldots, k\}$ that preserves the partial order. Let

$$\{b_1, \ldots, b_{j_1}\}, \{b_{j_1+1}, \ldots, b_{j_2}\}, \ldots, \{b_{j_{k-1}+1}, \ldots, b_{j_k}\} \subset X$$

be its "floors":

$$\left\{b_{j_{r-1}+1}, \ldots, b_{j_r}\right\} = F^{-1}(r),\ r = 1, \ldots, k.$$

Consider the face $f_F \in A(X, \preccurlyeq)$ defined by the equations:

$$x_{b_{j_{r-1}+1}} = \cdots = x_{b_{j_r}},\ r = 1, \ldots, k,$$

and inequalities:

$$x_{b_{j_1}} \le x_{b_{j_2}} \le \cdots \le x_{b_{j_k}}.$$

There is a one-to-one correspondence between faces and filters. Filters of the highest rank n, i.e., linear extensions of $\preccurlyeq$, correspond to the chambers.

The filter-product is defined as follows. For two filters F' and F'' on X, the filter $F = F''F'$ is uniquely determined by the properties:

- If $F''(u) < F''(v)$, then $F(u) < F(v)$.
- If $F''(u) = F''(v)$, then $F(u) < F(v)$ if and only if $F'(u) < F'(v)$.

Finally, consider the following construction. Let F be a filter on X, and let P be a filter of rank n (a linear order on X). The filter FP is again a filter of rank n. Consider the square matrix $M_X^F = M_X^{F \cdot P, Q}$ where P and Q are linear orders on X:

$$M_X^F[P, Q] = \begin{cases} 1 & \text{if } Q = FP \\ 0 & \text{if } Q \neq FP. \end{cases}$$

The operators M_X^F play a central role in our proof.

Remark 13.14. *These are non-commuting operators.*

Let us rewrite our pedestal matrix M_X as a sum over all monomials:

$$M_X = \sum_{r=1} \sum_{a_1, \ldots, a_r \ge 1,\, a_1 + \cdots + a_r = n} x_1^{a_1} \ldots x_r^{a_r} B_{a_1, \ldots, a_r},$$

where the entries of each matrix $B_{a_1, \ldots, a_r}$ are either 0 or 1.

This leads to an inclusion-exclusion formula:

Proposition 13.15. *If $B^{a_1, \ldots, a_r} \neq 0$, then the following inclusion-exclusion identity holds:*

$$B_{a_1, \ldots, a_r} = \sum_{F \in \mathcal{F}_{a_1, \ldots, a_r}} M_F - \left[\sum_{F \in \mathcal{F}_{a_1 + a_2, \ldots, a_r}} M_F \right] + \left[\sum_{F \in \mathcal{F}_{a_1 + a_2 + a_3, \ldots, a_r}} M_F \right] - \cdots$$

where the sums are taken over all possible mergers of neighboring indices a_i, and the signs are $(-1)^{\#mergers}$.

To explain this, consider an order Q from the row P on the left-hand side (LHS). This order agrees with P over the first $a_1 - 1$ locations, then disagrees once, then agrees again over the next $a_2 - 1$ locations, and continues in this pattern. On the right-hand side, an order Q corresponding to the first sum agrees with P over the first $a_1 - 1$ locations, then agrees or disagrees once, then agrees again over the next $a_2 - 1$ locations, and continues similarly. Therefore, we must eliminate those Q's which agree with P over the first $a_1 - 1$ locations, then agree once more, and so on for each a_i.

Now, consider our matrices $M_{F,X}$, which are of size $|\mathrm{Tot}_X| \times |\mathrm{Tot}_X|$. Let us now remove all order relations on X, resulting in the poset $\overline{X}$ with $|\mathrm{Tot}_{\overline{X}}| = n!$. Note that $M_{F,X}$ is a submatrix of $M_{F,\overline{X}}$, specifically an upper-left submatrix. All elements to the right of this submatrix are zero, so $M_{F,X}$ forms a block of $M_{F,\overline{X}}$. Each row of $M_{F,\overline{X}}$ contains exactly one 1, with the rest being zeros. Since each row of $M_{F,X}$ already contains one 1, it suffices to note that the spectrum of $M_{F,\overline{X}}$ consists of integers.

Let us briefly introduce tournaments. We will focus solely on the "totally unordered" poset $\overline{X}$. Consider a larger matrix, $M_{F,T}$, of size $2^{n(n-1)/2}$, where T represents the tournaments between n entries. A tournament is an assignment of the order $\preccurlyeq$ to each pair $i \neq j$ of the elements of the set $\{1, \ldots, n\}$, independently for each pair. Filters act on tournaments in the same way as they act on linear orders, so the same matrices hold. Given a tournament $\preccurlyeq$ and a filter F, we define a new tournament $\preccurlyeq_F$ as follows:

1. If $F(i) = F(j)$, then $i \preccurlyeq_F j$ if $i \preccurlyeq j$.
2. If $F(i) < F(j)$, then $i \preccurlyeq_F j$.

Any linear order defines a tournament in a straightforward way, so our matrices $M_{F,X}$ are blocks of $M_{F,T^{-s}}$. Thus, it suffices to study $M_{F,T^{-s}}$.

The key observation is that $M_{F,T}$ is a tensor product of $\frac{n(n-1)}{2}$ two-by-two matrices, corresponding to all pairs (i, j), since the orders $\preccurlyeq$ can be assigned to the pairs independently. Since the tensor product of upper triangular matrices is upper triangular, we only need to verify our claim for filters and tournaments in the case $n = 2$.

The three possible $M_{F,T^{-s}}$ matrices for this case are:

$$M_1 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Conjugating them by the discrete Fourier transform matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ transforms them into a triple of upper triangular matrices: $UM_1U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $UM_2U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $UM_3U^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. This completes the proof.

14 Ivan Cherednik: Q-zeta Revisited

Abstract

The fundamental feature of practically all zeta-functions and L-functions is that their meromorphic continuations to complex s provide a lot of information about the corresponding objects. However, complex values of s have generally no direct arithmetic/geometric meaning, and occur as a powerful technical tool. We will discuss the refined theory, which is basically the replacement of the terms $\frac{1}{n^s}$ by the invariants of lens space $L(n,1)$, certain q,t,a-series. One of their key properties is the superduality $q \leftrightarrow t^{-1}$, which is related to the functional equation of the Hasse-Weil zetas for curves, the symmetry $\epsilon_1 \leftrightarrow \epsilon_2$ of Nekrasov's instantons, and to other refined theories in mathematics and physics. These invariants have various specializations, including Rogers-Ramanujan identities and the topological vertex. We will begin the talk with the Riemann q-zeta-hypothesis in type A_1, in full detail.

Contents

14.1 Introduction

The fundamental property of practically all ζ-functions and L-functions is that their meromorphic continuations to complex s provide a lot of information about the corresponding objects. However, complex s have generally no direct arithmetic and geometric meaning, and occur as a powerful technical tool. We will discuss the refined theory, which is basically the replacement of n^{-s} by the invariants of lens space $L(n,1)$, certain $q,t,\mathfrak{a}$-series directly related to Elliptic Hall Polynomials. The superduality $q \leftrightarrow t^{-1}, a \to a$ is one of their key properties, corresponding to the functional equation of the Hasse-Weil zetas in the motivic approach. Their various specializations include topological vertex and Rogers-Ramanujan identities. We begin with the Riemann q-hypothesis for the root system A_1, when $a = t^2, t = q^k, k = s - \frac{1}{2}$.

14.2 Are Zeta's Geometric?

Zeta-functions are an important tool in mathematics almost everywhere. It's wide-ranging usefulness is comparable to partition functions in physics. An essential aspect of zeta-functions are the complex s, which we will use freely without understanding all of its details.

One reason why we care about them is because Selberg's zeta function (arising from closed geodesics) "almost" satisfies the Riemann Hypothesis for compact Riemann surfaces X. It is related to $\zeta_\Delta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}$ for $\mathrm{Spec}(\Delta_X)$. Furthermore, there are many geometric applications, mostly for $s = 0$.

These ideas are complementary to the theta-functions of objects found in Krichever's works: Riemann surfaces, KP, etc. We will focus on Δ, which has the following nice property:

Proposition 14.1 (Ray-Singer's Regularization)**.**

$$\det(\Delta) = \lim_{s \to 0} \exp(-\zeta'_\Delta(s))$$

Now, one of the key conjectures in number theory appears:

Conjecture 14.1 (Selberg's Conjecture)**.** *If* $X = \Gamma(N)\backslash\mathbb{H}$*,*

$$\lambda_1 \geq 1/4$$

Another reason why we care about them: Ising models with an external magnetic field are non-integrable (for $D > 1$). Furthermore, we have the following important result:

Theorem 14.2 (Lee-Yang Theorem)**.** *The corresponding partition functions* $\mathfrak{Z}_N$*, with complex fugacity* s*, satisfy the Riemann Hypothesis for any* D *gives that* $\lim_{N\to\infty} \log(\mathfrak{Z}_N)/N$ *has only one real phase transition.*

14.3 A Stirling Moak Formula

One way to generalize n^s is to take the Ueno-Nishizara transformation:

Definition 14.3 (Ueno, Nishizawa)**.**

$$n^s \mapsto [n]_s^q.$$

For example, we would have $x \mapsto \sinh(x)$.

We will do something different. Inspired by Double Affine Hecke Algebras, we replace x^{2k} by the Macdonald measure:

Definition 14.4. *The* ***Macdonald measure*** *is*

$$\delta_k(x;q) := \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j-2x})}{(1-q^{j+k+2x})(1-q^{j+k-2x})}.$$

What is connection to n^s? Let $q = \exp(-1/a)$ for $\mathbb{R} \ni a > 0$ and $u^s = \exp(s \log u), u \notin -\mathbb{R}_+$. There are two connections:

Theorem 14.5 (The Stirling Moak Limiting Formula)**.**

$$\lim_{a\to\infty} \left(\frac{a}{4}\right)^k \delta_k(\sqrt{az};q) = (-z)^k$$

if either $z \notin \mathbb{R}_+$ *or* $k \in \mathbb{Z}$*;*

Theorem 14.6 (The Straight Limit to Gamma-Functions)**.**

$$\lim_{a\to\infty} a^{2k}\delta_k(\sqrt{az};q) = \frac{\Gamma(k+2\sqrt{z})\Gamma(k-2\sqrt{z})}{\Gamma(2\sqrt{z})\Gamma(-2\sqrt{z})}.$$

Setting $\zeta_q(s) = \sum_{n=1}^{\infty} \delta_{-s}(\sqrt{ain};q)$, we can see that it is singular everywhere. Nevertheless, for generic sequences $a = a_m \to \infty$,

$$(-i/a)^s \zeta_q(s) \to \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1.$$

14.4 Q-Zeta As An Integral

Following the classical formula

$$\mathfrak{Z}(k) := 2\int_0^{\infty} (e^{x^2}+1)^{-1} x^{2k} dx = (1-2^{1/2-k})\Gamma\left(k+\frac{1}{2}\right)\zeta\left(k+\frac{1}{2}\right),$$

we can define a q-deformed analog:

Definition 14.7.

$$\beth_q(k) := (-i)\int_{\infty i}^{-\infty i}(q^{x^2}+1)^{-1}\delta_k dx.$$

Theorem 14.8.

$$\lim_{a\to\infty}\left(\frac{a}{4}\right)^{k-1/2}\beth_q = \beth$$

when $q = e^{-1/a}$.

Remark 14.9. *The proof for* $\Re(k) \leq \frac{1}{2}$ *is based on the shift operator.*

Remark 14.10. *Setting*

$$\beth_q^\dagger(k) := \frac{1}{i}\int_{\infty i}^{-\infty i}(q^{x^2}-1)^{-1}\delta_k dx$$

gives

$$\left(\frac{a}{4}\right)^{k-1/2}\beth_q^\dagger(k) \to \Gamma\left(k+\frac{1}{2}\right)\zeta\left(k+\frac{1}{2}\right), \Re(k) > \frac{1}{2}, a^{2k-1}\beth_q^\dagger(k) \to \tan(\pi k)\frac{\Gamma(k)}{2}$$

for $0 < \Re(k) < \frac{1}{2}$. *There are some interesting implications in analytic number theory arising from the appearance of* $\tan(\pi k)\Gamma(k)^2$.

Problem 14.11. *Find (numerically) zeros of* $\beth_q$ *in the right half-plane* $\Re(k) > 0$ *that are deformations of the classical zeros of* $\zeta(k+1/2)$.

For a zero $k = z$ of $\zeta(k+1/2)$ at (near) iR, the linear $(1/a)$-approximation $z_e(a)$ of the corresponding zero $z(a)$ of $\beth_q$ is given as follows:

$$\frac{z_e(a)}{z} = 1 + \frac{4(z+1/2)\zeta_+(z+3/2) - (z-1)\zeta_+(z-1/2)}{12a\zeta'(z+1/2)(1-2^{1/2-z})},$$

as $\zeta'(s) = \partial\zeta(s)/\partial s, \zeta_+(s) = (1-2^{1-s})\zeta(s)$.

The first zero z that might go to the right is $1977.27i$; it is exactly the first one "unusually" close to its neighbor (so the linear approximation cannot be trusted too much).

The limit $t = q^k \to 0$ has many applications, including Rogers-Ramanujan identities.

Theorem 14.12. *As* $\Re(k) \to +\infty$, $\beth_q$ *tends to*

$$\frac{1}{i}\int_0^\infty \frac{\prod_{j=0}^\infty (1-q^{j+2x})(1-q^{j-2x})}{q^{x^2}+1}dx.$$

14.5 Analytic continuation

Definition 14.13. *Define the* ***Truncated θ-function*** *as*

$$\Phi_{\varepsilon}^{k}(F) := \frac{1}{i}\int_{\varepsilon+iR} F\mu(x)dx.$$

where

$$\mu(x) = \prod_{i=0}^{\infty} \frac{(1-q^{i+2x})(1-q^{i+1-2x})}{(1-q^{i+k+2x})(1-q^{i+k+1-2x})}$$

Example 14.14.

- $\mathcal{E}_0 := q^{x^2}$
- $F = f(q^x)\mathcal{E}_0$
- $\mathcal{E}_- := (q^{x^2}-1)^{-1}$
- $\mathcal{E}_+ := (q^{x^2}+1)^{-1}$

Theorem 14.15. *For these F, $\Phi_{\varepsilon}^{k}(F)$ is analytic as $\Re k > \max\{-2\varepsilon, 2\varepsilon - 1\}$.*

Definition 14.16. *The* ***Q-Macdonald-Mehta*** *is:*

$$\Phi_{\frac{1}{4}}(\mathcal{E}_0) = \sqrt{\pi a \prod_{j=1}^{\infty} \frac{1-q^{k+j}}{1-q^{2k+j}}}, \Re k - \frac{1}{2}.$$

Remark 14.17.

1. *For analytic symmetric F,*

$$\Phi_{\varepsilon}^{k}(F) = 1 + q^{k/2i}\int_{\varepsilon+i\mathbb{R}} F\delta(x)\,dx, \Re k > 0.$$

2. *Why not for all k? There are bad k, such as $\{2C-1-\mathfrak{Z}_+, -2C-\mathfrak{Z}_+\}$, $C = \{\varepsilon + i\mathbb{R}\}$.*
3. *We have*

$$\lim_{a\to\infty} a^{k-1/2}\Phi_{\frac{1}{4}}(\mathcal{E}_0) = \sqrt{\pi}\frac{\Gamma(2k)}{\Gamma(k)}, q = e^{-\frac{1}{a}}.$$

14.6 Sharp Q-Zeta Function

Let $\varepsilon = \sqrt{\frac{\pi a}{2}}$, the integration path be a closed loop between $\infty - \varepsilon i$ and $\infty + \varepsilon i$ through zero. Then

$$\mathfrak{Z}_q^{\sqsubset}(k) := \frac{1}{2i}\int_{\infty+\varepsilon i}^{\infty-\varepsilon i} \frac{\delta_k(x;q)}{q^{-x^2+1}}\,dx$$

is analytic in the horizontal strip $K^\sharp = \{-2\varepsilon < \Im(k) < +2\varepsilon\}$ as $\Re k - \frac{1}{2}$. Its meromorphic continuation to all $k \in \mathbb{C}$ (via Cauchy's theorem), is called sharp zeta:

Definition 14.18. *The **sharp zeta function** is*

$$\mathfrak{Z}_q^\sharp(k) := -\frac{a\pi}{2}\prod_{j=0}^{\infty}\frac{(1-q^{j+k})(1-q^{j-k})}{(1-q^{j+2k})(1-q^{j+1})}\times\sum_{j=0}^{\infty}\frac{(1-q^{j+k})q^{-kj}}{(1-q^k)(q^{-(k+j)^2/4+1})}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l}$$

with poles in $K^\sharp$ at $\{-1/2-\mathbb{Z}_+\}$.

This strip is exactly between the first zeros of $q-\frac{k^2}{4}+1$.

Theorem 14.19. *For all k apart from the poles,*

$$\begin{aligned}&\lim_{a\to\infty}\left(\frac{a^{k-1/2}}{4}\right)\mathfrak{Z}_q^\sharp(k)\\ =&\sin(\pi k)\left(1-2^{1/2-k}\right)\Gamma\left(k+\frac{1}{2}\right)\zeta\left(k+\frac{1}{2}\right).\end{aligned}$$

Conjecture 14.2. *Given a classical zeta-zero $k=z$, $z^\sharp(a)$ is its $\sharp$-deformation, $\tilde{z}^\sharp(a)=z$:*

$$\tilde{z}^\sharp(a) = z\left(1-\frac{4(z+\frac{1}{2})\zeta_+(z+\frac{3}{2})-(z-1)\zeta_+(z-\frac{1}{2})}{12a\zeta'(z+\frac{1}{2})(1-2^{1/2-z})}\right).$$

We can proceed similarly for q^{-dx^2} and sharp L-functions.

Taking $z=14.1347i$ and $a=750$, $d=2$, $z^\sharp = 0.1304+14.1450i$, $\tilde{z}^\sharp = 0.1302+14.1465i$. The other zeros in $K^\sharp$ for $a=750$, $d=2$ are:

zeta	sharp-zeta	linear approximation
$21.0220i$	$0.3514+21.0702i$	$0.3504+21.0771i$
$25.0109i$	$0.5641+24.9586i$	$0.5745+24.9643i$
$30.4249i$	$0.9046+30.4014i$	$0.9134+30.4077i$
$32.9351i$	$1.1051+33.0341i$	$1.0998+33.0854i$
$37.5862i$	$1.6449+37.9660i$	$1.7675+38.1895i$
$40.9187i$	$1.9080+40.8119i$	$1.9141+40.7816i$
$43.3271i$	$2.2860+43.2485i$	$2.4497+43.3138i$
$48.0052i$	$2.9259+47.8424i$	$3.1103+47.5578i$

14.7 Riemann's Q-Hypothesis

The simplest sharp Dirichlet L-function is

$$L_q^\sharp(k) := \frac{1}{2}\sqrt{\pi a}\prod_{j=0}^{\infty}\frac{1-q^{j+1}}{1-q^{j+k+1}}\times\sum_{j=0}^{\infty}\frac{1-q^{j+k}}{1-q^k}\frac{q^{-kj}}{q^{(k+j)^2/4+1}+q^{-(k+j)^2/4}}\prod_{l=1}^{j}\frac{1-q^{l+2k-1}}{1-q^l}.$$

It converges to the analytic continuation of the Dirichlet L-function of conductor 3:

$$L(k) = \sum_{j=0}^{\infty} \left((3j+1)^{-k-1/2} - (3j+2)^{-k-1/2} \right),$$

where $k = s - \frac{1}{2}$, $\Re k > \frac{1}{2}$, and $a \to \infty$.

Proposition 14.20. *The function $L_q^\sharp(k)$ is regular in the horizontal strip*

$$K_3^\sharp = \left\{ -2\sqrt{\frac{\pi a}{3}} < \Im(k) < 2\sqrt{\frac{\pi a}{3}} \right\}.$$

The first negative $\Re$ of the $\frac{1}{a}$-linearization occurs at $z = 246.4149i$; this L-zero is very close to the previous one $246.3028i$. It is realistic to try to calculate the corresponding q-zero numerically.

Conjecture 14.3 (Q-Hypothesis)**.** *Let*

$$\hat{L}_b^\sharp(q)(k) := L_q^\sharp(k) - L_q^\sharp(-k).$$

All of its zeros in a strip a bit smaller than $K_3^\sharp$ are deformations of the zeros of $\hat{L}(k) := L(k) - L(-k)$ and they are all imaginary for $\Re(k)$ sufficiently small.

The following is the table of all zeros $z^\sharp$ of $\hat{L}_q^\sharp(k)$ corresponding to the classical zeros of $L(k)$ (z_1, z_2 etc.) for $a = 500$ inside $K_3^\sharp$. The condition $|\Re(z^\sharp)| < 0.3$ is imposed in this computation.

Let us give the first two sharp-zeros $z_{1,2}^\sharp$ with their linear approximations:

$$\begin{array}{lll} z_1 = 8.0397i, & z_1^\sharp = 8.0329i, & \tilde{z}_1^\sharp = 8.0324i \\ z_2^\sharp = 11.2492i, & z_2^\sharp = 11.3048i, & \tilde{z}_2^\sharp = 11.3062i \end{array}$$

The linear approximations are purely imaginary due to the antisymmetrization.

All zeroes in the strip in the following table:

z	$z^\sharp$
$8.0397i$	$8.0329i$
$11.2492i$	$11.3048i$
$15.7046i$	$15.6124i$
$18.2620i$	$18.2702i$
$20.4558i$	$20.7196i$
$24.0594i$	$23.9028i$
$26.5779i$	$26.3980i$
$28.2182i$	$28.6315i$
$30.7450i$	$31.0640i$
$33.8974i$	$33.5322i$
$35.6084i$	$35.7571i$
$37.5518i$	$37.9601i$
$39.4852i$	$40.1558i$
$42.6164i$	$42.3568i$
$44.1206i$	$44.4996i$

Table 1: Table of z and $z^\sharp$

14.8 Gauss Integrals and Zetas

Theorem 14.21 (The q-Mehta-Macdonald Formula)**.** *We can write the q-Mehta-Macdonald function as a product in terms of q-Gamma functions for* $\int_{i\mathbb{R}^n} \gamma(x)\mu(qx)\,dx$ *for the plus-Gaussian* $\gamma(x) = q^{-x^2/2}$ *and the mu-function*

$$\mu(q^x) = \prod_{\alpha, j\geq 0} \frac{\left(1 - q^{(x,\alpha)+\nu\alpha j}\right)\left(1 - q^{-(x,\alpha)+\nu\alpha(j+1)}\right)}{\left(1 - t^\alpha q^{(x,\alpha)+\nu\alpha j}\right)\left(1 - t^\alpha q^{-(x,\alpha)+\nu\alpha(j+1)}\right)}.$$

Here α are positive roots of a given reduced irreducible root system $R \subseteq \mathbb{R}^n$, normalized by the condition $(\alpha_{\text{sht}}, \alpha_{\text{sht}}) = 2$; we set $\nu_\alpha = 1$ for short α, and $\nu = 2, 3$ for long. Let $t_{\text{sht}} = q^{k_{\text{sht}}}$ and $t_{\text{lng}} = q^{k_{\text{lng}}}$; t^α depends only on $|\alpha|$. Also, $\rho_k = \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha$ and $\mu_1 = \frac{\mu}{\text{CT}(\mu)}$ for the constant term functional CT for Laurent series in terms of $X_b = q^{(x,b)}$ for $b \in P$. Let $0 < q < 1$ and $t^\alpha > 1$.

One of the key Double Affine Hecke Algebras formulas is given by

$$\int_{\mathbb{R}^n} P_\lambda(qx) P_\lambda(t^{-\rho}) f(x)\gamma(x)\mu(qx)\,dx \sim f(x = \lambda + \rho_k).$$

The integration can be replaced by CT.

Definition 14.22. *The* ***Jackson integration*** *is a variant of* $\int_{\mathbb{R}^n}$ *defined as*

$$J(f(x)) = \sum_{\hat{w}\in\hat{W}} f(\hat{w}(x)).$$

Definition 14.23. ***Elliptic Hall Polynomials*** *of level ℓ are*

$$J\left(\frac{P_\lambda}{P_\lambda(t^{-\rho})}q^{\ell x^2/2}\mu\left(q^x\right)\right).$$

Their number is the same as the number of Kac-Moody characters of level ℓ.

Let

$$\mathfrak{Z}_n^+(q,t)=\frac{\int_{\epsilon+iR^n}\frac{\gamma(x)}{1+\gamma(x)}\mu(qx)\,dx}{\int_{\varepsilon+iR^n}\gamma(x)\mu(q^x)\,dx}$$

and we set: $s=k_{sht}|R_+^{sht}|+k_{lng}|R_+^{lng}|+\frac{n}{2}$. Then one has:

$$\lim_{q\to 1_-}\mathfrak{Z}_+(q,t)=\eta(s):=(1-2^{1-s})\zeta(s)$$

for the Riemann's $\zeta(s)$. The translation by of ε combined with the usage of μ improves the range of k_ν where $\mathfrak{Z}_n^+$ is analytic. The basic range is $\Re k_\nu>0$, which is for $\varepsilon=0$. If $k_{lng}=k=k_{sht}$ and $\varepsilon=\rho/h$ for the Coxeter number, then $\mathfrak{Z}_+(q,t)$ is analytic for $\Re k>-\frac{1}{h}$, which corresponds to $\Re s>0$, i.e., our integral formula works for the whole critical strip $0<s<1$ in the limit, which can be potentially useful for "q-Lindelöf".

The convergence to $\eta(s)$ holds for any $s\in\mathbb{C}$ upon the analytic continuation. The justification is essentially the procedure of "picking up residues" due to [Weyl, Arthur, Heckman, Opdam].

Let's consider the type A_n setting. Set $v=n+1$ and $v^\circ=-kv$:

$$s=k\frac{v(v-1)}{2}+\frac{v-1}{2}=-\frac{1}{2}(v-1)(v^\circ-1).$$

The integral $\mathcal{I}_+(n)=\int_{\varepsilon+iR^n}\frac{\gamma(x)}{1+\gamma(x)}\mu(q^x)\,dx$ for $\varepsilon=\rho/\nu$ is an analytic function for $\Re k>-1/v$ and, accordingly, for $s>-\frac{1}{v}\frac{v(v-1)}{2}+\frac{v-1}{2}=0$.

In type A, there exists meromorphic $\mathfrak{Z}(q,t,\mathfrak{a})$ satisfying the superduality, which is $\mathfrak{Z}(t^{-1},q^{-1},\mathfrak{a})=\mathfrak{Z}(q,t,\mathfrak{a})$, and such that $\eta(s)$ is the limit $q\to 1_-$ of $\mathfrak{Z}(q,t=q^k,a=t^v)$. A similar stabilization is expected for Elliptic Hall Polynomials. Also, elliptic triply graded homology of torus knots are expected to be those of Lens spaces.

In terms of k,ν, the superduality becomes: $k\mapsto 1/k$, $\nu\mapsto -k\nu=\nu^\circ$. The corresponding s remains fixed under superduality. However, we have a non-trivial connection between the values of $\mathfrak{Z}$ at k and $1/k$ in the q-theory. For instance, the $\mathfrak{a}$-coefficients of $\mathcal{Z}$ are (conditionally) bounded as $|k|\to\infty$ and $\Re s>0$, or its values for superinvariant $\mathfrak{a}$, which is a variant of the Lindelöf hypothesis.

Algebraically, $\mathcal{Z}(q,t,\mathfrak{a})$ is a generating function of the simplest elliptic Hall polynomials, that of invariants of $L(n,1)$.

Geometrically:

Problem 14.24. *Find a conifold M with $L(n,1)$ as special fibers (for the localization in proper cohomology) such that $\mathbb{H}^*(M) = \mathcal{Z}$.*

15 Alexander Braverman: Introduction to Symplectic Duality and Coulomb Branches of 3D Quantum Field Theories

Abstract

I will give a survey of the series of my joint works with Finkelberg and Nakajima giving a mathematical construction of the so called Coulomb branches of 3D $\mathcal{N} = 4$ super-symmetric gauge theories (no knowledge of any of these words will be needed). I will also explain its connection with the (purely mathematical subject) of symplectic duality.

Contents

15.1 Introduction

We begin by presenting a survey of works on Coulomb branches of 3d gauge theories, before transitioning to a discussion of 4d gauge theories. The goal is to eventually address an open problem that connects to the lectures of Etingof and Nekrasov. This problem is intriguingly related to the geometric construction of various Hitchin integrable systems.

The motivation stems from physics, where people work with analytic objects, such as complex Lie groups. However, we will approach the problem algebraically, starting with some basic data: let G be a reductive connected algebraic group, and N be a finite-dimensional representation of G. Physics suggests that we should associate a piece of data to this structure, namely the 3d $\mathcal{N} = 4$ quantum gauge theory.

As mathematicians, we may not fully understand the physics, but we can begin attaching mathematical objects to this structure. One such object is the moduli space of vacua, which is a complicated structure that we do not yet fully understand. However, it contains special components with well-defined mathematical structures, specifically the Higgs branch ($\mathcal{M}_H$) and the Coulomb branch ($\mathcal{M}_C$) of the moduli space. From here, we will set aside the physical language and focus on the mathematical properties of these objects.

Physics suggests that these branches should be "hyper-Kähler manifolds" with additional structure. The quotation marks are due to the fact that these objects are often singular. Since it is unclear how to mathematically define a singular hyper-Kähler manifold, we will not fully address this, but we will fix a specific complex structure that allows us to work algebraically.

With the chosen complex structure, both the Higgs and Coulomb branches can be defined as affine algebraic varieties, which may be singular. These varieties are equipped with a Poisson structure and are generically symplectic, meaning that the Poisson structure on the smooth locus of the variety is derived from a symplectic structure.

Many questions arise: What is the mathematical definition of these varieties? While physics suggests their existence, we need to understand their precise mathematical nature. We will begin with the Higgs branch:

Problem 15.1. *What do we know about $\mathcal{M}_H$?*

Let us start with the simplest case, where the data consists of a group and a representation. For the Higgs branch, the definition is relatively straightforward, especially without delving too deeply into technicalities:

$$\mathcal{M}_H = T^*N = N \oplus N^*.$$

From the physics perspective, this should only depend on $N \oplus N^*$, meaning that replacing N with N^* should yield the same result. This is true for the Higgs branch, but less so for the Coulomb branch.

Next, we perform the Hamiltonian reduction by the group G. Since T^*N is a symplectic vector space, it carries a symplectic or Hamiltonian action of G, and thus has a moment map. The moment map is given by $T^*N \to \mathfrak{g}^*$.

Definition 15.2. *The **Higgs branch** is defined as*

$$\mathcal{M}_H = \mu^{-1}(0)/G = Spec(\mathbb{C}[\mu^{-1}(0)]^G).$$

This construction includes well-known examples: for instance, Nakajima quiver varieties are special cases of this construction, arising from quiver gauge theories.

Example 15.3. *Consider a graph Q with orientation. Attach to each vertex i two vector spaces V_i and W_i. Let $G = \prod GL(V_i)$ and $N = \prod_{i\to j} Hom(V_i, V_j) \times \prod_i Hom(V_i, W_i)$. After performing the Hamiltonian reduction, the Higgs branch $\mathcal{M}_H$ is called a **Nakajima quiver variety**. Nakajima related these varieties to representations of Kac-Moody algebras, but we will not delve into that topic here.*

Examples of such varieties include:

1. *$\mathbb{C}^2/\Gamma$, where $\Gamma \subset SL(2,\mathbb{C})$ is a finite subgroup.*
2. *$\mathfrak{sl}(n,\mathbb{C}) \supset \mathcal{N} \sqcup_\lambda \mathcal{O}_\lambda$, where $\mathcal{N}$ is the nilpotent cone. For each λ, μ, the Slodowy slice S^λ_μ to $\mathcal{O}_\mu$ inside $\overline{\mathcal{O}}_\lambda$ provides an important structure.*

There are many other examples, and several known symplectic varieties arise in this way.

Problem 15.4. *What do we know about $\mathcal{M}_C$?*

Next, we will define the Coulomb branch and explore its generalizations. Before doing so, we state some properties that any definition of $\mathcal{M}_C$ should satisfy, and we will then describe how to construct a definition that satisfies these properties.

Proposition 15.5.

1. *$\mathcal{M}_C$ is an affine normal Poisson "symplectic" variety of dimension $2\,rank G$. The quotation marks indicate that we aim for the structure to be as symplectic as possible—if the variety is smooth, we want the entire locus to be symplectic.*
2. *$\mathcal{M}_C$ comes equipped with a canonical integrable system: $\mathcal{M}_C \xrightarrow{\pi} \approx/W$, where $T \subset G$ is a maximal torus, $\approx = Lie T$, and W is the Weyl group. In classical integrable systems, the fibers are often tori in some sense. The term "torus" can refer to various concepts depending on context: in real manifolds, a torus is a product of circles; in affine algebraic geometry, it refers to $(\mathbb{C}^*)^{rank G}$. The open problem will become more interesting when considering fibers as generic fibers of abelian varieties.*
3. *$\mathcal{M}_C$ has a canonical C^*-action (and a canonical grading) under which the Poisson structure has degree -2.*

4. *There is a canonical quantization of these structures. Let $\mathcal{A}_C = \mathbb{C}[\mathcal{M}_C]$, which is a Poisson algebra, and since there is an integrable system, there exists a sufficiently large commutative subalgebra. A deformation $\mathcal{A}_{C,\hbar}$ of this algebra exists, and the integrable system also deforms.*

5. *Assuming $G_F \hookrightarrow Aut_G(N)$, the above structure has a (canonical) deformation over $\approx_F / W_F = \mathfrak{g}_F /$ Adjoint action.*

However, if we take a sufficiently generic group G and representation N, the result may not make sense, as it could produce varieties we do not wish to describe. Thus, only in some special cases can we explicitly describe the Coulomb branches, as "explicit" means that the varieties correspond to familiar ones. In most cases, however, we will obtain new varieties that have not been encountered before, although some familiar varieties may emerge. Here's a simple example:

Example 15.6. *Take any G and let $N = \mathfrak{g}$, the adjoint representation. In this case, we can explicitly describe the Coulomb branch:*

$$\mathcal{M}_C = \mathbb{T}^* T^V / W,$$

where T is the maximal torus and $\mathbb{T}^*$ is the tangent bundle. Note that $T^* T^\vee = T^\vee \times \approx$, and we have two canonical maps: $T^\vee \times \approx \to \approx$ and $(T^\vee \times \approx)/W \to \approx /W$. It's not difficult to verify that the symplectic structure is indeed well-defined.

Here is a more complex example:

Example 15.7. *Take any G with $N = 0$. This corresponds to* ***pure gauge theory*** *in physics. The variety we obtain is symplectic, but its symplectic structure is not immediately evident from the construction we present, as it requires a different approach.*

We obtain the ***universal centralizer****:*

$$\{(x \in (\mathfrak{g}^\vee_{reg})^*, g \in G^\vee \text{ such that } ad_g(x) = x)\}/\text{conjugation},$$

where "reg" indicates that the characteristic polynomial equals the minimal polynomial. In general, "regular" means the minimal possible dimension of the centralizer in the group.

Given G, we can attach its Langlands dual group $G^\vee$, whose root datum is dual to that of G.

The map we want is:

$$\pi : \{(x \in (\mathfrak{g}^\vee_{reg})^*, g \in G^\vee \text{ such that } ad_g(x) = x)\}/\text{conjugation} \to (\mathfrak{g}^\vee_{reg})^* / G^\vee = \approx /W.$$

The map $\pi(x, g) = g$ only remembers the conjugacy class of x, and the moduli space of regular conjugacy classes is the same as the moduli space of all conjugacy classes. If $G = GL_n$, then $\approx /W$ represents the spectrum of a matrix, and for each spectrum, there is a unique conjugacy class of regular matrices.

Exercise 15.8. *Define a canonical symplectic structure on $\approx$ so that π becomes an integrable system.*

This integrable system is related to the open Toda integrable system, though we will not discuss it here.

15.2 Construction

We will present the construction for the $N = 0$ case, and there is a relatively straightforward way to generalize this.

We aim to construct an algebraic variety, which involves constructing an algebra of functions on this variety. We define $\mathcal{A}_C = H_*(\text{something})$. It is natural to ask: why is this an algebra? Typically, homology does not have an algebra structure, but in some cases, it does. For example, if we take the homology of a Lie group, multiplication on the Lie group induces multiplication in homology, and this is analogous to what happens here.

To be more precise, we need to introduce the Grassmannian. Let $K = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[\![t]\!]$. Then the Grassmannian is

$$\mathrm{GR}_G := G(K)/G(\mathcal{O}).$$

This is defined as a set, but to define it properly in algebraic geometry, we should describe its functor of points to specify what it means to map an affine scheme into it (which we will not do). However, the claim is that this is an **ind-scheme**, and in fact, an **ind-project**, which is the direct limit of projective algebraic varieties. This is slightly imprecise because it leads to G being semisimple, while we desire it to be reductive. However, this complication can be ignored for simplicity.

Another way to write this is

$$\mathrm{GR}_G = \bigsqcup_{\lambda} \mathrm{GR}_G^{\lambda},$$

where λ represents the dominant coweights, and GR_G^{λ} are the $G(\mathcal{O})$-orbits. Then $\overline{\mathrm{GR}}_G^{\lambda}$ is a projective variety, and $\mathcal{A}_C = H_*^{G\mathcal{O}}(\mathrm{GR}_G) = H_*(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$. This formulation clarifies that $\mathcal{A}_C$ indeed has an algebra structure. However, the key result is as follows:

Lemma 15.9. *In this case, $\mathcal{A}_C$ is commutative.*

For general N, one can define a similar space $R_{G,N} \to \mathrm{GR}_G$, where $R_{G,N} = \{g \in G(K)/G(\mathcal{O}), n \in N(\mathcal{O}) \mid g^{-1}(n) \in N(\mathcal{O})\}$. Then

$$\mathcal{A}_C = H_{*,\mathrm{BM}}^{G(\mathcal{O})}(R_{G,N}),$$

is a commutative algebra, where BM refers to the Borel-Moore homology. This space is infinite-dimensional, so we need to modify the construction slightly, but this is not difficult.

Physics tells us that the functions on the Coulomb branch are given by monopole operators, and it is not hard to verify that these monopole operators correspond exactly to the Borel-Moore homology classes.

Now, let's discuss the non-commutative deformation. We have

$$H_{*,\mathrm{BM}}^{G(\mathcal{O})\rtimes\mathbb{C}^*}(R_{G,N}),$$

where a new loop rotation by $\mathbb{C}^*$ is introduced. For instance, when $N = 0$, then $R_{G,N} = G(K)/G(\mathcal{O})$ and $\mathbb{C}^*$ rotates t. We stated that it was a lemma that the homology algebra is commutative, but the proof no longer holds when this rotation is added, as everything becomes a module. However, the loop relation becomes an algebra over $H^*_{\mathbb{C}^*}(\mathrm{pt}) = \mathbb{C}[\hbar]$.

The integrable system is

$$H_{*,\mathrm{BM}}^{G(\mathcal{O})}(R_{G,N}) \supset H^*_{G(\mathcal{O})}(\mathrm{pt}) = H^*_G(\mathrm{pt}) = \mathbb{C}[\approx/W].$$

Exercise 15.10. *Show that this Poisson structure commutes.*

15.3 Generalization to 4d $\mathcal{N} = 2$ Gauge Theories

Previously, in 3d, we worked over 3-manifolds such as $\mathbb{R}^3$. Now, the claim is that the same data defines a 4d gauge theory. If we look at the physics literature, we find that in order for a physical theory to exist, there must be some very strong condition on $\mathcal{N}$. For example, if G is a torus, only the 0 representation will work. We will ignore this condition here, but the point is that if we wish to define certain analytic objects—such as the hyperkähler metric—requiring specific conversions, we need the physical theory to exist. However, if we are only concerned with the algebra, we do not need the physical theory to exist.

Formally, we will work on $\mathbb{R}^3 \times S^1$, and $\mathcal{M}_C$ is a "hyperkähler manifold" (the quotations are included because it can sometimes be singular). In the 3d story, we only discussed this in one complex structure because we considered a trivial hyperkähler manifold where it looks the same in all complex structures.

The hyperkähler manifolds corresponding to 4d gauge theories will differ depending on the complex structure. For example, a generic complex structure will correspond to an affine algebraic variety, and we have $\pi^{\mathrm{gen}} : \mathcal{M}_C^{\mathrm{gen}} \to T/W$. However, some special complex structures will correspond to projective varieties, and we have $\pi^{\mathrm{sp}} : \mathcal{M}_C^{\mathrm{sp}} \to \approx/W$, where the distinction lies in the fact that the second map is a projective morphism. A generic fiber of π^{gen} is isomorphic to $(\mathbb{C}^*)^{\mathrm{rank}\ G}$, while the generic fiber of π^{sp} is an abelian variety that should vary when we vary the base point.

The difference between an affine algebraic torus and an algebraic torus in the sense of abelian varieties is that all affine algebraic tori are isomorphic, but not all abelian varieties are isomorphic. Therefore, the question is how to properly define these objects.

For $\mathcal{M}_C^{\text{gen}} = \text{Spec}K^{G(\mathcal{O})}(R_{G,N})$, we are doing exactly the same thing as before, but replacing homology with K-theory. For $\mathcal{M}_C^{\text{sp}}$, there is currently no definition. This space is expected to have a projective homogeneous coordinate ring because the fibers are supposed to come with a canonical projective embedding, which leads us to the open problem:

Problem 15.11. *The definition for $\mathcal{M}_C^{sp}$ should be formulated in the "same" way as in 3d, but replacing G by G_{aff}, the affine Kac-Moody algebra.*

This approach should provide a way to define many integrable systems of Hitchin type.

16 Paul Wiegmann: Peierls Phenomenon via Bethe Ansatz

Abstract

In the 1930s Rudolf Peierls argued that the one-dimensional electrons interacting with phonons undergo an instability, leading to the formation of a periodic structure known as an electronic crystal. Peierls's instability stands in a short list of major phenomena of condensed matter physics.

From a mathematical perspective, a comprehensive solution to the Peierls problem was given in papers by Igor Krichever and co-authored by Natasha Kirova, Sergei Brazovski, and Igor Dzyaloshinsky In the early 80's. It was found that electronic crystals are periodic solutions of soliton equations, falling within the framework of Krichever-Novikov's theory of finite-gap potentials.

The Peierls phenomenon also emerges as a limiting case of models of interacting fermions, such as Gross-Neveu models with a large rank symmetry group when the rank of the group tends to infinity. These models are solvable by the Bethe Ansatz for finite rank groups. The talk presents the result of a recent paper co-authored by Konstantin Zarembo, Valdemar Melin, and Yoko Sekiguchi, where Krichever's finite-gaps solutions of soliton equations were obtained as a singular large rank limit of the Bethe Ansatz solution of models with Lie group symmetry.

Contents

16.1 Introduction

This talk focuses on Krichever's works on the Peierls model. He published 3 major works:

- Spin states in the Peierls model, and finite-band potentials
- Discrete Peierls models with exact solutions
- Sound and charge-density wave in the discrete Peierls model

16.2 The Peierls Phenomenon

A one-dimensional crystal can be considered as a chain of ions (or atoms) that undergo slight vibrations. These vibrations cause the atoms to oscillate back and forth, but the displacement is much smaller than the distance between adjacent atoms. This atomic movement gives rise to what we refer to as "sound" within the crystal.

Now, let us introduce electrons into this crystal. For simplicity, assume that the number of electrons is equal to the number of ions. Each ion creates a potential well, attracting the electrons towards it. This can be described in the context of Schrödinger's equation, where the electrons fall into these potential wells but are also capable of vibrating in sync with the ions.

In an idealized scenario, the ions would form a periodic lattice with minimal vibrations. When electrons are introduced, they would settle on top of each ion, resulting in a periodic lattice formed by both the ions and electrons. This configuration characterizes an insulator.

However, the Peierls phenomenon challenges this idealized picture. It demonstrates that the above configuration is unstable and possesses higher energy than an alternative configuration. In the more stable configuration, the ions rearrange themselves. By labeling the ions as even and odd, the odd-numbered ions may shift slightly to the right, while the even-numbered ions may shift slightly to the left. This displacement of ions drags the electrons along, effectively doubling the period of the lattice. This instability is known as **Peierls instability**.

The energy in this new configuration, resulting from the interaction between electrons and the potential wells, is lower because the potential wells can be distorted or displaced. The configuration with a doubled period is more stable than the homogeneous, single-period configuration.

To understand this in more detail, consider the shift in position of an electron at location x. The amount of movement to the right is termed a distortion, denoted as $U(x)$. This distortion is periodic, alternating between positive and negative values for different ions, thus creating a periodic wave. This wave-like distortion is central to the Peierls phenomenon.

The situation becomes more complex if the number of electrons is not exactly equal to the number of ions, but instead is a fractional ratio (e.g., $\frac{1}{3}, \frac{2}{5}$). This results in a more intricate structure. Nevertheless, the main takeaway is that the simple, naive configuration where the lattice period matches the ion period is unstable.

The phenomenon is counterintuitive: why would the electrons cause the ions to shift and double the period? The explanation lies in the Pauli exclusion principle and the effects of electronic statistics, which manifest in this macroscopic effect.

In some cases, the period may not merely double; it could result in multiple periods, leading to a more complex structure. Interestingly, this can be explained through arithmetic. When dealing with large numbers, such as the number of electrons and ions (both on the order of 10^6), dividing one by the other yields a ratio (a rational number) that determines the periods of the lattice. This provides a simple way to explain the phenomenon, although the underlying details can become quite complicated.

16.3 History

The Peierls phenomenon was first introduced in the 1930s, although it was not formally published until 1954. It explains that a one-dimensional, equally spaced chain with one electron per ion is inherently unstable. This discovery gave rise to the field of **electronic crystals**, in which electrons themselves form periodic structures.

To illustrate, consider a simple model where black dots represent ions, and the line represents the distortion of the electron positions. In theory, no distortion should occur, but in practice, the electrons oscillate back and forth, creating a periodic wave.

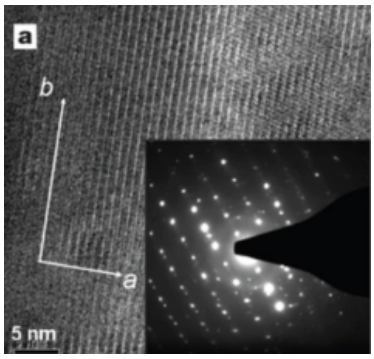
When electrons move within this self-induced periodic potential, the result is a spectrum with both bands and gaps—regions where certain energies are either forbidden or allowed. This distortion opens a gap in the electronic spectrum, which is a characteristic feature of the Peierls phenomenon. If we use optical techniques such as spectroscopy to observe this gap, it confirms that the electrons have indeed formed a periodic structure.

Although Peierls predicted this phenomenon in the 1930s, it was not experimentally confirmed until 1946 by W. Little of Stanford University.

This provides a clear demonstration of what we refer to as an electronic crystal - a striking example of theory meeting experiment.

16.4 Peierls Problem: Discrete Version

Consider the Schrödinger equation: $c_n\psi_n + c_{n-1}\psi_{n-1} = \epsilon_n\psi_n$. Find the spectrum as a functional of $C = \{c_1, \ldots\}$: $\epsilon[C]$, and then compute the energy by



summing over all eigenvalues below μ:

$$E[C] = \sum_{\epsilon<\mu} \epsilon[C] + \sum_n c_n^2$$

.

Problem 16.1 (Peierls Problem, Discrete Version)**.** *Find C that minimizes the energy:*

$$\min_C E[C]$$

Let's break down the problem in a more mathematical way: Imagine an electron sitting on a lattice, where each lattice site is labeled by n ($n = 1, 2, 3, 4, 5$, etc.).

We have the Schrödinger equation:

$$c_n\psi_n + c_{n-1}\psi_{n-1} = \epsilon_n\psi_n$$

The wave function of these electrons is influenced by a coefficient c, known as the hopping amplitude. Next, we find the spectrum as a functional of $C = \{c_1, \ldots\}$: $\epsilon[C]$. The equation is called a discrete Schrödinger equation or a linear difference equation, where we need to determine how the wave function (ψ) depends on n and what the spectrum is.

In this equation, the c_n tells us where the ions are located, and since they can move, the central question becomes:

Problem 16.2 (Peierls Problem)**.** *Find C that minimizes the energy:*

$$\min_C E[C]$$

To solve this, we need to sum over all eigenvalues below μ:

$$E[C] = \sum_{\epsilon<\mu} \epsilon[C] + \sum_n c_n^2$$

and then find the value of C that gives the minimum energy.

In summary, the problem is to assume an arbitrary C, solve the equation, compute the spectrum, find the total energy, and then minimize that energy with respect to C. The value of C that gives the minimum energy is the solution we're looking for. This is the essence of the problem, which is quite simple in its statement.

Proposition 16.3 (Krichever)**.** *The extrema are given by the finite-gap solutions of the Toda chain. The minimum is given by the one-gap solution.*

The solution to this problem leads to a finite gap in the spectrum, which is given by the periodic solutions of the Toda chain. Although the specific equation isn't written here, finding a periodic solution for the Toda chain will give you a C that is periodic in n. This periodic solution corresponds to a local extremum of the energy function.

The Toda chain has many periodic solutions—one gap, two gaps, three gaps, and so on. Each of these corresponds to a local extremum, but only one represents the ground state, which is the solution with the lowest energy. Typically, this lowest energy solution has one gap.

The electron density will determine the period of this solution. The period is determined by the ratio of certain factors. If this ratio is 1 over an odd number, you get a one-gap solution. If the ratio is more complex, like a rational number, you could have a multi-periodic solution.

The surprising part of this problem is that such a simple problem gives rise to these periodic solutions of integral equations.

16.4.1 Peierls Problem and the Lax Operator

The key reason is because this Schrödinger equation is actually the Lax operator. Krichever showed that the Schrödinger equation with a variable hopping

$$L\psi = c_n\psi_{n+1} + c_{n-1}\psi_{n-1} = \epsilon\psi_n$$

is identified with the Lax operator, and then proved that the extrema of energy are identified with finite-gap periodic solutions.

So, when you approach problems like solving the Toda chain or the KdV equation, you might wonder why we focus on finding periodic solutions when there are many other possible solutions that aren't periodic. The reason is that these periodic solutions form the extrema of a functional provided by the electrons themselves. Essentially, there's a natural functional whose extrema correspond to these periodic, finite-gap solutions.

16.5 Peierls Problem: Continuous Version

For the purposes of this talk, it is more convenient to work with a continuous version of the problem, even though the discrete version can also be solved.

Consider the $(1+1)$ Dirac equation:

$$\begin{cases} -i(\partial_x - \Delta(x))\psi_+ = \epsilon\psi_- \\ -i(\partial_x + \Delta(x))\psi_- = \epsilon\psi_+ \end{cases}$$

where $\Delta(x)$ is some potential.

We can reformulate this simple differential equation in terms of the Hamiltonians, specifically the Dirac Hamiltonian:

$$H = \overline{\psi}\sigma_1(i\partial_x + \sigma_3\Delta)\psi + \frac{1}{2\lambda}\Delta^2$$
$$E = \mathrm{Tr}_{H<\mu}(H).$$

The goal is to find Δ such that the energy, which is the trace (or sum of all eigenvalues) of the operator, is minimized. The solution depends on the system parameters, particularly the chemical potential μ, which must be determined from the ratio of the number of particles in the system to the number of available states.

I'll explain how this works. You have two key quantities: the total number of particles (electrons) and the number of lattice sites. The ratio of these two numbers, known as the filling factor, plays a crucial role in determining the solution.

Interestingly, the solution turns out to be a special case of the modified Korteweg - de Vries (mKdV) equation. The mKdV equation has many periodic solutions, and in this case, we fix the period based on the system parameters.

Proposition 16.4. *The minimum energy is achieved if Δ is a periodic solution of the mKdV equation:*

$$\Delta_t - 6\Delta^2\Delta_x + \Delta_{xxx} = 0, \qquad \Delta = \textit{function}(x - ct).$$

16.5.1 Cnoidal Wave

Let's discuss the KdV equation and its periodic solutions. The KdV equation is closely related to another equation through a transformation. The solution that minimizes the energy and "makes the electrons happiest" is a well-known solution of the KdV equation called the **cnoidal wave.**

The cnoidal wave is expressed using hyperbolic sine functions. The wave's period depends on a parameter k, which is related to the number of particles in the system or the chemical potential μ. While the relationship between k and the number of particles might seem complex, it is essential for understanding the solution.

The cnoidal wave is the most familiar solution of the KdV equation, and its period is determined by the number of particles in the system. By providing the number of electrons and the number of lattice sites (referred to as the filling factor), we can calculate k, which in turn gives us the period of the wave.

Equation	Cnoidal Wave Properties
mKdV:	$\Delta_t - 6\Delta^2\Delta_x + \Delta_{xxx} = 0$
Miura:	$q^{\pm} = \Delta^2 + \Delta_x$
KdV:	$q_t - 6qq_x + q_{xxx} = 0$
Cnoidal wave:	$\Delta(x) = \Delta_0 k^{1/2}\,\mathrm{sn}(x\|k)$
Number of Particles = Period:	$N/N_0 = 2k^{1/2}K(k)$
Gap:	$\Delta_0 = \Lambda e^{-\pi/\lambda}$

Table 2: Equations and properties of cnoidal waves

16.5.2 Spectral Curve

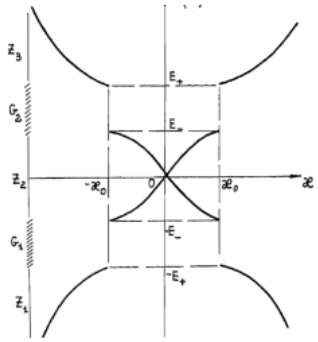
Let's use the cnoidal wave:

$$-i(\partial_x \pm \Delta(x))\psi_{\pm} = \epsilon(p)\psi_{\pm}$$

which makes electrons "happy." The energy of this periodic wave depends on momentum, and momentum determines how many states or eigenstates exist per unit of energy:

$$\frac{dN(\epsilon)}{N_0} = dp$$

From this, we can derive a spectral curve.



In this case, the spectral curve is an elliptic curve with the following properties:

$$dp = \frac{|\epsilon^2 - S|}{\sqrt{R}}\,d\epsilon$$

$$R(\epsilon) = (\epsilon^2 - E_+^2)(\epsilon^2 - E_-^2)$$

$$2S = -\Delta_0^2 + E_+^2 + E_-^2$$

The edges of the spectrum are given by:

$$E_{\pm} = \frac{\Delta_0}{2}\left(k^{-1/2} \pm k^{1/2}\right)$$

16.6 Quantization of the Spectral Curve

Problem 16.5 (Melin, Sekiguchi, Wiegmann, Zarembo)**.** *How do we obtain periodic solutions of classical integrable equations from quantum integrable models?*

The KdV equation and its periodic solutions have been widely studied. At the same time, we know that nonlinear integrable equations can be quantized. In lucky cases, their solutions can be obtained using methods from quantum field theory, specifically through Bethe's ansatz.

The key question is whether the simple periodic solutions of classical equations can be derived as a limit of the quantum Bethe ansatz solution for the corresponding quantum problem. The bridge between the quantum and classical worlds is provided by the semi-classical limit, where Planck's constant approaches zero. In this limit, some underlying mathematical structures in quantum integrable systems should reduce to the periodic solutions of the KdV equation. This is a specialized but interesting question.

Integrable equations have many solutions, with periodic solutions being just one type. Solitons are another type of solution that corresponds to particles in the quantum world. There is a semi-classical procedure where one considers quantum solitons and then lets Planck's constant approach zero to obtain classical solitons with periodic solutions. This story is quite complex, and that's what we will focus on.

Another way to think about it is in terms of algebraic curves. For the Peierls problem or the periodic solutions of the KdV equation, we get an algebraic curve known as a spectral curve, such as an elliptic curve. In quantum problems, however, we don't have such curves. Instead, there are structures that are not yet fully identified, but hints suggest that they might be related to algebraic curves in the limit.

These structures are not the same as Riemann surfaces, but they appear to become Riemann surfaces when Planck's constant approaches zero. These might be deformations of Riemann surfaces, and in the limit, they become algebraic curves. There could be interesting structures underlying them, which might include additional parameters or constants.

We might also explore a correspondence with spin chains, such as the Heisenberg spin chain. For spin-1/2 quantum problems, we know quite a bit about the solutions. If the spin is very large, the problem becomes classical and is solved by integrable equations that are classically integrable. There is a correspondence between these classical and quantum problems, although it is not fully understood yet.

16.6.1 Quantum Version: Gross-Neveu Model

What does quantization means in this context? In the Peierls problem, we treat electrons as quantum objects, meaning they obey the Pauli exclusion principle

and cannot occupy the same state as each other. We have the equation

$$H = \overline{\psi}\sigma_2(i\partial_x + \sigma_3\Delta)\psi + \frac{1}{2\lambda}\Delta^2.$$

However, the parameter Δ, which describes how far the ions move left and right, is treated classically. Specifically, we fix Δ at some value, solve the problem to find the energy, and then minimize the energy with respect to Δ. In this setup, Δ acts as a parameter, and once it is fixed, we can find the energy and solve for Δ, leading to a periodic solution of the KdV equation.

Now, imagine if Δ were not fixed but could also vary dynamically - it would then be considered a quantum field with its own dynamics. This is a more complex scenario compared to the adiabatic approximation, where Δ is determined by the extremum of $\mathrm{Tr}(H)$. Δ was assumed to be nearly constant while electrons moved very quickly. In this simpler case, we can find Δ by minimizing the trace of the Hamiltonian, but if Δ is fluctuating, things become more complicated.

To handle this, we can integrate over Δ to account for its fluctuations. This leads to interacting fields and results in a more complex quantum problem. This model is known as the Gross-Neveu model, which has been studied in quantum field theory since the 1970s. It describes interacting fermions in one dimension and is a challenging problem that bridges the gap between classical and quantum theories.

In the quantum version, Δ itself is a quantum field, and we have the Gross-Neveu model

$$H = \overline{\psi}\sigma_2(i\partial_x\Delta)\psi + \frac{\lambda}{2}\left(\overline{\psi}\psi\right)^2.$$

We consider large N as a semiclassical parameter: $\psi \to (\psi_1, \ldots, \psi_N)$ with

$$H = \sum_{1\le k\le N} \overline{\psi}_k\sigma_2(i\partial_x + \sigma_3\Delta)\psi_k + \frac{\lambda}{2}\left(\sum_{1\le k\le N} \overline{\psi}_k\psi_k\right)^2.$$

We aim to recover the Peierls model in the limit large of a large N.

To make the transition between the classical and quantum descriptions, a useful trick is to introduce multiple species of electrons. Instead of just one electron, consider many electrons, labeled by n. In the limit as $n \to \infty$, this large n acts as a semi-classical parameter, and the problem becomes easier to handle. In this large n limit, the quantum problem simplifies to resemble a classical problem.

We solve this problem with many species (large n), which allows us to approximate the problem in the semi-classical limit. By expanding in terms of $\frac{1}{n}$, we can recover the Peierls problem in this limit. Eventually, this leads us to periodic solutions of the KdV equation, which are related to elliptic curves or tori in classical systems. Although quantum problems don't have tori, they can exhibit structures that approach tori in the semi-classical limit.

16.6.2 Lie Group

We have

$$H = \sum_{1\le k\le N} \overline{\psi_k}\sigma_2(i\partial_x + \sigma_3\Delta)\psi_k + \frac{\lambda}{2}\left(\sum_{1\le k\le N} \overline{\psi}_k\psi_k\right)^2.$$

Before solving the problem, we need to identify its symmetry. The problem is invariant under the $O(2N)$ Lie group, where N represents the number of species, and the 2 refers to the plus/minus symmetry. So, the relevant symmetry group here is $O(2N)$, which governs the interactions and overall structure of the problem. Thus, we can say that the integrable model is controlled by its global symmetry $O(2N)$.

In fact, we don't necessarily need to write down the Hamiltonian explicitly. We can simply state that the fermions interact according to a global $O(2N)$ symmetry. Once we declare that the problem is governed by this global symmetry, the rest will follow from that declaration.

16.6.3 Mass Spectrum

These fermions form particles that, due to their interactions, bind together to form mesons, which then bind together to form baryons, which bind to form nuclei. There are many states, and these states are remarkably organized according to the representations of the corresponding Lie algebra. For example, with $O(2N)$, we obtain the Dynkin diagram D_n for $n > 3$. The particle content is all of the fundamental representations.

The mass spectrum of the model governed by this symmetry group has

- n-th tensor:
$$m_n = m\sin\frac{\pi n}{2n-2}$$
- spinors:
$$m_s = m_{\overline{s}} = \frac{m}{2\sin\frac{\pi}{2n-2}}$$

In this framework, there are vector particles, antisymmetric tensor particles, and others, each corresponding to a fundamental representation. Every circle on the Dynkin diagram represents a new particle. Among these particles, there are two types of spinors: spinor and anti-spinor (or conjugate spinors). Their masses differ, and in the large N limit, how these masses depend on N becomes very important.

Proposition 16.6 (Ogiovetski, Reshetikhin, Wiegmann)**.** *The scattering matrices, the mass spectrum, and the Bethe Ansatz are known for all simple Lie groups.*

16.6.4 Quantum Integrable Systems: Scattering Matrices and TBA

When we talk about a quantum integrable system, we mean that the scattering process is factorized. Specifically, the scattering matrix for multiple particles can be reduced to the scattering matrices of pairs of particles. If we know how two particles scatter, we can construct the scattering matrix for any number of particles. This is known as the **factorized theory of scattering**.

The scattering matrix depends on a parameter called rapidity, which is related to the difference in the rapidity between two particles, say particle A and particle B. For example, particle A could be a vector particle, and particle B could be a spinor. Rapidity is defined in terms of momentum, parameterized by θ. As θ approaches infinity, the formula for rapidity emerges.

Once we have the S-matrix, we can define an object called the scattering phase or scattering phase shift. This is an operator, essentially a matrix that describes how particle A scatters with particle B as a function of θ.

From the S-matrix, we can derive integral equations—one for momentum and another for energy. While we won't go into the details of where these equations come from, they form the foundation of an integrable quantum system. If you provide me with the scattering matrix, we can use these integral equations to determine the momentum and energy of a system with many particles in a field theory.

The scattering matrix is factorized into a product of two-particle scattering

$$S_{ab}(\theta), \theta = \theta_a - \theta_b, p_a(\theta \to \infty) \sim m_a \sinh\theta.$$

Once we know this, we can can write the thermodynamic Bethe-Ansatz equations for the "spectral curve" $K_{ab} = \dfrac{1}{2\pi i}\dfrac{d}{d\theta}\log S_{ab}$:

$$\int K_{ab}(\theta_a - \theta_b)\, dp_b = m_a \sinh\theta_a \qquad \int K_{ab}(\theta_a - \theta_b)\epsilon_b = \mu_a - m_a \cosh\theta_a$$

where we sum over the particle content (along the Dynkin diagram). In fact, K and S are made out of Cartan matrices. Given a Lie algebra, we can look up the Cartan matrix, use it to form the S-matrix, derive K, and then use K to construct these integral equations.

Solving these equations gives us the momentum and energy of the system. By summing them up, we get the total momentum and energy of the state. The energy depends on the momentum, and this relationship gives us the spectral curve, which tells us how the energy depends on the momentum for elementary particles. This process ultimately allows us to determine the spectrum of the system.

16.6.5 Scattering Matrix D_N (In Momentum Space)

For system D, we have the following:

$$\hat{K}_{ab} = \begin{cases} \delta_{ab} + \frac{1}{2} e^{\frac{|k|}{2N-2}} \frac{\sinh\left(\frac{|a-b|-N+1}{2N-2}k\right) - \sinh\left(\frac{a+b-N+1}{2N-2}k\right)}{\sinh\left(\frac{k}{2N-2}\right)\cosh\left(\frac{k}{2}\right)}, & a,b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \frac{\sinh\left(\frac{|N-1-b|-N+1}{2N-2}k\right) - \sinh\left(\frac{b|k|}{2N-2}\right)}{\sinh\left(\frac{k}{2N-2}\right)\cosh\left(\frac{k}{2}\right)}, & a > N-2, b \leq N-2 \\ \delta_{ab} + \frac{1}{4} e^{\frac{|k|}{2N-2}} \frac{\sinh\left(\frac{|N-1-a|-N+1}{2N-2}k\right) - \sinh\left(\frac{a|k|}{2N-2}\right)}{\sinh\left(\frac{k}{2N-2}\right)\cosh\left(\frac{k}{2}\right)}, & a \leq N-2, b > N-2 \\ \delta_{ab} - \frac{1}{4} e^{\frac{|k|}{2N-2}} \frac{\sinh\left(\frac{k}{2}\right)}{\sinh\left(\frac{k}{2N-2}\right)\cosh\left(\frac{k}{2}\right)} - \frac{1}{4} \frac{(-1)^{a+b} e^{|k|/2N-2}}{\cosh\left(\frac{k}{2N-2}\right)}, & a,b > N-2 \end{cases}$$

We don't need to analyze this in depth, and we presented it only to show how concrete and precise everything is. There's a shorter way to write this complex formula using the Cartan matrix, which would simplify it to one line. The key point is that everything is well-established and known.

16.6.6 Ground State

Now, to find the energy and momentum of a particular state, we first need to identify the state. We'll focus on the ground state, which is the state with the lowest energy. This ground state is primarily made up of spinors, while other particles are formed from bound states of these spinors. This isn't a trivial result - it may not be immediately obvious, but spinors are like elementary building blocks, similar to quarks, from which we can build all other particles through tensor products.

When focusing on the ground state, the complex equations simplify. we only need to consider how spinors scatter with other spinors, how they scatter with anti-spinors, and how tensors interact with spinors. The TBA are reduced to

$$\int_{-B}^{B} (K_{ss} + K_{s\bar{s}})(\theta - \theta')\, dp_s = m_s \sinh\theta, \quad 2\int_{-B}^{B} K_{as}(\theta - \theta')\, dp_s = m_a \cosh\theta, \quad \frac{N}{N_0} = \int_{-B}^{B} dp_s$$

Given this, we assume there is one periodic solution, meaning the momentum spectrum is confined to a single interval, from some unknown value $-B$ to B. This single interval is reminiscent of what you might find in a hyperelliptic curve, where there would be multiple intervals or cuts, but here, we assume only one cut.

To solve these equations, we know K, but we don't know B. We find B by imposing a condition that the system contains a certain number of electrons. Once we determine B, we can find the momentum as a function of B and then integrate to get the relationship between momentum and energy. It turns out that K are not monstrous at all, and are given by:

$$K_{ss} + K_{s\bar{s}} = \frac{\tanh\frac{|k|}{2}}{2\left(1 - e^{-\frac{|k|}{N-1}}\right)}, \quad K_{as} = -\frac{e^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}}$$

16.6.7 Singular Large N Limit

The K operators and kernels are not as complex as they might seem at first glance. They are familiar kernels, often referred to as singular kernels. In the large N limit (where N is the rank of the Lie algebra), some terms can be ignored, simplifying the Fourier transform and the resulting kernels. This reduction allows the complex equations to simplify into singular integral equations, which are easier to manage:

$$K_{ss}+K_{s\bar{s}} = \frac{\tanh\frac{k}{2}}{2\left(1-e^{-\frac{|k|}{N-1}}\right)} \underset{N\to\infty}{\to} -\frac{N}{\pi^2}\log\coth\frac{\theta}{2}, \quad K_{as} = -\frac{e^{\frac{\pi|k|}{2N-2}}}{2\cosh\frac{\pi k}{2}} \underset{N\to\infty}{\to} \frac{1}{2\pi\cosh\theta}$$

Analyzing these simplified equations, Wiegmann discovered that they are closely related to the same problem that gives rise to the spectral curve of the KdV equation, which describes periodic solutions. Although the connection appears when solving the equations and taking limits, it's important to note that these monstrous equations reduce to something much simpler in the large N limit. Specifically, they become a Riemann-Hilbert problem for an elliptic curve, which is also the solution to KdV. More specifically, they degenerate to the Riemann-Hilbert problem

$$\int_{-B}^{B}\ln\coth\frac{\theta-\theta'}{2}\frac{dp_s}{\pi} = m\cosh\theta, \quad \int_{-B}^{B}\ln\coth\frac{\theta-\theta'}{2}\epsilon(\theta')\frac{d\theta'}{\pi} = m\cosh\theta-\frac{\mu}{2}$$

This elliptic spectral curve, associated with soliton waves, is expressed in terms of elliptic functions, which define the edges of the spectrum:

$$dp = \frac{\epsilon^2+\overline{\Delta}^2-E_+^2-E_-^2}{\sqrt{(\epsilon^2-E_+^2)(\epsilon^2-E_-^2)}}d\epsilon, \quad \frac{N}{N_0} = 2k^{1/2}K(k), \quad E_\pm = \frac{\Delta_0}{2}(k^{-1/2}\pm k^{1/2})$$

These equations can also be reformulated as a boundary value problem for an analytic function in the upper and lower half-planes, connected by specific jump conditions on the real axis. These jump conditions link the real and imaginary parts of a holomorphic function, which can be expressed as integral equations.

However, if N is not infinite, we cannot do this. But as N approaches infinity, the problem simplifies, reducing to classical integral equations, transitioning from a quantum to a classical context as certain terms become negligible.

16.6.8 Comments

We conclude with two comments. First, we explained earlier that if we are given a series D_n algebra (such as $O(2n)$), then as we send the rank $\to\infty$, we obtain a classical KdV hierarchy. Similarly, if we do this for A_n, where we also send the rank to infinity, and this leads to the nonlinear Schrödinger equation (NLS). However, for B_n and C_n, we don't know what the result will

be. There's an interesting relationship between different algebras and various integrable hierarchies. For some reason, KdV corresponds to series D - we don't know why. Similarly, the nonlinear Schrödinger equation corresponds to A_n, again at large ranks. We don't fully understand what underpins this correspondence, and we also don't know what happens for B_n and C_n.

Another question that we don't have the answer to, involves the algebraic-geometric construct of the curve, which we use to solve classical integrable equations. This structure is important from both algebraic-geometric and topological perspectives. Apparently, there is also a quantum version of this construction, and it's quite explicit. We don't fully understand what we're doing with it, but we know how to work with it. What this construction actually means conceptually, we aren't sure. But from a practical or engineering standpoint, we can take the large N limit and see what happens.

17 Andrei Marshakov: Krichever Tau-Function: Basics and Perspectives

Abstract

I plan to start with the definition of quasiclassical tau-function, introduced by Igor Krichever in 1992, formulate its main properties with some simple proofs, and discuss certain particular cases, which include the Seiberg-Witten prepotentials, matrix models etc. Then I am going to turn to certain modern developments, related with this object, which include the relation with instanton partition functions, isomonodromic tau-dunctions and even some unexpected relations with other famous relations in mathematical physics.

Contents

17.1 Plan

We present two stories: the old story and the new story. Let's start with the old story.

This story originates from a paper by Krichever: *The τ-function of the universal Whitham hierarchy, matrix models, and topological field theories* from 1994. There are two main "issues" from this paper: the paper defines everything though $\mathcal{F} = \log \tau$ instead of τ, and the terminology has changed since 2007 and since has been standarized.

We will state the definition of the Krichever τ-function and then look at some applications:

- Matrix models and Topological strings;
- Seiberg-Witten theory and integrable systems;

From this second bullet point, we will reach some beautiful and essential formulas, such as:

- Residue formula;
- WDVV equations.

Finally, we will discuss the "new story", which deals with topics such as:

- Nekrasov functions, 2d conformal theories, and isomonodromic deformations;
- 2d gravity and Verlinde formula.

17.2 Notation

Consider the topology of a compact oriented Riemann surface with genus g. Then $\dim H_1(\Sigma_g) = 2g$, with the (symplectic) intersection form given by $A_\alpha \circ B_\beta = \delta_{\alpha\beta}$.

The dual basis in $H^1(\Sigma_g)$ consists of holomorphic first-kind Abelian differentials $\overline{\partial}(d\omega_\alpha) = 0$, normalized to the A-cycles:

$$\oint_{A_\beta} \mathrm{d}\omega_\alpha = \delta_{\alpha\beta}.$$

The period matrix is defined as:

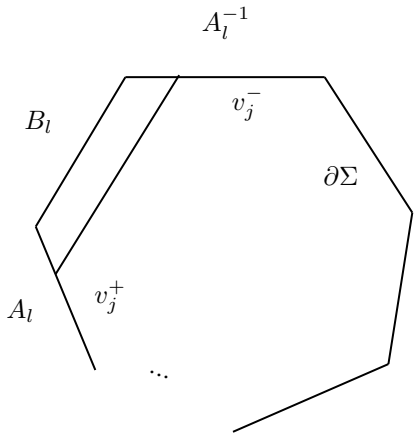
$$\oint_{B_\alpha} \mathrm{d}\omega_\beta = T_{\alpha\beta}.$$

17.3 Riemann Bilinear Relations

The period matrix is symmetric, as shown by the following calculation:

$$\begin{aligned} 0 &= \oint_{\Sigma} d\omega_\beta \wedge d\omega_\gamma \\ &= \oint_{\partial\Sigma} \omega_\beta d\omega_\gamma \\ &= \sum_{\alpha} \left(\oint_{A_\alpha} d\omega_\beta \oint_{B_\alpha} d\omega_\gamma - \oint_{A_\alpha} d\omega_\gamma \oint_{B_\alpha} d\omega_\beta \right) \\ &= T_{\beta\gamma} - T_{\gamma\beta}. \end{aligned}$$

Proof. We present a proof sketch via Stokes' theorem on the cut Riemann surface.



Cut a Riemann surface (4g-gon) with boundary $\partial\Sigma$. The boundary values of Abelian integrals $v_\alpha^\pm = \omega_\alpha^\pm$ on the two boundaries of the cut differ by the period integral of the corresponding differential $d\omega_\alpha$ over the dual cycle. □

Similarly, we can write identies for other abelian differentials. In the meromorphic case, for second-kind Abelian differentials,

$$d\Omega_k \overset{P\to P_0}{\sim} \frac{d\zeta}{\zeta^{k+1}} + \ldots, \quad \oint_A d\Omega_k = 0, \quad k \geq 1.$$

For third-kind Abelian differentials $d\Omega_\pm$:

$$d\Omega_0 = d\log\frac{E(P,P^+)}{E(P,P^-)} \sim \frac{d\zeta_\pm}{\zeta^\pm} + \ldots, \quad \oint_A d\Omega_0 = 0.$$

For the first and third-kind Abelian differentials:

$$\begin{aligned} 0 &= \oint_\Sigma d\omega_\beta \wedge d\Omega_0 \\ &= \sum_\alpha \Big(\oint_{A_\alpha} d\omega_\beta \oint_{B_\alpha} d\Omega_0 - \oint_{A_\alpha} d\Omega_0 \oint_{B_\alpha} d\omega_\beta \Big) \\ &\quad + 2\pi i \sum_{P_\pm} \mathrm{res}_{P_\pm} \omega_\beta d\Omega_0 = \oint_{B_\beta} d\Omega_0 - 2\pi i \int_{P^-}^{P^+} d\omega_\beta. \end{aligned}$$

17.4 Krichever Data and Prepotentials

Consider a complex curve Σ_g with a pair of meromorphic differentials (dx, dy), and fixed periods.

The subfamily of curves $\{\Sigma_g\}$ has dimension:

$$(3g-3) - (2g-3) = g.$$

Intuitively, the Krichever data should be an integrable system (back to the Liouville theorem) on the g-dimensional family of Σ_g. One can choose g independent functions (Hamiltonians), while the coordinates on the Jacobian of Σ_g serve as complexified angle variables.

More rigorously,

Definition 17.1. *The **Krichever data** is a g-parametric family of Riemann surfaces Σ, endowed with a generating differential and connection ∇_{mod} on moduli space:*

$$dS \propto y\, dx, \quad \nabla_{mod} dS = \textit{holomorphic},$$

where $y(P) = \oint_P dy$, and $P \in \Sigma$.

Now, we define the prepotential, which is a particular case of the Krichever-tau function.

Definition 17.2. *The **prepotential** is given by*

$$a = \frac{1}{2\pi i} \oint_A dS, \quad a_D = \oint_B dS := \frac{\partial \mathcal{F}}{\partial a},$$

where A and B are dual cycles in $H_1(\Sigma)$.

The prepotential is defined locally on the moduli space of Σ.

By the integrability from the Riemann Bilinear relations:

Proposition 17.3.

$$\frac{\partial a_D^\alpha}{\partial a^\beta} = T_{\alpha\beta} = T_{\beta\alpha} = \frac{\partial a_D^\beta}{\partial a^\alpha}.$$

Proof. Let ∇_{mod} be the connection via covariantly constant coordinates on Σ, e.g., $x = 0$ (with potential problems at $dx = 0$). Then:

$$\nabla_{\mathrm{mod}} dS = (\nabla_{\mathrm{mod}} y)\, dx,$$

where $\nabla_{\mathrm{mod}} y$ is defined from the equation of $\Sigma \subset \mathbb{C}^2$.

Using this and the fact that

$$\delta_{\alpha\beta} = \frac{\partial a^\alpha}{\partial a^\beta} = \frac{1}{2\pi i} \oint_{A_\alpha} \frac{\partial dS}{\partial a^\beta},$$

one finds that

$$\frac{\partial dS}{\partial a^\beta} = d\omega_\beta,$$

which is a normalized holomorphic differential.

Then,

$$\frac{\partial a_D^\alpha}{\partial a^\beta} = \oint_{B_\alpha} \frac{\partial dS}{\partial a^\beta} = \oint_{B_\alpha} d\omega_\beta = T_{\alpha\beta}.$$

□

Remark 17.4. *For the second derivatives:*

$$\frac{\partial^2 \mathcal{F}}{\partial a^\alpha \partial a^\beta} = T_{\alpha\beta}.$$

Let's consider an example of prepotential from Seiberg-Witten theory:

Example 17.5. *Consider the curve (Σ, dx, dy), defined by:*

$$\frac{w + \Lambda^{2N}}{w} = P_N(z) = z^N + \sum_{k=0}^{N-2} u_k z^k, \quad dx = \frac{dw}{w}, \quad dy = dz,$$

since obviously:

$$\oint_{(A,B)} dz = 0, \quad \oint_{(A,B)} \frac{dw}{w} \in 2\pi i \mathbb{Z}.$$

From $\nabla_{mod} w = 0$ and $\nabla_{mod} z P_N(z) = \sum_{k=0}^{N-2} \delta u_k z^k$, we have:

$$\nabla_{mod} dS = \nabla_{mod} z \frac{dw}{w} = \sum_{k=0}^{N-2} \delta u_k z^k P_0 N(z) \frac{dw}{w},$$

which is holomorphic on Σ.

Now we can finally define the Krichever τ-function.

To complete the definition using the time variables associated with the second-kind Abelian differentials with singularities at a point P_0, we define

$$t_k = \frac{1}{k}\mathrm{res}_{P_0}\xi^{-k}dS, \quad k > 0,$$

and

$$\frac{\partial \mathcal{F}}{\partial t_k} := \mathrm{res}_{P_0}\xi^k dS, \quad k > 0,$$

where ξ is an inverse local coordinate at P_0, with $\xi(P_0) = \infty$.

The consistency condition for the above is ensured by

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial t_k} = \mathrm{res}_{P_0}(\xi^k d\Omega_n),$$

which is symmetric due to $(\Omega_n)^+ = \xi^n$ for the main singular part at P_0.

Additionally,

$$\frac{\partial^2 \mathcal{F}}{\partial t_n \partial a^\alpha} = \int_{B_\alpha} d\Omega_n = \mathrm{res}_{P_0}\xi^n d\omega_\alpha,$$

which again follows from the Riemann bilinear relations.

Remark 17.6.

- *Definition from RBI;*
- *Can be defined for any set of Abelian differentials* $\{dH_I\} = \{d\omega_\alpha, d\Omega_n, d\Omega_0, \dots\}$ *and corresponding flat coordinates* $\{T_I\} = \{a_\alpha, t_n, t_0, \dots\}$*;*
- *pq-duality:* $dx \leftrightarrow dy$ *is generally a nontrivial subtle point;*
- *Prepotentials:* $\nabla_x^{mod} \leftrightarrow \nabla_y^{mod}$*;*
- *dKP: A nontrivial relation (e.g., a Fourier transform for a matrix integral);*
- *A nontrivial relation for residue formulas;*
- *Starting point for the "topological recursion".*

17.5 Residue Formula

Theorem 17.7.

$$\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = res_{dx=0}\left(\frac{dH_I dH_J dH_K}{dx\, dy}\right)$$

The proof idea is to take one mode derivative of a second-derivative formula.

In the prepotential case,

$$\frac{\partial T_{\alpha\beta}}{\partial a_\gamma} \equiv \partial_\gamma T_{\alpha\beta} = \int_{B_\beta} \partial_\gamma d\omega_\alpha = -\int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha .$$

Further,

$$\partial_\gamma T_{\alpha\beta} = -\int_{\partial\Sigma} \omega_\beta \partial_\gamma d\omega_\alpha = \int_{\partial\Sigma} \partial_\gamma \omega_\beta d\omega_\alpha = \sum \mathrm{res}_{dx=0} \left(\partial_\gamma \omega_\beta d\omega_\alpha\right)$$

since the expression acquires poles at $dx = 0$.

Now we prove it:

Proof. Use expansions where $dx = 0$:

$$\omega_\beta(x) \overset{x\to x_\alpha}{\sim} \omega_{\beta a} + c_{\beta a}\sqrt{x - x_\alpha} + \cdots , \quad d\omega_\beta \overset{x\to x_\alpha}{\sim} \frac{c_{\beta a}}{2\sqrt{x - x_\alpha}} + \cdots$$

For the moduli connection ∇_{mod}, we have:

$$\partial_\gamma \omega_\beta \equiv \partial_\gamma \omega_\beta|_{x=\mathrm{const}} = \left(-\frac{c^a_\beta}{2\sqrt{x - x_a}} \partial_\gamma x_a + \text{regular terms} \right).$$

Then,

$$\begin{aligned} \mathrm{res}\left(\partial_\gamma \omega_\beta d\omega_\alpha\right) &= \sum_a \mathrm{res}\left(\frac{c^a_\beta \partial_\gamma x_a}{2\sqrt{x - x_a}} d\omega_\alpha \right) \\ &= \sum_a \mathrm{res}\left(\frac{d\omega_\beta}{dx} d\omega_\alpha \partial_\gamma x_a \right) \\ &= \sum_a \mathrm{res}\left(\frac{d\omega_\alpha d\omega_\beta d\omega_\gamma}{dx\, dy} \right), \end{aligned}$$

where the last equality similarly follows from the expansions for $y(x)$:

$$y(x) = \begin{cases} y_a\sqrt{x - x_a} + \ldots , & \text{as} \quad x \to x_a, \\ dy = \left\{ \frac{y_a}{2\sqrt{x - x_a}} dx + \ldots , \quad \text{as} \quad x \to x_a. \right. \end{cases}$$

Also, for $d\omega_\gamma = \partial_\gamma dS$, we have:

$$d\omega_\gamma = \left\{ -\frac{y_a \partial_\gamma x_a}{2\sqrt{x - x_a}} dx + \text{regular terms}, \quad \text{as} \quad x \to x_a. \right.$$

□

17.6 Landau-Ginzburg Topological Theories to WDVV

The topological theories defined by a polynomial superpotential (generally of several complex variables) are given by:

$$W(\lambda) = \lambda^N + \sum_{k=0}^{N-2} u_k \lambda^k.$$

The primaries are given by the dKP equation:

$$\varphi_k(\lambda) := \frac{\partial W}{\partial t_k} = \left(\frac{d}{d\lambda} W^{k/N}\right)_+$$

where the flat times are given by:

$$t_k = \frac{1}{k} \mathrm{res}_{P_0} \xi^{-k} dS = -\frac{N}{k(N-k)} \mathrm{res}_\infty \left(W^{1-k/N} d\lambda\right)$$

for $(\Sigma, dx, dy) = (\Sigma_0, dW, d\lambda)$ with $\xi = W(\lambda)^{1/N}$.

The derivatives of the Krichever tau-function are given by:

$$\frac{\partial \mathcal{F}}{\partial t_k} = \mathrm{res}_{P_0} \xi^k dS = \frac{N}{N+k} \mathrm{res}_\infty \left(W^{1+k/N} d\lambda\right).$$

Together with:

$$\mathcal{F}_{ik} = \frac{\partial^2 \mathcal{F}}{\partial t_i \partial t_k} = \mathrm{res}_\infty \left(W^{k/N} \frac{\partial W}{\partial t_i}\right) = \mathrm{res}_\infty \left(W^{k/N} \partial\lambda W_+ i/N\right).$$

And (the Grothendieck residue):

$$\mathcal{F}_{ijk} = -\mathrm{res}_\infty \frac{\partial_\lambda W_+^{i/N} \partial_\lambda W_+^{j/N} \partial_\lambda W_+^{k/N}}{W'} = \mathrm{res}_{W'=0} \frac{\phi_i(\lambda)\phi_j(\lambda)\phi_k(\lambda)}{W'}$$

The Landau-Ginzburg (LG) primaries satisfy the associative algebra (a polynomial ring modulo $W_0(\lambda)$):

$$\varphi_i(\lambda)\varphi_j(\lambda) = \sum_{k=1}^{N-1} C_{ij}^k \varphi_k(\lambda) + R_{ij}(\lambda) W_0(\lambda),$$

and therefore, we have:

$$[C_i, C_j] = 0 \quad \text{for the matrices} \quad C_{ij}^k := C_{ij}^k.$$

for the matrices $\|C_i\|_j^k := C_{ij}^k$. In terms of matrices,

$$\|F_i\|_{jk} := \mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_k},$$

which leads to the overdetermined system of differential equations:

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i \quad \text{for all} \quad i,j,k,$$

for the Krichever tau-function.

Theorem 17.8. *Let $F = F(T)$ be the Krichever tau-function, i.e., the residue formula*

$$\frac{\partial^3 \mathcal{F}}{\partial T_I \partial T_J \partial T_K} = res_{dx=0} \left(\frac{dH_I dH_J dH_K}{dxdy} \right)$$

holds. Then it satisfies the WDVV equations once the matching relation

$$\#\{T\} = \#\{dx = 0\}$$

is fulfilled.

Remark 17.9.

- *The number of critical points $\#\{dx = 0\}$ is counted modulo possible involution.*
- *Under non-degeneracy conditions, the proof is straightforward.*
- *The constant "metric" $\eta = \mathcal{F}_1$ is not necessary.*

We present the proof idea for the theorem:

Proof. Proof idea: Consider the finite-dimensional ring at $dx = 0$:

$$\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k \phi_k(\lambda_\alpha), \quad \forall \lambda_\alpha.$$

This equation is solved for

$$C_{ij}^k = \sum_\alpha \phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha)(\phi_k(\lambda_\alpha))^{-1}$$

upon $\#\{i\} = \#\{\alpha\}$ and $\det_{i\alpha} \|\phi_i(\lambda_\alpha)\| \neq 0$. The modification (assuming $\xi(\lambda_\alpha) \neq 0$) gives

$$\phi_i(\lambda_\alpha)\phi_j(\lambda_\alpha) = \sum_k C_{ij}^k(\xi)\phi_k(\lambda_\alpha) \cdot \xi(\lambda_\alpha), \quad \forall \lambda_\alpha.$$

which leads to a redefinition

$$\eta_{kn} = \mathcal{F}_{kn1} \quad \longrightarrow \quad \eta_{kn}(\xi) = \sum_a \xi_a \mathcal{F}_{kna},$$

with $\xi_a = \sum_\alpha \xi(\lambda_\alpha) \left(\phi_a(\lambda_\alpha)\right)^{-1}$. □

17.7 2d Minimal Gravity

For each (p, q)-th point, take a pair of polynomials:

$$X = \lambda^p + \ldots, \quad Y = \lambda^q + \ldots$$

of degrees p and q, respectively. For the Landau-Ginzburg case, $(p, q) = (N, 1)$.

A dispersionless version of the Lax and Orlov-Shulman operators from KP theory is given by:

$$\langle \hat{X}, \hat{Y} \rangle = \hbar, \quad \hat{X} = \frac{\partial}{\partial \lambda^p} + \ldots, \quad \hat{Y} = \frac{\partial}{\partial \lambda^q} + \ldots.$$

An invariant way to express this is through an algebraic equation:

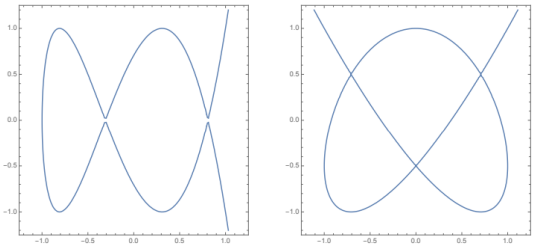
$$Y^p - X^q - X f_{ij} X^i Y^j = 0,$$

with some constants $\{f_{ij}\}$.

Generally, this defines a smooth curve of genus:

$$g = \frac{(p-1)(q-1)}{2} = \#\text{primaries}.$$

For example, the degenerate curves of Yang-Lee and Ising models of $g = 2$ and $g = 3$ can be pictured as:



17.8 Solution to dKP

On a rational curve, the solution is given by:

$$S = \sum_{k=1}^{p+q} t_k H_k = \sum_{k=1}^{p+q} t_k X_k / p(\lambda)^+, \quad k \mod p,$$

and the differential is:

$$dS \overset{\xi \to \infty}{=} \sum_k \left(t_k \xi^{k-1} d\xi + \frac{\partial \mathcal{F}}{\partial t_k} \frac{d\xi}{\xi^{k+1}} \right).$$

The dependence of $X(\lambda) = \lambda^p + \sum_{k=0}^{p-2} X_k \lambda^k$ on $\{t\}$ is given by dS.

For $dX = 0$, the system of equations known as the "hodograph" equations is given by:

$$\frac{dS}{d\lambda} = 0 \quad \text{at the} \quad p-1 \quad \text{roots of} \quad X'(\lambda) = 0.$$

Any Hamiltonian $H_k(\lambda) = \frac{\partial S}{\partial t_k} = \xi_k(\lambda)_+$ is a polynomial in the variable λ. In particular, H_1 corresponds to the dispersionless Hirota equations, where all second derivatives are expressed in terms of the first derivatives, such as:

$$\frac{\partial^2 \mathcal{F}}{\partial t_3 \partial t_3} = 3\left(\frac{\partial^2 \mathcal{F}}{\partial t_1^2}\right)^3.$$

17.9 Singularities and Series Expansions for (p, q) Reductions

17.9.1 Singularity for $(p, q) = (3, 4)$

Let:

$$X = \lambda^3 + X_1\lambda + X_0, \quad Y = \lambda^4 + Y_2\lambda^2 + Y_1\lambda + Y_0.$$

For the flat times $\{t_1, t_2, 0, 0, t_5, 0, t_7 = \text{const}\}$, we have the following relations:

$$t_1 = -\frac{2}{3}X_0^2 + \frac{4}{27}X_1^3 + \frac{5}{9}t_5 X_1^2,$$

$$t_2 = -\frac{2}{3}X_0 X_1 - \frac{5}{3}t_5 X_0.$$

Solving for X_0:

$$t_1 = -6\frac{t_2^2}{(2X_1 + 5t_5)^2} + \frac{4}{27}X_1^3 + \frac{5}{9}t_5 X_1^2 \overset{t_2=0}{=} \frac{4}{27}X_1^3 + \frac{5}{9}t_5 X_1^2$$

which gives the Boulatov-Kazakov equation.

17.9.2 Series for $(p, q) = (2, 2K + 1)$

For $p = 2$ KdV reduction, the relations are given by:

$$X = \lambda^2 + 2u, \quad \xi = \sqrt{X} = \sqrt{\lambda^2 + 2u}.$$

The action S is:

$$S = \sum_{k=0}^{K+1} t_{2k+1} X^{k+\frac{1}{2}}(\lambda).$$

The dependence of $u = u(t)$ from $dS|_{dX=0} = 0$ gives:

$$P(u) \equiv \frac{1}{2}\frac{dS}{d\lambda}\Big|_{\lambda=0} = \sum_{k=0}^{K+1}(2k+1)!!\,\frac{t_{2k+1}}{k!}u^k = 0.$$

The explicit formula for the tau-function F is:

$$F = \frac{1}{2}\sum_{k,l=0}^{K+1} t_{2k+1}t_{2l+1}\,\frac{(2k+1)!!(2l+1)!!}{k!l!(k+l+1)}u^{k+l+1}.$$

Alternatively, this can be written as:

$$F = \frac{1}{2}\int_0^u P_2(v)\,dv.$$

17.9.3 Further Series for $(p, q) = (2, 2K + 1)$

In order to compare with world-sheet gravity, we need to consider resonances and analytic terms.

- **Resonances:** Absent for $(2K + 1)$-reduction.
- **Residue Formula:** Contributions from infinity? The residue formula is crucial for understanding the contributions at infinity.
- $p - q$ **or** $X - Y$ **Duality:** Considerations of duality relations between p and q, or equivalently between X and Y, provide insights into the model's structure.
- **Verlinde Formula:** This formula, developed with A. Artemev and P. Gavrylenko, plays an important role in understanding the relation between certain algebraic structures in the theory.

When $\mu \neq 0$, we encounter the Chebyshev background.

17.10 Chebyshev Curves, Ground Rings, and Tachyons

For the case when the cosmological constant $\mu \neq 0$, the Chebyshev curves are defined as follows:

$$T_p(Y) = T_q(X),$$

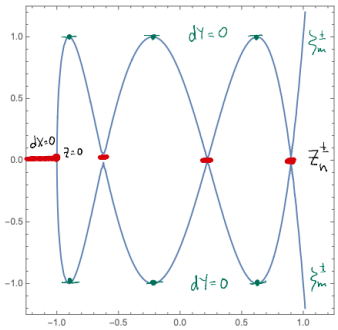
parameterized by $z \in \mathbb{C}$.

The corresponding values of X_n and Y_n are:

$$X_n = T_2(z_n^{\pm}) = \cos\left(\frac{\pi(2n-1)}{2K+1}\right),$$

$$Y_n = T_{2K+1}(z_n^{\pm}) = \pm\cos\left(\frac{\pi\left(n-\frac{1}{2}\right)}{2K+1}\right) = 0.$$

For example, here is the Chebyshev curve for $(2, 2K+1)$-series, with (degenerate) cuts $\{z_n^{\pm}\}$ marked in red, critical points $\{\zeta_m^{\pm}\}$ where $dY = 0$ - in green, and the point $z = 0$ where $dX = 0$.



The ground ring of minimal $(2, 2K+1)$ gravity is isomorphic to:

$$U_k(x)U_l(x) = U_{k+l}(x) + U_{k+l-2}(x) + \cdots + U_{|k-l|}(x), \quad k, l = 0, 1, \ldots$$

modulo $U_{2K}(x) = 0$.

The KP Hamiltonians are given by:

$$H_{2n+1}(z) = T_{2n+1}(z) \sim Y(z)^{\frac{2n+1}{2K+1}} + \cdots$$

For the non-faithful "tachyonic" module, we have:

$$T_n = \frac{dH_{2K+1-2n}}{dz} = U_{2(K-n)}(z), \quad n = 1, \ldots, K.$$

The ground ring acts as:

$$T_n(z) = U_{n-1}(X)T_1(z) = U_{n-1}(T_2(z)), \quad n = 1, \ldots, 2K.$$

The tachyonic operators $T_n \sim T_{2K-n}$ are identified up to a sign, due to the "reflection relations":

$$U_{2K+l}(x) + U_{2K-l}(x) = 0.$$

Proof. Indeed, we have the following relations:

$$T_1 \sim U_{2K-2}(z)$$

and

$$\begin{aligned} T_n \sim U_{n-1}(T_2(z))U_{2K-2}(z) &= \frac{1}{z}U_{2n-1}(z)U_{2K-2}(z) \\ &\overset{\text{ring } U}{=} \frac{1}{z}\left(U_{2K+2n-3}(z) + U_{2K+2n-1}(z) + \cdots + U_{2K-2n+1}(z) + U_{2K-2n-1}(z)\right) \\ &\overset{\text{reflection}}{=} \frac{1}{z}\left(U_{2K-2n+1}(z) + U_{2K-2n-1}(z)\right) \\ &\overset{\text{ring } U}{=} U_{2(K-n)}(z). \end{aligned}$$

The identification $T_n \sim T_{2K-n}$ is due to:

$$\begin{aligned} U_{2K-n}(X) &= U_{2K-n}(T_2(z)) \\ &= \frac{1}{z}U_{4K-2n-1}(z) \\ &\overset{\text{reflection}}{=} -\frac{1}{z}U_{2n-1}(z) \\ &= -U_{n-1}(T_2(z)) \\ &= -U_{n-1}(X). \end{aligned}$$

□

The residue formula for the third derivative of the tau-function on a Chebyshev curve is given by:

$$\begin{aligned} \frac{\partial^3 \mathcal{F}}{\partial t_i\, \partial t_j\, \partial t_k} &= -\mathrm{res}_{dY=0}\frac{dH_i\, dH_j\, dH_k}{dX\, dY} \\ &= -\frac{1}{2K+1}\,\mathrm{res}_{U_{2K}(z)=0}\frac{U_{2i}U_{2j}U_{2k}(z)}{2zU_{2K}(z)}\,dz \\ &= -\frac{1}{2K+1}\sum_{m=1}^{K}\frac{U_{2i}U_{2j}U_{2k}(\zeta_m)}{\zeta_m U_0^{2K}(\zeta_m)}, \end{aligned}$$

where $\zeta_m = \pm\cos\left(\frac{\pi m}{2K+1}\right)$, for $m = 1, 2, \ldots, K$.

We have:

$$\begin{aligned} X_m = T_2(\zeta_m) &= \cos\left(\frac{2\pi m}{2K+1}\right), \\ Y_m^{\pm} = \pm T_{2K+1}(\zeta_m) &= \pm\cos\left(\frac{\pi m}{2}\right) = \mp(-1)^m. \end{aligned}$$

Using the relation $T_{n+1}(x) = xU_n(x) - U_{n-1}(x)$ and

$$\zeta_m^{\pm} U_{2K-1}(\zeta_m^{\pm}) = (-1)^m \cos\left(\frac{\pi m}{2K+1}\right), \quad m = 1, 2, \ldots, K,$$

at $U_{2K}(z) = 0$, the third derivative of the tau-function is given by:

$$\begin{aligned}
\frac{\partial^3 \mathcal{F}}{\partial t_i \, \partial t_j \, \partial t_k} &= -\frac{1}{2K+1} \sum_{m=1}^{K} \frac{U_{2i} U_{2j} U_{2k}(\zeta_m)}{\zeta_m U_0^{2K}(\zeta_m)} \\
&= -\frac{1}{(2K+1)^2} \sum_{m=1}^{K} \frac{U_{2i} U_{2j} U_{2k}(\zeta_m)(1-\zeta_m^2)}{\zeta_m U_{2K-1}(\zeta_m)} \\
&= \frac{2}{(2K+1)^2} \sum_{m=1}^{K} \frac{\sin\left(\frac{2\pi m i}{2K+1}\right) \sin\left(\frac{2\pi m j}{2K+1}\right) \sin\left(\frac{2\pi m k}{2K+1}\right)}{\sin\left(\frac{2\pi m}{2K+1}\right)} \\
&= \frac{1}{2(2K+1)} (-1)^{1+i+j+k} N_{ijk},
\end{aligned}$$

where N_{ijk} is the Verlinde expression on the right-hand side.

17.11 Verlinde Formula: Basics

The S-matrix satisfies the following transformation property:

$$\chi_a\left(-\frac{1}{\tau}\right) = \sum_b S_{ba} \chi_b(\tau),$$

where unitarity is required:

$$S^\dagger S = 1.$$

The Verlinde formula is related to the fusion algebra through the following expression for the fusion coefficients N_{ab}^c:

$$N_{ab}^c = \sum_m S_{ma} S_{mb} (S^\dagger)_{mc} S_{m1}.$$

For the minimal (p, q)-model, the S-matrix for $S^2 = 1$ is given by:

$$S_{rs,\rho\sigma} = 2\sqrt{\frac{2}{pq}} (-1)^{1+s\rho+r\sigma} \sin \pi \frac{p}{q} r\rho \sin \pi \frac{q}{p} s\sigma,$$

where p, q denote the parameters of the model. For the specific case of $(p, q) = (2K+1, 2)$, the formula simplifies to:

$$S_{r,\rho} = \frac{2}{\sqrt{2K+1}} (-1)^{1+\rho+r+K} \sin\left(\frac{2\pi r\rho}{2K+1}\right).$$

This represents the fusion coefficients for the case $s = \sigma = 1$.

Theorem 17.10. *For the specific case $(p, q) = (2K+1, 2)$, the Verlinde formula is given by:*

$$\frac{4}{2K+1}(-1)^{1+i+j+k}\sum_{m=1}^{K}\frac{\sin\left(\frac{2\pi mi}{2K+1}\right)\sin\left(\frac{2\pi mj}{2K+1}\right)\sin\left(\frac{2\pi mk}{2K+1}\right)}{\sin\left(\frac{2\pi m}{2K+1}\right)} = N_{ijk},$$

where $N_{ijk} \in \{0, 1\}$.

Proof. A nontrivial proof of this formula is:

$$\begin{aligned}
N_{ijk} &= (-1)^{1+i+j+k}\frac{4}{2K+1}\sum_{m=1}^{K}\frac{\sin\left(\frac{2\pi mi}{2K+1}\right)\sin\left(\frac{2\pi mj}{2K+1}\right)\sin\left(\frac{2\pi mk}{2K+1}\right)}{\sin\left(\frac{2\pi m}{2K+1}\right)} \\
&= (-1)^{i+j+k}\,\mathrm{res}_{z=0}\left(\frac{U_{2i}U_{2j}U_{2k}(z)\,dz}{zU_{2K}(z)}\right) \\
&= (-1)^{1+i+j+k}\left(\mathrm{res}_{z=0}+\mathrm{res}_{z=\infty}\right)\left(\frac{U_{2i}U_{2j}U_{2k}(z)\,dz}{zU_{2K}(z)}\right) \\
&= (-1)^{1+i+j+k}N_{ijk} = \left(\mathrm{res}_{z=0}+\mathrm{res}_{z=\infty}\right)\frac{U_{2i}U_{2j}U_{2k}(z)\,dz}{zU_{2K}(z)}.
\end{aligned}$$

Substituting $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$, we have:

$$\begin{aligned}
(-1)^{1+i+j+k}N_{ijk} &= \left(\mathrm{res}_{w=i}+\mathrm{res}_{w=0}\right)\frac{w^{2i+1}-w^{-(2i+1)}}{(w^2-w^{-2})(w^{2K+1}-w^{-(2K+1)})}\,dw \\
&= (-1)^{1+i+j+k}+\mathrm{res}_{w=0}\frac{w^{2i+1}-w^{-(2i+1)}}{(w^2-w^{-2})(w^{2K+1}-w^{-(2K+1)})}\,dw \\
&= 1-\sum_{l=0}^{K-2}\delta_{i+j+k+2l,2K}-\sum_{l=0}^{\lfloor\frac{K-3}{2}\rfloor}\left(\delta_{i+j+2l+1,k}+\delta_{i+k+2l+1,j}+\delta_{k+j+2l+1,i}\right) \\
&= \sum_{l=0}^{\min(i,j)-1}\delta_{|i-j|+2l+1,k}+\sum_{l=0}^{\lfloor\frac{K-2}{2}\rfloor}\delta_{i+j+k,2(K+l+1)}.
\end{aligned}$$

□

17.12 Non-Algebraic Generalization

The "continuous" theory, as studied by Collier, Eberhardt, Mühlmann, and Rodriguez, is generalized by the expression:

$$N(p_1, p_2, p_3) = 2b\sum_{m=1}^{\infty}(-1)^m\frac{\sin 2\pi mbp_1\sin 2\pi mbp_2\sin 2\pi mbp_3}{\sin\pi mb^2},$$

where $b^2 = \frac{p}{q}$ in the minimal theory. Here, we have the identity:

$$(-1)^m \sin \pi m b^2 = \sin(\pi m b^2 + \pi m) = \sin 2\pi m b p_0 = S^0_m,$$

with $p_0 = \frac{\frac{1}{b}+b}{2}$, corresponding to the $h_0 = 0$ or "unity" operator.

Zamolodchikov's formula comes from the residue formula for a non-algebraic curve defined by:

$$x(z) = \cos(\pi b^{-1} z), \quad y(z) = \cos(\pi b z),$$

with the relation $b^2 \in \mathbb{R}$.

Indeed, we have

$$N(p_1, p_2, p_3) = \sum_{dx=0} \frac{dH_{p_1} dH_{p_2} dH_{p_3}}{dx\, dy} = \sum_{x_0(z)=0} \phi(p_1 z)\phi(p_2 z)\phi(p_3 z) \frac{x''(z)y'(z)}{x'(z)y''(z)}$$

since

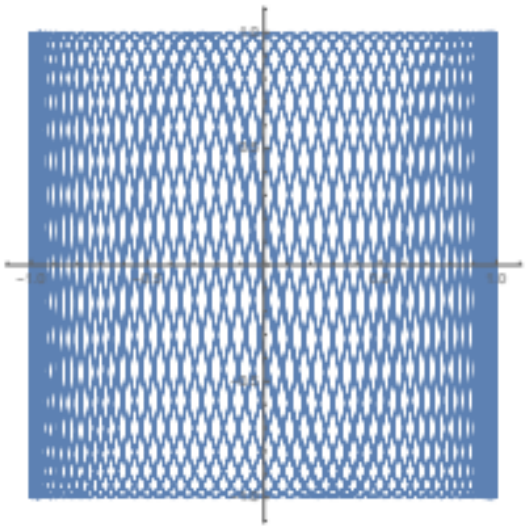
$$x'(z) \sim \sin \pi b^{-1} z = 0 \quad \text{at} \quad z_m = bm, \quad m \in \mathbb{Z},$$

and

$$y'(z_m) \sim b \sin \pi b z_m = b \sin \pi b\,(2m)\,, \quad x''(z_m) \sim b^{-2} \cos \pi b^{-1} z_m = b^{-2}(-1)^m.$$

Finally, the rest of the derivation follows from the identification $\phi(pz) = \sin 2\pi p z$.

Here is an image of a non-algebraic curve giving rise to an infinite sum over $dx = 0$:



17.13 Many Other Developments

In addition to the topics discussed above, there have been numerous other developments in the field, including:

- **Instanton Partition Functions:** These functions arise in the study of gauge theories and are particularly important in the context of supersymmetric gauge theories and topological field theories. They capture the contributions of instantons (non-perturbative solutions) to the partition function of the system.
- **2D Conformal Field Theories:** Two-dimensional conformal field theories (CFTs) are central in string theory, statistical mechanics, and condensed matter physics. The study of their correlation functions, symmetries, and representation theory provides deep insights into quantum field theory and critical phenomena.
- **"Relativistic" (qt)-deformations:** These deformations generalize the quantum groups and affine Lie algebras, which are crucial in the study of integrable systems. In the relativistic limit, these deformations lead to new symmetries and deformed versions of CFTs and topological field theories.
- **"Topological Vertices", Cluster Algebras, and Double-loop Algebras:** The study of topological string theory has led to the development of "topological vertices" which are used to compute partition functions of 3D manifolds. Cluster algebras and double-loop algebras provide mathematical structures that help in understanding the combinatorics and representation theory of such systems, further linking mathematical physics with algebraic geometry.
- **Isomonodromic Deformations:** These deformations are used in the study of integrable systems, particularly in the context of singular solutions of differential equations. The theory of isomonodromic deformations has applications in mathematical physics, including the study of quantum groups and exactly solvable models.
- **And more...**

18 Alexander Veslov: Harmonic Locus and Calogero-Moser Spaces

Abstract

The harmonic locus consists of the monodromy-free Schroedinger operators with rational potential quadratically growing at infinity. It is known after Duistermaat and Grunbaum that in the multiplicity-free case the poles $z_1, ..., z_N$ of such potentials satisfy the following algebraic system

$$\sum_{j\neq i}^{N} \frac{2}{(z_i - z_j)^3} - z_i = 0, \quad i = 1, ..., N,$$

describing the complex equilibriums of the corresponding Calogero-Moser system. Oblomkov proved that the harmonic locus can be identified with the set of all partitions via Wronskian map for Hermite polynomials. We show that the harmonic locus can also be identified with the subset of the Calogero-Moser spaces introduced by Wilson, which is invariant under a natural symplectic action of $\mathbb{C}^\times$ As a corollary, for the multiplicity-free part of the locus we effectively solve the inverse problem for the Wronskian map by proving that the spectrum of Moser's matrix coincides with the set of contents of the corresponding Young diagram. We also compute the characters of the $\mathbb{C}^\times$-action at the fixed points, proving a conjecture of Conti and Masoero. The talk is based on a joint work with Giovanni Felder.

Contents

18.1 Monodromy-free Operators and Locus Problem in 1D

Let $V(z)$, where $z \in \mathbb{Z}$, be a meromorphic potential and $L = -D^2 + V(z)$ be the corresponding Schrödinger operator.

Problem 18.1 (Locus Problem)**.** *Describe the potentials $V(z)$ such that the corresponding equation*

$$(-D^2 + V(z))\psi = \lambda\psi, \quad D = \frac{d}{dz}$$

has all solutions ψ, which are meromorphic in $z \in \mathbb{C}$ for all λ.

Example 18.2 (Novikov 1974, Its and Matveev 1975, Krichever 1976)**.** *All finite-gap operators are monodromy-free.*

There is a long history in the 1870 and 1880's on differential equations in the complex domain with "uniform" solutions (Hermite, Picard, Halphen, Darboux, ...).

Theorem 18.3 (Duistermaat and Grünbaum, 1986)**.** *The operator $L = -D^2 + V(z)$ is monodromy-free if and only if the Laurent series expansion of its potential near every pole z_0*

$$V = \sum_{i=-2}^{\infty} c_i (z - z_0)^i$$

satisfies the locus (quasi-invariance) conditions:

$$c_{-2} = m(m+1), \quad m \in \mathbb{N}, \quad c_{2k-1} = 0, \quad k = 0, \ldots, m.$$

18.2 Terminology and Modern Motivation

Proposition 18.4 (Airault, McKean, Moser, 1977)**.** *Poles of the rational solutions*

$$V(z) = \sum_{i=1}^{N} \frac{2}{(z - z_i)^2}$$

of the KdV equation

$$u_t = 6uu_x - u_{xxx}$$

belong to the locus given by the algebraic system

$$\sum_{j \neq i} \frac{1}{(z_i - z_j)^3} = 0, \quad i = 1, \ldots, N,$$

describing the (complex) equilibriums of the Calogero-Moser system with Hamiltonian

$$H = \sum_{i=1}^{N} p_i^2 + \sum_{j \neq i} \frac{2}{(z_i - z_j)^2}.$$

Additionally,

- *If $N \neq \frac{m(m+1)}{2}$ is not a triangular number, the locus is empty.*
- *If $N = \frac{m(m+1)}{2}$, it has dimension m and consists of zeros of the Burchnall-Chaundy (Adler-Moser) polynomials:*

$$P_1 = z, \quad P_2 = \frac{1}{3}(z^3 + \tau_2), \quad P_3 = \frac{1}{45}(z^6 + 5\tau_2 z^3 + \tau_3 z - 5\tau_2^2),$$

$$P_4 = \frac{1}{4725}(z^{10} + 15\tau_2 z^7 + 7\tau_3 z^5 - 35\tau_2\tau_3 z^2 + 175\tau_3^2 z - \frac{7}{3}\tau_3^2 + \tau_4 z^3 + \tau_4\tau_2), \ldots$$

18.3 Harmonic Locus

Definition 18.5. *The **harmonic locus** $\mathcal{HL}$ consists of monodromy-free potentials of the form*

$$V = z^2 + \sum_{i=1}^{N} \frac{m_i(m_i+1)}{(z-z_i)^2}.$$

Proposition 18.6. *When all multiplicities $m_i = 1$ (simple part), the poles satisfy the algebraic system*

$$\sum_{j \neq i} \frac{2}{(z_i - z_j)^3} - z_i = 0, \quad i = 1, \ldots, N,$$

describing the (complex) equilibria of the Calogero-Moser system with Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + U(q), \quad U(q) = \frac{1}{2}\sum_{i=1}^{N} q_i^2 + \sum_{1 \leq i < j \leq N} \frac{1}{(q_i - q_j)^2}.$$

Proposition 18.7 (Oblomkov, 1999)**.** *All such potentials can be explicitly described via the Wronskians of the Hermite polynomials*

$$V(z) = z^2 - 2D^2 \log W\left(H_{k_1}(z), \ldots, H_{k_n}(z)\right), \quad k_1 > k_2 > \cdots > k_n > 0,$$

where $H_k(z)$ is the k-th Hermite polynomial.

18.4 Hermite Polynomials

Hermite polynomials $H_n(z)$ are the classical orthogonal polynomials with Gaussian weight $w(z) = e^{-z^2}$:

$$H_k(z) = (-1)^k e^{z^2} \frac{d^k}{dz^k} e^{-z^2},$$

with specific values:

$$H_0(z) = 1, \quad H_1(z) = 2z, \quad H_2(z) = 4z^2 - 2, \quad H_3(z) = 8z^3 - 12z,$$

$$H_4(z) = 16z^4 - 48z^2 + 12, \quad H_5(z) = 32z^5 - 160z^3 + 120z, \ldots$$

These polynomials satisfy the recurrence relation:

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z).$$

The operators $\tilde{L} = -D^2 + x^2 - 2D^2 \log W\left(H_{k_1}(x), \ldots, H_{k_n}(x)\right)$ are the integrable (Darboux) transformations of the harmonic oscillator operator $L = -D^2 + x^2$ with the eigenfunctions $\psi_n = H_n(x)e^{-x^2/2}$:

$$L\psi_n = \left(n + \frac{1}{2}\right)\psi_n, \quad n = 0, 1, \ldots$$

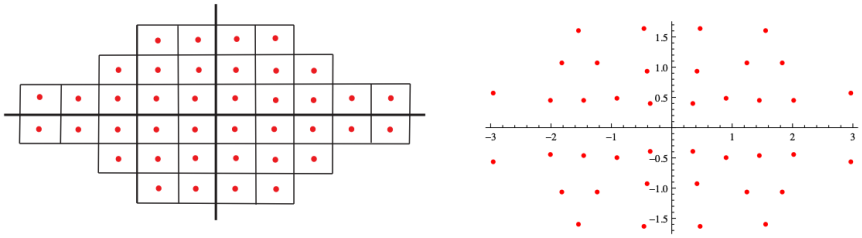
Proposition 18.8 (Felder, Hemery, Veselov, 2012)**.** *Label these Wronskians by the partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ as*

$$W_\lambda(z) := W\left(H_{\lambda_1+l-1}, H_{\lambda_2+l-2}, \ldots, H_{\lambda_l}\right),$$

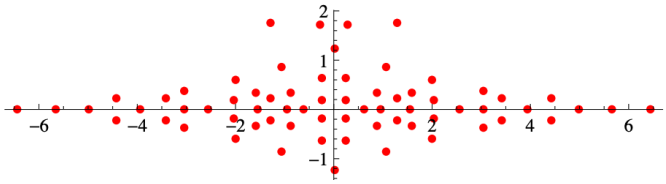
then the following properties hold:

1. *$W_\lambda(z)$ is a polynomial in z of degree $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$,*
2. *$W_\lambda(-z) = (-1)^{|\lambda|}W_\lambda(z)$,*
3. *$W_{\lambda^*}(z) = (-i)^{|\lambda|}W_\lambda(iz)$, where λ^* is the conjugate of λ.*

For the doubled partitions $(\lambda_1, \lambda_1, \ldots, \lambda_l, \lambda_l)$, there is a surprising (empirical) relation between the Young (Ferrers) diagram and the zero set of $W_\lambda(z)$:



However, in general, this relation is not so clear. For example, for $\lambda = (28, 16, 10, 6, 4, 4, 3, 1)$, we have the following:



18.5 Inverse Problem for Harmonic Locus

Problem 18.9. *Given the zero set of W_λ, how can we recover the partition λ?*

For the simple zeros case with $m_i = 1$, we have the following answer (conjectured in 2012).

Recall that the content $c(\square)$ of the box $\square = (i, j)$ from the Young diagram λ is defined as $c(\square) = j - i$. The multiset $C(\lambda) := \{c(\square), \square \in \lambda\}$ determines λ uniquely:

0	1	2	3
−1	0	1	
−2			

Proposition 18.10 (Felder and Veselov, 2024)**.** *For a simple locus configuration $(z_1, \dots, z_n)$, the corresponding partition λ is uniquely determined by the property that the contents of λ coincide with the eigenvalues of Moser's matrix M:*

$$C(\lambda) = Spec(M), \quad M_{ij} = \begin{cases} -\frac{1}{(z_i - z_j)^2}, & \text{for } i \neq j, \\ \sum_{k \neq j} \frac{1}{(z_k - z_j)^2}, & \text{for } i = j. \end{cases}$$

The proof uses the theory of Calogero-Moser systems and Calogero-Moser spaces.

18.6 Calogero-Moser Systems

We present a brief history of Calogero-Moser Systems.

Proposition 18.11 (Moser 1975)**.** *Lax form $\dot{L} = [L, M]$ for the (now called CM) system with*

$$H_{CM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{1 \leq i < j \leq n} \frac{\gamma^2}{(q_i - q_j)^2},$$

where

$$L_{ij} = p_i \delta_{ij} + \frac{i\gamma}{q_i - q_j}(1 - \delta_{ij}), \quad M_{ij} = \sum_{k \neq i} \frac{i\gamma}{(q_k - q_i)^2} \delta_{ij} + \frac{i\gamma}{(q_i - q_j)^2}(1 - \delta_{ij}).$$

Proposition 18.12 (Kazhdan, Kostant, and Sternberg 1978)**.** *The CM system as a symplectic reduction of free motion on the Lie algebra of $U(n)$, with the moment map*

$$\mu : (P, Q) \mapsto [P, Q] = i\gamma(1 - \delta_{ij}), \quad (P, Q) \in T^* u(n),$$

$$Q = q_i \delta_{ij}, \quad P = L, \quad H_{CM} = \frac{1}{2} tr(P^2) = \frac{1}{2} tr(L^2).$$

Proposition 18.13 (Perelomov 1978)**.** *Harmonic version* $\dot{L}_\pm = [L_\pm, M] \pm L_\pm$*, with*

$$L_\pm = L \pm Q, \quad H_{CM} = \frac{1}{2} tr(L^2) + \frac{1}{2} tr(Q^2) = \frac{1}{2} p^2 + \frac{1}{2} q^2 + \sum_{1 \le i < j \le n} \frac{\gamma^2}{(q_i - q_j)^2}.$$

Proposition 18.14 (Wilson 1998)**.** *The Calogero-Moser space* $\mathcal{C}_n$ *is the quotient space*

$$\mathcal{C}_n = \{(X, Z, v, w) : [X, Z] + I = vw\}/GL_n(\mathbb{C}),$$

where X *and* Z *are* $n \times n$ *complex matrices, and* v *and* w *are an* n*-dimensional vector and covector (considered as* $n \times 1$ *and* $1 \times n$ *matrices, respectively). An element* $g \in GL_n(\mathbb{C})$ *acts on* (X, Z, v, w) *as*

$$(X, Z, v, w) \mapsto (gXg^{-1}, gZg^{-1}, gv, wg^{-1}).$$

Proposition 18.15 (Wilson)**.** $\mathcal{C}_n$ *is a smooth irreducible affine algebraic variety of dimension* $2n$*, which can be viewed as a quantisation of the Hilbert scheme of* n *points in* $\mathbb{C}^2$*.*

There is a natural symplectic action of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ on $\mathcal{C}_n$ defined by

$$X \mapsto \mu X, \quad Z \mapsto \mu^{-1} Z, \quad v \mapsto v, \quad w \mapsto w, \quad \mu \in \mathbb{C}^\times.$$

Let $\mathcal{C}_n^{\mathbb{C}^\times}$ be the fixed point subset of $\mathcal{C}_n$ under this action. Wilson identified it with the set $\mathcal{P}_n$ of all partitions of n.

18.7 Modified Calogero-Moser Spaces

Proposition 18.16 (Felder and Veselov, 2024)**.** *The modified Calogero-Moser space* $\mathcal{CM}_n$ *is the quotient*

$$\mathcal{CM}_n = \{\Pi = (L, Q, M, v, w)\}/GL_n(\mathbb{C}),$$

where L*,* Q*, and* M *are* $n \times n$ *complex matrices, and* v *and* w *are a vector and covector, as before, which satisfy the following relations:*

1. $[L, Q] = I - vw$*,*
2. $[M, Q] = L$*,*
3. $[M, L] = Q$*,*
4. $Mv = 0, \quad wM = 0.$

The group GL_n *acts by conjugation on* L*,* Q*, and* M*, and on* v*,* w *as before.*

Proposition 18.17 (Felder and Veselov, 2024)**.** *The modified Calogero-Moser space* $\mathcal{CM}_n$ *can be identified with the harmonic locus and with the set of partitions of* n *via the map*

$$\chi : \mathcal{CM}_n \to \mathcal{HL}_n, \quad \chi(\Pi) = z^2 - 2D^2 \log\det(zI - Q).$$

Proof. We present a proof sketch:

Step 1. The modified CM space $\mathcal{CM}_n$ can be identified with the fixed set $\mathcal{C}_n^{\mathbb{C}^\times}$:

$$X = \frac{1}{2}(L+Q), \quad Z = L - Q.$$

Step 2. M is the generator of the $\mathbb{C}^\times$-action:

$$[M, X] = X, \quad [M, Z] = -Z.$$

Step 3. Use Wilson's identification of $\mathcal{C}_n^{\mathbb{C}^\times}$ with the set of partitions $\mathcal{P}_n$ and his formula

$$\det(X_\lambda - \sum_{i \geq 1} p_i (-Z_\lambda)^{i-1}) = B(\lambda) s_\lambda,$$

where p_i and s_λ are power sums and Schur symmetric functions, respectively.

Step 4. Use the theory of Appell polynomials and the generating function of Hermite polynomials to derive that, up to a constant multiple,

$$W_\lambda(z) = \det(X_\lambda - zI - \frac{1}{2} Z_\lambda) = \det(Q_\lambda - zI).$$

Step 5. Use Oblomkov's theorem to link with the harmonic locus.

□

18.8 Inversion of the Wronskian Map

Proposition 18.18 (Felder and Veselov, 2024)**.** *The subset of $\mathcal{CM}_n$ with diagonalisable Q with simple spectrum can be identified with the simple part of the harmonic locus $\mathcal{HL}_n$. The spectrum of the corresponding Moser's matrix M is integer and coincides with the content multiset $C(\lambda)$ of the corresponding partition λ.*

Indeed, Moser's matrices obviously satisfy the relations (1), (2), and (4) (with $w = (1, 1, \ldots, 1) = v^T$), while (3) is equivalent to the locus conditions:

1. $[L, Q] = I - vw$,
2. $[M, Q] = L$,
3. $[M, L] = Q$,
4. $Mv = 0, \quad wM = 0$.

The proof of the formula $\mathrm{Spec}(M_\lambda) = C(\lambda)$ follows from an explicit description of X_λ, Z_λ from Wilson (1998), who used the Frobenius parametrisation of λ. This agrees with

Proposition 18.19 (Calogero et al, 1970s)**.** *The matrix M defined by the zeros of the Hermite polynomial $H_n(z)$ has eigenvalues $0, \ldots, n-1$ (which are the contents of $\lambda = (n)$).*

18.9 Application: Proof of Conti-Masoero Conjecture

Consider now the matrix

$$K_{ij}(\lambda) = \delta_{ij}\left(1 + \sum_{l\neq j} \frac{6}{(z_l - z_j)^4}\right) - (1-\delta_{ij})\frac{6}{(z_i - z_j)^4},$$

where $z_i = z_i(\lambda)$ are the roots of the corresponding Hermite Wronskian $W_\lambda(z)$, which are assumed to be simple. This is the Hessian matrix of

$$U(q) = \frac{1}{2}\sum_{i=1}^{n} q_i^2 + \sum_{1\leq i<j\leq n} \frac{1}{(q_i - q_j)^2}$$

at the equilibrium points $q_i = z_i$.

Proposition 18.20 (Felder and Veselov, 2024)**.**

$$Spec(K(\lambda)) = \left\{(\lambda_l(\square) + 1 - c(\square))^2\,, \quad \square \in \lambda\right\},$$

This is equivalent to a conjecture of Conti and Masoero, 2021.

Proposition 18.21 (Perelomov, 1978)**.** *The frequencies of small oscillations of the CM system near the equilibrium given by the zeros of the Hermite polynomial $H_n(z)$ are $1, 2, \ldots, n$ (which are also the exponents of the Lie algebra $\mathfrak{u}(n)$).*

18.10 Concluding Remarks

There are many related questions still open. Here are some of them:

- Inverse problem for the non-simple part of the harmonic locus and for the trigonometric locus for

$$V(z) = \sum_{i=1}^{N} m_i(m_i+1)\sin^{-2}(z - z_i).$$

- Description of the elliptic locus for

$$V(z) = \sum_{i=1}^{N} m_i(m_i+1)\wp(z - z_i),$$

  where $\wp(z)$ is the Weierstrass elliptic function.

- Description of the monster potentials

$$V(z) = Lz^2 + z^{2\alpha} - 2D^2\sum_{k=1}^{n}\log(z^{2\alpha+2} - z_k),$$

  introduced by Bazhanov, Lukyanov, and Zamolodchikov (2003).

- Multidimensional case in relation to the Hadamard problem and Huygens' principle (Chalykh, Feigin, Veselov 1999).

19 Stanislav Smirnov: Coulomb Gas and Lattice Models

Abstract

Even before the introduction of Conformal Field Theory by Belavin, Polyakov and Zamolodchikov, it appeared indirectly in the work of den Nijs and Nienhuis using Coulomb gas techniques. The latter postulate (unrigorously) that height functions of lattice models converge to the Gaussian Free Field, allowing to derive many exponents and dimensions of 2D lattice models.

This convergence is in many ways mysterious, in particular it was never formulated in the presence of a boundary, but rather pn a torus or a cylinder. We will discuss possible formulations on general domains or Riemann surfaces and their relations to CFT, SLE and conformal invariance of critical lattice models. Interestingly, new objects in complex geometry and potential theory seem to arise.

Contents

19.1 Introduction

Based on joint work with Hongler, Kemppainen, Khristoforov, Glazman, Chelkak, Izyurov, and more.

We start by discussing the famous 40 year old paper *Infinite Dimensional Conformal Symmetry in Two-Dimensional Quantum Field Theories* by Belavin, Polyakov, and Zamolodchikov. From complex analysis, there are a lot of objects which have infinite conformal symmetry. The exciting part of this paper is that the authors came up with natural objects from other areas which exhibit infinite-dimensional conformal symmetry.

In the context of a 2D quantum field theory (QFT) at criticality, it was already established prior to this paper (by Stueckelberg, Wilson, Kapranov, Fisher, and others) that we have renormalization group theory. Field theories and critical points, in this sense, can be viewed as fixed points of the renormalization group flow. Therefore, we expect them to be invariant under rotations and translations, and is supported by mathematics. If the theory is not invariant under rotation, scaling, or translation, then we can simply rotate it, and it will map to another fixed point.

At this point, the authors of the paper made a significant leap of faith. They proposed that, in 2D, the theory should also be invariant under inversion. This idea was not previously considered: unlike in 3D, where Brownian motion is not invariant under inversion. Smirnov had to independently discover this idea multiple times, always finding himself surprised by the distinction between dimensions two and three.

They also suggested that the theory should be invariant under conformal maps. From a complex analysis perspective, this introduces some challenges. The paper claims that there are infinite-dimensional semi-groups of conformal maps that can be applied to domains. By extending this group to the whole plane, one can obtain interesting results. However, while this approach may work for physicists, it is not acceptable within a mathematics department without rigorous justification. Cardy, in the same year (1984), was the first to formalize how to handle boundaries properly in this context. Nevertheless, Cardy also provided a counterexample where a 2D field theory, invariant under rotation, scaling, and translation, does not lead to a conformal field theory. This example has logical and physical implications, particularly in the context of elasticity theory.

The specific issue that puzzles complex analysts lies in the application of these concepts. As Smirnov ventured into probabilistic methods, Smirnov realized that the correct question is not "What is the model?" but rather "What phenomenon is the model describing?" At criticality, we assume that as the lattice spacing tends to zero, the system becomes conformally invariant. This leads to the surprising conclusion that the lattice model, in the continuum limit, exhibits conformal invariance, a key feature of conformal field theory. The question, how-

ever, remains: why should this occur? While Cardy's counterexample provides some insight, the rigorous proof of this phenomenon had eluded us until recently. A related result shows that random walks converge to Brownian motion, which is conformally invariant in 2D up to time scaling. But the central question remains: how can we prove this rigorously?

Before proceeding further, we should clarify what we mean by the lattice model with criticality. To illustrate this concept, we have drawn a few lattice models, though the square lattice is much simpler to depict compared to the more complex hexagonal lattice. We will briefly discuss two or three canonical models that cover many of the most classical models in statistical physics.

19.2 The $O(n)$ Model

One such model is the loop representation of the $O(n)$ model. The $O(n)$ model is essentially a generalization of the Ising model, where, instead of using binary spins (± 1), one assigns vectors on an n-dimensional sphere. A well-known example is the XY model, which was introduced by Heisenberg, who was motivated by the Ising model in his thesis. Ising himself had conjectured that no phase transition occurred in the Ising model, so Heisenberg devised a new model to explore this idea further. Both the Ising and XY models fall under this broader framework. In the loop representation, one can draw loops on a hexagonal lattice, and the partition function is written as a sum over all configurations, weighted by n raised to the power of the number of loops:

$$Z = \sum_w n^{\#\text{loops}} x^{\text{length}}.$$

There are two primary ways to derive this model from the Ising model:

- Start with spins on a hexagonal lattice and then introduce domain walls. For $n = 1$ (Ising model), we do not count the number of clusters, and x functions as the inverse temperature, with $T = -\frac{1}{\log x}$.
- Alternatively, one can place spins at the vertices of the hexagonal lattice and apply a high-temperature expansion. In this case, loops do not separate into distinct classes and tend to remain within the same class. This approach also leads to several conjectural bijections in the field.

Key observations include:

- For $n = 1$, we do not count the loops, recovering the Ising model. The critical value of x_c is $\frac{1}{\sqrt{3}}$, as shown in a paper by one of the researchers. This corresponds to percolation at $\tilde{x}_c = 1$.
- For $n = 0$, we get a self-avoiding walk. Here, there are no loops, but boundary conditions can enforce the existence of a source and a sink, with a curve running between them. A small value of x minimizes the curve's length, making it a straight line, while a large x results in a space-filling

curve. Two critical values of x_c emerge, with the most interesting being $x_c = \frac{1}{\sqrt{2+\sqrt{2}}}$.

- For $n = 2$, we obtain the XY Heisenberg model, with the critical value $x_c = \frac{1}{\sqrt{2}}$.

Although there is a general formula for the critical behavior in terms of n, we will not state it here.

An interesting aspect of this reformulation is that while the original $O(n)$ model requires n to be an integer (since n represents the dimension of the sphere on which the spin vectors reside), the reformulation allows for non-integer values of n, thus broadening the scope of the model.

That said, there are challenges with this model. In particular, when $n \neq 1$, consider covering half of the model and encountering five half-loops. It becomes unclear how these half-loops contribute to the overall weight. Some may be parts of the same loop, while others might belong to different cycles. This introduces a form of nonlocality, which poses a significant challenge. The question then becomes: How should we handle this nonlocality?

A second model worth mentioning is the Fortuin-Kasteleyn model, or the random cluster model. There's a bit of humor in the naming of models in statistical physics, as many are named after the researchers' students. For instance, Lenz invented the Ising model and named it after his student, while Domb invented the Potts model and named it after his student, and so on. The Fortuin-Kasteleyn model is very similar to the $O(n)$ model but uses dense loops on a square lattice. The partition function is typically written as $\sqrt{q}$, and for $n = \sqrt{q}$, it becomes:

$$Z = \sum n^{\#\text{loops}}.$$

Here, there is no x term because the loops are dense, so the perimeter is always constant.

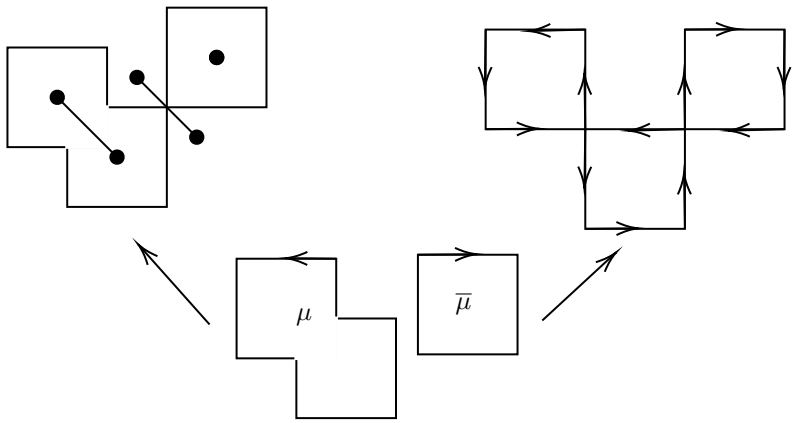
This model is related to the random cluster model on a lattice rotated by 45°, essentially the dual of the original lattice. Notable cases include:

- For $n = 1$, the model corresponds to percolation, as $n = 1$ simply equates the loop collection probabilities with the bond collection probabilities on the rotated lattice.
- The random cluster model can be viewed as a loop FK model. In this model, there is an x term, but it vanishes when applying the bijection because x would count clusters, which is unnecessary for critical models. When working with loops, it counts the number of loops, raised to the power of the number of loops. This leads to an interesting simplification.
- For $n = \sqrt{2}$, the model corresponds to FK-Ising, a version of the Ising model in which the clusters represent the Ising spins.

- For $n = 0$, the model describes a uniform spanning tree, and similarly, we can apply the FK-Ising framework to describe this case.

19.3 The Six Vertex models

Now, the question remains: How should we address the nonlocality inherent in these models? One solution, introduced by Baxter (unrelated to the well-known Yang-Baxter equations), is to randomly orient the loops.



In this formulation, loops are considered to be oriented either clockwise or counterclockwise. This necessitates a new partition function, and a natural temptation arises to assign the factors $n/2$ to clockwise loops and $n/2$ to counterclockwise loops. However, a more effective approach is to introduce a unit vector μ, which satisfies $2\mu = n + im$ and lies on the unit circle. In this case, $|\mu| = 1$ and $\overline{\mu} = \mu^{-1}$.

Thus, the partition function is modified as follows:

$$Z_{\mathbb{C}} = \sum \mu^{\#\text{loops}} z^{\text{length}} = \sum x^{\#\text{turns}} x^{\text{length}}.$$

For instance, in the center of the previous diagram, the number of oriented loops is zero because there is one clockwise loop and one counterclockwise loop. The weight of each of these loops is μ and μ^{-1}, respectively. Each loop is counted according to the power of its length.

Therefore, the weight for each loop depends on whether it is oriented clockwise or counterclockwise. This approach allows the loops to be counted based on their orientations, providing a clear method to handle the nonlocality issue.

The term λ raised to the power of the number of turns corresponds to x raised to the power of the length. Essentially, in the left-hand diagram, whenever a left turn occurs, the weight λ is assigned, and for a right turn, the weight $\overline{\lambda}$ is assigned.

Now, consider the effect of completing a cycle. The total turn of the cycle is $360°$. In steps of $60°$, the total turn is either $+6$ or -6, as left and right turns cancel out λ and $\overline{\lambda}$. If it is chosen that $\lambda^6 = \mu$, then this weight is the same, but it locally counts the number of left turns and right turns.

Thus, half of the diagram can be covered and the result fully observed. The same process can be applied to the right-hand diagram. The only difference is that on the right, we now consider the left turn λ, the right turn $\overline{\lambda}$, and $\lambda^4 = \mu$.

This leads to what can be termed a complex loop model. The partition function is then given by:

$$Z_{\mathbb{C}} = \sum u^{\#\text{oriented loops}}.$$

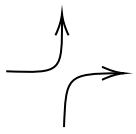
What is gained in this formulation is a partition function that is locally defined, which is a more convenient structure. However, as is often the case, something is lost in the process. Specifically, instead of a probability measure, a complex partition function is obtained. Nevertheless, by neglecting the orientation, we project to the previous case. Thus, while the total mass of the complex measure remains the same, its variation grows exponentially with the volume. As a result, the measure has infinite total variation but unit total mass. Due to significant cancellations, it becomes challenging to make accurate estimates.

There are additional challenges as well. For example, $\mathbb{E}\overline{f} \neq \overline{\mathbb{E}f}$. Instead, the expectation of $\overline{f}$ is given by $\mathbb{E}_{\overline{\mu}}\overline{f} = \overline{\mathbb{E}_{\mu} f}$.

The expectation of $\overline{f}$ no longer equals the expectation of $\overline{f}$, which is typically true for integrals with respect to this measure. The integral of $\overline{f}$ is not equal to the complex conjugate of the integral due to the complex nature of the measure. Consequently, the expectation with respect to the measure $\overline{f}$ must be written explicitly, which introduces additional complexity. This is a subtle point that can lead to repeated errors, especially in the early stages of understanding the formalism.

On the right side of the previous image, another projection is presented where the loops are forgotten, but the arrows are preserved. In this case, the structure of the loops is ignored, leaving only the arrows. This projection results in a diagram that can be analyzed further.

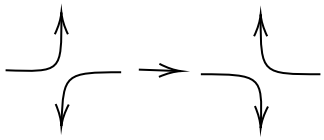
Interestingly, there are eight distinct ways in which the loops connect at a point. One of these configurations, repeated four times, is oriented northeast.



This is one of the four ways to do it in southwest, southeast, northwest, or northeast directions.

What is the weight of this? There is one left turn and one right turn, λ and $\overline{\lambda}$, which cancel out, so $\lambda \times \overline{\lambda} = 1$. Therefore, the weight of this vertex is 1 because there is one right and one left, and no other way to split the turns.

But then, there are also two other configurations where the turns come from sideways and go vertically. When we forget the turns, it becomes the same picture. In the left picture, there are two left turns, so the weight is λ^2. In the right picture, there are two right turns, so the weight is $\overline{\lambda}^2$.



So, while both weights are complex, the total weight is given by $\lambda^2 + \overline{\lambda}^2 = c_{6M}$, which is real because λ is a unitary number, where c_{6M} is the 6-model constant.

Typically, we denote

$$\lambda\overline{\lambda} = 1 = a, b$$

and

$$\lambda^2 + \overline{\lambda}^2 = c,$$

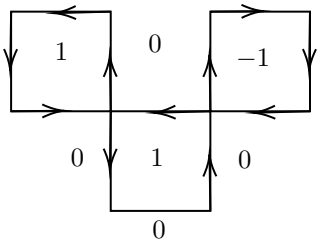
which corresponds to the well-known six-vertex model.

In the particular case of the square lattice, there is a projection that ensures the weights remain real, resulting in a probability measure. However, there is an important caveat: we lose some information. Specifically, the boundary values no longer match. In the model on the right, for instance, the interior values are real, but the boundary values are complex. The corner on the left represents a left turn, indicating a complex boundary value, which introduces a potential issue. This is one of the reasons why this model has posed challenges in the past.

As mentioned earlier, we should clarify this issue. We encountered significant difficulties with it, but there is an interesting earlier piece of work that indirectly touches on Conformal Field Theory. This work, by two Dutch scientists, Nienhuis and den Nijs, from 1982, is commonly referred to as the Coulomb-gas model.

Their approach began with the following reasoning: suppose we have this model and apply a bijection. Now, we have a set of lines with a real measure. When we place arrows on these lines, we can interpret them as a geographical map. Consider a map where every 10 meters or 100 meters we have Lebesgue curves. To determine whether we are climbing or descending, we need to orient them. If the arrows point counterclockwise, we are climbing; if they point clockwise, we are descending.

This orientation allows us to define a height function. A key property of the height function, H, for instance:



We move from 0 to 1 if we cross on the right and from 0 to -1 if we cross on the left. With that in mind, we can make a few arguments. Everything we've said so far is not a proof; these are insightful physics arguments from brilliant minds. They posited that there should be a scaling limit when we take the mesh of the lattice to zero. Thus, we have a random height function with some measure applied to it. Even though the measure is real, there should still be a limit. The authors argued that when we take the height function H_{mesh} as the mesh approaches zero, $H_{\text{mesh}} \to h$, we obtain a random height function. The partition function can then be written as:

$$Z_h = \exp\left(-\int dx\, dy\, \frac{g}{4\pi}|\nabla h|^2 + |\nabla h|^4\right)$$

We expect everything to be rotationally invariant because we are working with a renormalization group. It's clear that it's rotationally invariant with respect to 90°. However, if we perform a diffusion-limited aggregation, we still observe the structure, even though it's a physical model like this one. The system is rotationally invariant, so everything depends on this principle, with some coefficients, one of which we denote as $\frac{g}{4\pi}$.

We'll return later to explain why there's a π here, but first, there is an argument suggesting that we should not include any of these terms. The argument is that we should first examine the simplest possible case. On the other hand, if we compare four nearby points, we can recover the bar or a by making a strong effort. This might lead to higher-order operators, which somewhat makes sense.

Now, what happens is that the parameter $\frac{g}{4\pi}$ changes things. When we rescale, it's no longer a scale-free model. As we've already stated, with each lattice step, we jump by $+1$. In their paper, they actually use $+\pi$, employing a slightly different normalization. Thus, g should differ accordingly.

What is remarkable in their paper is that they write this form and make this ansatz. This object is called the Gaussian free field (or free boson) in the physics literature. It is conformally invariant, well-studied, and is the two-dimensional analog of the Brownian graph on a line. Unfortunately, it's a random distribution, and mathematicians usually avoid taking exponents of random distributions.

At its core, this boils down to calculating the dimension of critical exponents. This calculation involves the Gaussian free field, an object that mathematicians

know how to handle. The only remaining question is determining g. Once we calculate one quantity, which we already know from other methods, we can then deduce g and calculate other quantities. This approach predates BPZ.

19.4 Conformal Invariance

Smirnov's primary interest lies in the foundational aspects of conformal field theory, where the simplest theories are situated. This field is particularly exciting to him due to his background in complex analysis and the way it connects to elegant discrete structures.

Naturally, one may pose the same questions: why and how? Specifically, in their papers, they consistently focus on cylinders or tori. The reason for this lies in the complexity of handling intricate boundary conditions. A few years ago, two weeks were spent with Cardy in an attempt to solve this problem, but the challenge remained unresolved. Today, however, an attempt will be made to explain a potential approach to addressing it. It is hoped that, with time, greater insight has been gained. This approach is applicable to any Riemann surface.

An intriguing claim follows: the same logic is applied when a complex measure is involved. It is asserted that this measure is a complex-valued field, which, in the limiting case, becomes real-valued, taking values on the real line. However, even when this is executed correctly, the imaginary component persists.

So, let's ask: How do we achieve conformality? Suppose we want to show that a discrete object has a conformally invariant limit. How do we proceed?

There are many approaches to conformal invariance: extremal lengths, harmonic functions, and others. Most of them can be discretized, but the problem lies in the fact that they are defined by global conditions, making it difficult to verify that certain properties hold for a model.

The central charge C, which differs from the constant c in our 6-vertex model, ranges from -2 to 1. Specifically, -2 corresponds to a uniform spinning tree. Let me show you the uniform spinning tree. Here's the uniform spinning tree. Meanwhile, 1 corresponds to the double dimer model, or the model where $XY = 1$ and Ising is $\frac{1}{2}$.

We have:

$$g = 1 + \frac{\epsilon_0}{\pi} = \frac{4}{\kappa}.$$

From this, we can deduce a quadratic relationship with c. Interestingly, the two occurrences of c coincide, but this raises an important question: why does c map to the GFF (Gaussian Free Field)? This is not how it is typically expected to behave.

In many cases, people introduce the concept of screening charges. Some have even argued that if you consider a loop wrapping around a donut, the total

rotation is not 360° but 0°. Therefore, its weight should be 1 rather than μ. This argument resembles the idea of a screening charge that modifies the central charge. However, this interpretation is not entirely accurate. It's more of a contribution to the system, rather than a straightforward replacement. After 15 years of searching for an answer, Smirnov presents a potential explanation.

The easiest starting point is to focus on conformal invariance, which applies to holomorphic or harmonic functions subject to given boundary conditions. The Dirichlet problem for the Laplace equation is conformally invariant. In two dimensions, we can map the domain to a disk or a half-plane, where there is an exact formula for the Poisson kernel. We can solve the problem in this domain and then map the solution back to the original geometry.

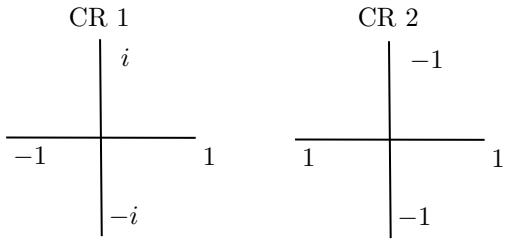
This method works for other domains as well, and we can either prescribe the real part of the solution or use more sophisticated techniques. The beauty of this approach is its ease of discretization, especially for harmonic functions.

A function is harmonic if, at every point on a graph, its value is the average of its neighboring values. Discretizing harmonic functions is straightforward, but unfortunately, it is not canonical. For example, if we define a harmonic function as the average of all first-order neighbors or all second-order neighbors, these are different definitions that lead to distinct function spaces. However, in the limit, both definitions converge to the same operator, denoted by Δ, which is often referred to as the Laplacian. This is an example of universality: when we take the limit of a random walk, it behaves like Brownian motion in two dimensions.

There are only four parameters involved in this description: the speed (which is a metric for correlations) and whether there is drift. In two dimensions, we focus on this scenario, but Smirnov plan to extend the discussion to three dimensions when he publishes this work.

Typically, when we express the Cauchy-Riemann equations, we use complex derivatives. In simpler cases, we might write something like $u_x = -v_y$ and $u_y = v_x$. There could be a sign mistake here, but this formulation is equivalent to stating that we have a flow that is both divergence-free and curl-free. We have seen these differential forms before: one-forms, two-forms—and their relations to the operators we are working with.

The beauty of two dimensions lies in its special properties when dealing with analytic functions. However, we can approach these problems in different ways. For instance, in the Ising model, we used a method where we summed the function values at four points with coefficients 1, i, -1, and $-i$ to construct an operator. This is one of the two types of operators we commonly use. Upon closer inspection, we find that if a function is defined on edges and has one relation per vertex, we end up with twice as many unknowns as there are relations.



The second setup is nicely adapted to the Ising model, and this approach also appears in contexts such as random surfaces and dimers. Morally speaking, if we find something that satisfies the CR1 properties, we would assume that it's holomorphic. This assumption hinges on the belief that the renormalization group flow exists and that a well-defined limit of the system can be found. While the relation alone doesn't constitute a proof, it serves as a strong indication.

Let me add one more important point: boundary value problems. There's a key insight here: covariance is equivalent to the boundary value problem.

The boundary values of a function are closely tied to its behavior at the edges of a domain. For example, consider the relation $M(z) = N(\varphi(z))$. Here, we have two domains connected by a map φ, and we're interested in how the fields or observables transform from one domain to the other. The crucial step in solving this is the preservation of boundary values. Once boundary values are preserved, we can map everything onto a well-behaved model—such as a disk—and this ensures invariance.

These relations can also be manipulated further. For instance, we might consider:

$$M(z) = N(\varphi(z))\varphi'(z),$$

where σ is a parameter, and examine how this alters boundary behavior. In the Ising model, this is directly related to analytic functions. Specifically, the transformation of boundary values follows a particular rule under such mappings. In one instance, Smirnov observed that a certain analytic function in the Ising model behaves like $\sqrt{dz}$, which naturally leads to a boundary value problem in which the boundary values transform according to a square root relation. In this case, $M \parallel dz^6$.

But there are additional covariances to consider:

$$M(z) = N(\varphi(z)) + \alpha \log \varphi'(z).$$

This expression suggests the existence of an invariant quantity with a small complex twist. For example, starting with a real boundary value on a disk, the mapping introduces a complex component. You begin with a real Gaussian Free Field (GFF), but then, oops, you're compelled to incorporate complex elements. From there, one can differentiate, leading to further structure, again connected

to the GFF. Consider the form:

$$M(z) = N(\varphi(z)) \cot \varphi'(z) + \beta \frac{\varphi''(z)}{\varphi'(z)}.$$

This essentially introduces a metric element and related geometric considerations.

If you're interested in working with the GFF, this expression captures the gradient of the field. Another important formulation is:

$$M(z) = N(\varphi(z))\varphi'(z)^2 + \gamma S\varphi(z),$$

where

$$S\varphi(z) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2$$

is the Schwarzian derivative.

This is a particularly interesting exercise because these constructions are essentially the only ones that satisfy the chain rule. If you want to express something that respects the chain rule, it must be in this form. Once you begin applying the chain rule, other related structures begin to emerge—but within this framework, this is essentially the only way it works.

Now, let's return to what we had achieved earlier. We were able to formulate observables for the Ising model, percolation, and other systems. These observables fall within this general class and are distinguished by their spins: for percolation, we had spin $\frac{1}{2}$, while in quantum field theory, logarithmic terms appeared.

Turning now to the six-vertex model, we encounter something remarkable: by using the same parameters for two different models, we observe different outcomes when applying the same observations. The reason is subtle—complex terms play a hidden role in one case. As a result, part of the "spin" is concealed within those complex contributions.

19.5 Parafermions

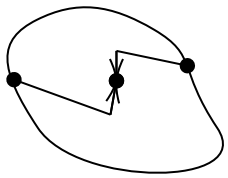
We consider the six-vertex model, characterized by the parameter

$$c = \sqrt{u} + \sqrt{\overline{\mu}} = \lambda^2 + \overline{\lambda}^2.$$

Our approach is as follows: we begin with our six-vertex model subject to certain boundary conditions. Suppose we introduce a disorder operator. Intuitively, this corresponds to identifying an edge in the six-vertex model where there is a "hole in the floor": the plumbing is faulty, and everything is draining through this point.

Next, we introduce two sources on the boundary:

- Two half-sources, each pumping 1 gallon per minute from the boundary;
- One double sink, which extracts 2 gallons of water per minute.



We then examine whether this setup satisfies the Cauchy–Riemann relations via the partition function:

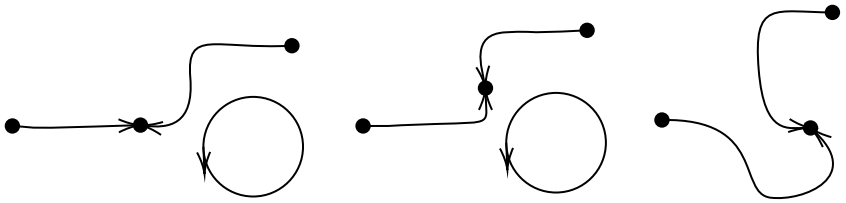
$$F(z) = Z \oplus \nu^{\text{winding}}.$$

The motivation behind this construction lies in the special nature of these curves — they determine how the "water" flows into the system. The horizontal curves, in particular, are easier to control as we vary the position of the point. Moreover, this curve is uniquely defined: for any given configuration, one can uniquely trace this specific curve.

The remaining curves, by contrast, are not uniquely traceable — they may follow different paths depending on the configuration. However, regardless of how they are traced, they all share the same winding number.

Exercise 19.1. *Show that $F(z)$ is well defined.*

We may now proceed to sketch possible configurations in a neighborhood of a given point. Suppose we fix such a point. Then, figures might look like:



If we want to satisfy the Cauchy-Riemann equations, we must solve

$$-1 \cdot 1 + i \cdot c \cdot \nu^2 + 1 \cdot 1 \cdot \nu^4 = 0.$$

The following lemma tells us that there exists a solution.

Lemma 19.2. *If $c = \sqrt{\mu} + \sqrt{\overline{\mu}}$, then*

$$(\nu^2 + i\sqrt{\mu})(\nu^2 + i\sqrt{\overline{\mu}}) = 0$$

so $\nu^2 = i\sqrt{\mu}$ gives holomorphicity.

If we take the two pictures:

$h = 0$ $\qquad\qquad$ $h = 0$

They have weights 1 and v^8. However, the height functions are:

$$1 = \rho^{h(z)}$$

and

$$v^8 = \rho^{h(z)}.$$

We should check that:

$$\rho^2 = v^8.$$

Of course, the situation is slightly more complicated because we must also account for cases where things are rotated by 90 degrees. So, strictly speaking:

Lemma 19.3.

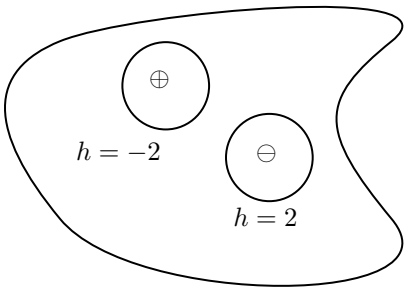
$$f(z) = \sum \oplus \cdot \rho^{h(z)} \cdot c(\textit{type } z)$$

where $c = 1$ *or* $c = \sqrt{\rho}$.

Now, what's the advantage of this approach?

- First, we can handle any boundary conditions because earlier, we referenced a diagram with two sources and a unique connection. This allows for local reconstructions without any issues.
- Additionally, we should be able to manage the operations $\ominus\rho^h$ and $\oplus\rho^{-h}$.
- $h(z)$ is defined up to monodromy.

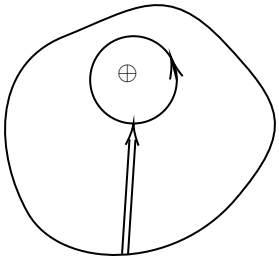
For instance, if we have an operator with both a plus and minus sign, when we traverse around the minus operator, two curves end up at it. As we move around, my height increases by +2. Thus, my height in this domain is not well-defined. This is analogous to a geographic map where two contour lines meet at a pole. Walking around the pole, you might be 2 meters higher on one side, but 2 meters lower on the other side.



So, what's the trick here? The trick is that the height function is situated in the ground, so that when we walk around it, we experience a height change of +2 meters on one side and -2 meters on the other. This occurs because the height function moves in the opposite direction on the other side. To make this situation valid, there's an easy way to handle it.

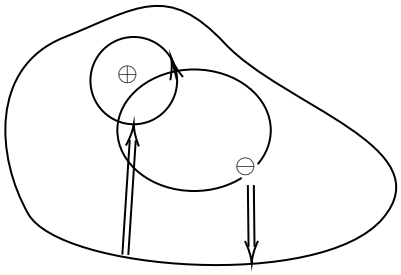
One way to make this legal is to introduce cross-cuts, which is a common technique when studying Riemann surfaces. When you think about it, you can define the function independently of the cross-cuts. You can move them later, but during calculations, it's important to fix the cross-cuts in place. However, keep in mind that walking around them causes the height function to change. This is actually a good thing because every complex analyst loves monodromy.

So, what does this mean? Let's clarify with an example. Suppose we have a plus operator, and we rotate it by 360°. What will happen is that it will become tangled with its own cross-cut. When we rotate it, it will cross its own cut.



The result of rotating by 360° is that we multiply by ρ^2. Similarly, if you rotate by $-360°$, we also multiply by ρ^2. Why? Because we cross a line moving in the opposite direction, so the height doesn't decrease: it increases. However, your weight (or height) is also negative, so the multiplication is the same.

Now, here's the interesting part. Suppose you move a plus operator around a minus operator. What happens?



When we move the plus operator around the minus operator, the minus operator intersects the line of the plus operator. These operators remain horizontal. Thus, when we move them, the line is intersected, leading to multiplication by ρ^{-2}. Furthermore, the plus operator crosses the line of the minus operator, which introduces an additional factor of ρ^{-2}. Consequently, the total multiplication becomes ρ^{-4}.

Now, let us consider two points, u and v, and examine how $F(u, v)$ behaves given boundary conditions. One assumption is that there exists a model Γ to which we map ρ. A useful observation here is that since we multiply by ρ, and since $\rho^2 = e^{2\pi i\sigma}$ and $\rho^{-4} = e^{-4\pi i\sigma}$, we can express $F(u, v)$ as:

$$F(u, v) = \frac{\varphi'(u)^6\varphi'(v)^\sigma}{(\varphi(u) - \varphi(v))^{2\sigma}}$$

Notably, if $\sigma = \frac{1}{2}$, there is no monodromy.

Finally, consider the following interesting scenario: Suppose we send v to u and investigate where this expression converges. This is straightforward to calculate. The expression will indeed converge, so assume $v = u + \epsilon$. Upon normalizing ϵ, we observe that it results in $\frac{1}{\epsilon^{2\sigma}}$. Typically, we would expect a gradient, leading to the following form:

$$\frac{1}{\epsilon^{2\sigma}}\left[1 + \frac{\epsilon^2\sigma}{\sigma}S\varphi\right]$$

Here, $\frac{\sigma}{\sigma} = \frac{C}{12}$, where C denotes the central charge.

19.6 Conclusion

There are two main takeaways:

- Covariance and Boundary Value Problems: Covariance is equivalent to the boundary value problem. Starting with a Gaussian free field on a disk, when this field is mapped to another domain and different covariances are imposed, the results can vary significantly.

  For different values of the central charge, the Gaussian free field satisfies the following form:

$$M(z) = N(\varphi(z)) + \alpha \log \varphi'(z)$$

where α is a function of the central charge. This relationship has already been observed in the context of random surfaces, as discussed in the work of Scott Sheffield.

- Monodromies in Potential Theory: Monodromies are a powerful tool for differentiation. Essentially, they act as the Green's function for the $\overline{\partial}$ operator, but with monodromy. This concept has not been previously encountered in the context of potential theory.

20 Grigori Olshanski: Macdonald-Level Extension of Beta Ensembles and Multivariate Hypergeometric Polynomials

Abstract

A beta ensemble (or log-gas system) on the real line is a random collection of N point particles $x_1, ..., x_N$ whose joint probability distribution has a special form containing the Vandermonde raised to the power $\beta > 0$. I will survey results related to some discrete analogs of beta ensembles, which live on q-lattices, and large-N limit transitions.

Contents

20.1 Introduction

In this section, I will describe several models of random particle systems. These models originate from the representation theory of infinite-dimensional classical groups, though my focus will be on the algebraic-combinatorial aspects of the theory.

To begin, let us define some basic terminology:

A collection X of N points on the real line $\mathbb{R}$ is referred to as an N-particle configuration. The space of all N-particle configurations is denoted by $\mathrm{Conf}_N(\mathbb{R})$. An N-particle ensemble on $\mathbb{R}$ is specified by a probability measure M on $\mathrm{Conf}_N(\mathbb{R})$, which allows us to discuss random configurations.

The central problem to be addressed in the context of concrete models is: how can we construct ensembles containing infinitely many particles? This is a nontrivial issue, as dealing with probability measures on "large" spaces, such as $\mathrm{Conf}_\infty(\mathbb{R})$, is generally quite challenging. One possible approach is to consider a large-N limit transition.

20.2 Preliminaries: Discrete beta-ensembles on $\mathbb{Z}$ of representation-theoretic origin

20.2.1 Dyson's circular beta-ensembles

Let $\mathbb{T} \subset \mathbb{C}$ denote the unit circle centered at 0. We define $\mathrm{Conf}_N(\mathbb{T})$ as the set of N-particle configurations $(u_1, \dots, u_N)$ on $\mathbb{T}$. Let $\mathrm{Prob}(\cdot)$ represent the set of probability measures on a given space. We are interested in probability measures $M_N \in \mathrm{Prob}(\mathrm{Conf}_N(\mathbb{T}))$. Given such a measure, we may refer to an ensemble of random N-particle configurations on $\mathbb{T}$.

Let $\beta > 0$ be a parameter. The N-particle Dyson's circular beta-ensemble is given by the probability measure

$$M_{N,\beta}(du) := \frac{1}{C_{N,\beta}} \prod_{1 \le i < j \le N} |u_i - u_j|^\beta \cdot \mu_{\mathbb{T}^N}(du),$$

where $u = (u_1, \dots, u_N) \in \mathrm{Conf}_N(\mathbb{T})$, $\mu_{\mathbb{T}^N}(du)$ is the Lebesgue measure on the torus $\mathbb{T}^N = \mathbb{T} \times \cdots \times \mathbb{T}$, and $C_{N,\beta}$ is the normalization constant.

This concept originates from Dyson *J. Math. Phys. 1962.*

The origin. For the three special values $\beta = 1, 2, 4$ (corresponding to $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$), the Dyson ensembles admit a simple matrix/Lie group interpretation. Specifically, we consider three infinite series of compact symmetric spaces $G(N)/K(N)$:

$$U(N)/O(N), \quad U(N) \times U(N)/\mathrm{diag}\, U(N), \quad U(2N)/Sp(N).$$

Consider the double cosets $K(N)gK(N)$, where $g \in G(N)$. In the case $\beta = 2$, the double cosets correspond to conjugacy classes in $U(N)$.

In all three cases, the double cosets are parametrized by configurations $u = (u_1, \dots, u_N) \in \mathrm{Conf}_N(\mathbb{T})$.

Thus, we have a natural projection

$$G(N) \to K(N)\backslash G(N)/K(N) = \mathrm{Conf}_N(\mathbb{T}).$$

It turns out that the pushforward, under this projection, of the normalized Haar measure is exactly $M_{N,\beta}$ for $\beta = 1, 2, 4$ (a particular case of Élie Cartan's formula).

For general $\beta > 0$, the situation can be viewed as an extrapolation.

20.2.2 Dual picture: problem of harmonic analysis for ∞-dimensional symmetric spaces

The "dual picture" refers to the consideration of the Hilbert space $\mathcal{L}^2\left(\frac{G(N)}{K(N)}\right)$ instead of the compact symmetric space $G(N)/K(N)$, along with the natural unitary representation T_N of the group $G(N)$ on this space.

Its decomposition is well-known: this is an example of a (relatively simple) problem of spherical noncommutative harmonic analysis.

Problem 20.1 ($\beta = 1, 2, 4$)**.**

1. *Is it possible to give a sense to the large-N limit*

$$T_\infty := \lim_{N\to\infty} T_N$$

   as a unitary representation of the direct limit group $G(\infty) := \bigcup_N G(N)$?

2. *How can we decompose T_∞ into irreducibles (harmonic analysis)?*

The second part is nontrivial because there is no invariant measure on $G(\infty)/K(\infty)$, meaning that one cannot extend the definition of $\mathcal{L}^2(G(N)/K(N))$ directly.

Answer:

1. Yes, the limit representation T_∞ can be defined. It turns out that its construction involves additional continuous parameters (which is not a defect but a bonus!).
2. The decomposition of T_∞ into irreducibles is governed by an ensemble with infinitely many particles on $\mathbb{R}$.
3. This ensemble is obtained as the large-N limit of certain discrete ensembles on the lattice $\mathbb{Z}$, which resemble a discrete analog of Dyson beta ensembles.
4. Moreover, the entire construction admits a purely combinatorial interpretation, which is valid for all $\beta > 0$.

Thus, we can reformulate the problem of harmonic analysis for infinite-dimensional symmetric spaces as a problem of algebraic combinatorics with a slight probabilistic flavor.

20.2.3 Discrete beta-ensembles of the lattice $\mathbb{Z}$

In what follows, we assume

$$\beta = 2\tau, \quad \tau \in \{1, 2, 3, \dots\},$$

so that β is a positive even integer. This assumption is made for simplicity; the results hold for any $\beta > 0$.

The discrete ensembles in question live on the lattice $\mathbb{Z}$ (which is dual to the circle $\mathbb{T}$!). Thus, instead of the continuous space $\mathrm{Conf}_N(\mathbb{T})$, we are now dealing with the countable set $\mathrm{Conf}_N(\mathbb{Z})$. Its elements are N-particle configurations on $\mathbb{Z}$:

$$\mathscr{L} = (\ell_1 > \cdots > \ell_N) \subset \mathbb{Z}.$$

These are, in fact, veiled highest weights $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$:

$$\lambda \to \mathscr{L}, \quad \ell_i = \lambda_i + (N-i)\tau, \quad 1 \leq i \leq N.$$

We now introduce a probability measure

$$M_{N,2\tau}^{z,z',w,w'} \in \mathrm{Prob}(\mathrm{Conf}_N(\mathbb{Z})), \qquad \tau \in \{1, 2, 3, \dots\}.$$

Here, (z, z', w, w') is a quadruple of continuous parameters subject to some constraints. For instance, sufficient conditions are

$$z, w \in \mathbb{C}, \qquad z' = \overline{z}, \qquad w' = \overline{w}, \qquad \mathrm{Re}(z+w) > -\frac{1}{2}.$$

Definition 20.2 (Probability Measure on $\mathrm{Conf}_N(\mathbb{Z})$)**.** *We define the weight of a configuration $\mathscr{L} = (\ell_1, \dots, \ell_N) \in Conf_N(\mathbb{Z})$ as*

$$M_{N,2\tau}^{z,z',w,w'}(\mathscr{L}) := \frac{1}{C_{N,2\tau}} \prod_{i=1}^{N} F_N(\ell_i) \cdot V_{N,2\tau}(\mathscr{L}),$$

where

$$F_N(\ell) := \frac{\Gamma(-z-(N-1)\tau+\ell)\Gamma(-z'-(N-1)\tau+\ell)}{\Gamma(w+\ell+1)\Gamma(w'+\ell+1)}, \quad \ell \in \mathbb{Z}$$

and

$$V_{N,2\tau}(\mathscr{L}) := \prod_{1 \leq i < j \leq N} (\ell_i - \ell_j)^2 \times \left| \prod_{1 \leq i \neq j \leq N} (\ell_i - \ell_j - 1)(\ell_i - \ell_j - 2) \cdots (\ell_i - \ell_j - (\tau - 1)) \right|.$$

Remark 20.3.

1. *For $\tau = 1$, the second double product disappears.*

2. *If $\tau > 1$, $V_{N,2\tau}(\mathscr{L})$ vanishes whenever $\ell_i - \ell_{i+1} < \tau - 1$. Thus, the measure $M_{N,2\tau}^{z,z',w,w'}$ lives on the subset of τ-sparse configurations: any two particles are separated by at least $\tau - 1$ holes.*

3. *For large distances between the ℓ_i's,*

$$V_{N,2\tau}(\mathscr{L}) \approx \prod_{i<j} (\ell_i - \ell_j)^{2\tau}.$$

4. *As in Dyson's context, $V_{N,2\tau}(\mathscr{L})$ is responsible for pair interactions between the particles, of the log-gas type. Only now, we have a lattice model.*

Theorem 20.4. *Let the parameters τ and z, z', w, w' be fixed. After a scaling and yet another transformation, the measures $M_{N,2\tau}^{z,z',w,w'}$ converge, as $N \to \infty$, to a probability measure $M_{\infty,2\tau}^{z,z',w,w'}$ that lives on a space of infinite particle configurations on the real line $\mathbb{R}$.*

20.3 Macdonald-level hypergeometric ensembles

20.3.1 Notation

Denote q and t as the two parameters of Macdonald polynomials. We assume:

$$0 < q < 1, \quad 0 < t < 1, \quad t = q^\tau, \quad \tau = 1, 2, 3, \dots$$

The last assumption is made for simplicity.

Now, we add two additional parameters, $\zeta_\pm$, satisfying

$$\zeta_- < 0 < \zeta_+.$$

The two-sided q-lattice $\mathbb{L} = \mathbb{L}_- \cup \mathbb{L}_+ \subset \mathbb{R}$ is defined as:

$$\mathbb{L}_- := \zeta_- q^{\mathbb{Z}} = \{\zeta_- q^m : m \in \mathbb{Z}\}, \quad \mathbb{L}_+ := \zeta_+ q^{\mathbb{Z}} = \{\zeta_+ q^m; m \in \mathbb{Z}\}.$$

Since $0 < q < 1$, the lattice nodes accumulate near 0 and diverge in the direction of $\pm\infty$:

$$\dots \quad \zeta_- q^{-1} \quad \zeta_- \quad \zeta_- q \quad \dots \quad \zeta_+ q \quad \zeta_+ \quad \zeta_+ q \quad \dots$$

Finally, let $\mathrm{Conf}_N(\mathbb{L})$ is the set of N-particle configurations on $\mathbb{L}$, where the configurations are τ-sparse (this means that any two particles are separated by at least $\tau - 1$ holes).

20.3.2 N-particle hypergeometric ensembles

Below we use a standard notation from q-calculus:

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad x \in \mathbb{C}.$$

It is closely related to the notion of the q-Gamma function:

$$\Gamma_q(A) := \frac{(q;q)_\infty}{(q^A;q)_\infty} \cdot (1-q)^{1-A}.$$

We fix a quadruple $(\alpha, \beta, \gamma, \delta)$ of parameters, subject to the conditions specified below. The following formula defines a "hypergeometric" probability measure $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}$ on the set $\mathrm{Conf}_N(\mathbb{L})$.

Definition 20.5. *If $X = (x_1 > \cdots > x_N) \in Conf_N(\mathbb{L})$, then the measure is given by:*

$$M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(X) := \frac{1}{C(N;q,t;\alpha,\beta,\gamma,\delta)} \prod_{i=1}^{N} F^{\alpha,\beta,\gamma,\delta}_{N;q,t}(x_i) \cdot V_{N;q,t}(X),$$

where $C(N;q,t;\alpha,\beta,\gamma,\delta)$ is a normalization constant, and

$$V_{N;q,t}(X) := \prod_{1\le i\ne j\le N} \prod_{r=0}^{\tau-1} |x_i - x_j q^r|,$$

is a (q,t)-analog of $\prod_{i<j}(x_i - x_j)^{2\tau}$ (or rather its version on $\mathbb{Z}$). Also,

$$F^{\alpha,\beta,\gamma,\delta}_{N;q,t}(x) := (1-q)^{|x|} \frac{(\alpha x;q)_\infty (\beta x;q)_\infty}{(\gamma t^{1-N} x;q)_\infty (\delta t^{1-N} x;q)_\infty}, \quad x \in \mathbb{L},$$

is a (q,t)-analog of the previously defined function:

$$F_N(\ell) = \frac{\Gamma(-z-(N-1)\tau+\ell)\Gamma(-z'-(N-1)\tau+\ell)}{\Gamma(w+\ell+1)\Gamma(w'+\ell+1)}, \quad \ell \in \mathbb{Z}.$$

The parameters $(\alpha, \beta, \gamma, \delta)$ should satisfy the following conditions for each N:

- $F^{\alpha,\beta,\gamma,\delta}_{N;q,t}(x) \ge 0$ for any $x \in \mathbb{L}$,
- the normalization exists.

20.3.3 Conditions on $(\alpha, \beta, \gamma, \delta)$: principal and degenerate series

There are two variants of sufficient conditions on $(\alpha, \beta, \gamma, \delta)$ that guarantee the well-definedness of the hypergeometric measures for all N:

1. Principal series:

$$\alpha = \overline{\beta} \in \mathbb{C} \setminus \mathbb{R}, \quad \gamma = \overline{\delta} \in \mathbb{C} \setminus \mathbb{R}, \quad \alpha\beta < \gamma\delta q.$$

   In this case, $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(X) > 0$ for all $X \in \mathrm{Conf}_N(\mathbb{L})$.

2. Degenerate series:

$$\beta < 0 < \alpha, \quad \alpha^{-1} \in \mathbb{L}_+, \quad \beta^{-1} \in \mathbb{L}_-, \quad \gamma = \overline{\delta} \in \mathbb{C} \setminus \mathbb{R}.$$

Here, $M_{N;q,t}^{\alpha,\beta,\gamma,\delta}(X) > 0$ only for configurations X contained in the truncated lattice

$$L[\beta^{-1}q, \alpha^{-1}q] := \{x \in \mathbb{L} : \beta^{-1}q \le x \le \alpha^{-1}q\}.$$

This occurs because the conditions $\alpha^{-1} \in \mathbb{L}_+$ and $\beta^{-1} \in \mathbb{L}_-$ imply that the product $(\alpha x; q)_\infty (\beta x; q)_\infty$ vanishes for all $x \in \mathbb{L}$ outside the lattice interval $[\beta^{-1}q, \alpha^{-1}q]$.

20.3.4 Large-N limit

Let $\mathrm{Conf}_\infty(L)$ denote the set of particle configurations $X \subset L$ such that:

- $|X| = \infty$,
- X is bounded away from $\pm\infty$,
- If $\tau > 1$, then X is τ-sparse.

Note that the spaces $\mathrm{Conf}_N(\mathbb{L})$ are countable, while the space $\mathrm{Conf}_\infty(L)$ has the cardinality of the continuum. It is a totally disconnected topological space.

Here is our main result:

Theorem 20.6. *Let $(\alpha, \beta, \gamma, \delta)$ be in the principal or degenerate series. We still assume $t = q^\tau$ with $\tau \in \{1, 2, 3, \dots\}$. Then there exists a limit:*

$$\lim_{N\to\infty} M_{N;q,t}^{\alpha,\beta,\gamma,\delta} = M_{\infty;q,t}^{\alpha,\beta,\gamma,\delta} \in \mathit{Prob}(\mathit{Conf}_\infty(L)).$$

In other words, there exists a limit probability measure that determines a particle ensemble on the two-sided q-lattice L, with infinitely many particles accumulating at $0 \notin L$.

A similar claim holds for any $t \in (0, 1)$, although the description of the configurations becomes more involved.

20.3.5 The Special Case $\tau = 1$ (i.e., $q = t$)

In this case, more can be said.

Theorem 20.7. *If $q = t$, then the limit ensemble on L defined by the measure $M_{\infty;q,q}^{\alpha,\beta,\gamma,\delta}$ is determinantal, meaning its correlation functions are:*

$$\rho_n(x_1, \dots, x_n) = \det\left[K_q^{\alpha,\beta,\gamma,\delta}(x_i, x_j)\right]_{1\le i,j\le n},$$

where $K_q^{\alpha,\beta,\gamma,\delta}(x, y)$ is a kernel on $L \times L$, independent of n.

This kernel has the explicit form:

$$K_{\alpha,\beta,\gamma,\delta,q}(x,y) = \frac{A(x)B(y) - A(y)B(x)}{x-y}, \quad x,y \in L,$$

where $A(x)$ and $B(x)$ are certain functions on L, expressed through the q-hypergeometric function ${}_2\phi_1$.

Because the correlation functions are explicitly computable, we conclude that the model with $q = t$ is exactly solvable.

Theorem 20.8. *The limit measure $M^{\alpha,\beta,\gamma,\delta}_{\infty;q,q}$ is diffuse, meaning it has no atoms.*

This kernel admits an explicit expression of the form

$$K^{\alpha,\beta,\gamma,\delta}_{q}(x,y) = \frac{A(x)B(y) - A(y)B(x)}{x-y}, \quad x,y \in L,$$

where $A(x)$ and $B(x)$ are certain functions on L, expressed through the q-hypergeometric function ${}_2\phi_1$.

20.3.6 Degeneration $M^{\alpha,\beta,\gamma,\delta}_{N;q;t} \rightsquigarrow M^{z,z',w,w'}_{N;2\tau}$

Recall that we started with the lattice $\mathbb{Z}$ and proceeded to the two-sided q-lattice L. The formulas in these two cases are similar. Furthermore, for each fixed N, the ensemble on L can be degenerated to the ensemble on $\mathbb{Z}$.

For simplicity, assume $\zeta_\pm = \pm 1$, so that $L = -q\mathbb{Z} \cup q\mathbb{Z}$. There is a natural bijection $\mathbb{Z} \leftrightarrow q\mathbb{Z} : \ell \leftrightarrow q^\ell = x$. Likewise, we have a natural bijection:

$$\mathrm{Conf}_N(\mathbb{Z}) \leftrightarrow \mathrm{Conf}_N(\mathbb{L}^+), \quad \mathscr{L} \leftrightarrow X$$

$$\mathscr{L} = (\ell_1 > \cdots > \ell_N) \leftrightarrow (q^{\ell_1} < \cdots < q^{\ell_N}) = X.$$

Fix a quadruple (z, z_0, w, w_0) from the principal series and set

$$\alpha = q^{w+1}, \quad \beta = q^{w_0+1}, \quad \gamma = q^{-z}, \quad \delta = q^{-z_0}.$$

We now consider the limit regime as $q \nearrow 1$, where the q-lattice $\mathbb{L} \subset \mathbb{R}$ becomes increasingly dense.

Theorem 20.9. *In this limit, the random configurations on $\mathbb{L}$ governed by $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}$ tend to concentrate near the point 1. Specifically, they tend to move away from the negative part $\mathbb{L}^-$ of the lattice. More precisely, for any fixed small $\epsilon > 0$,*

$$\lim_{q \nearrow 1} \sum_{X \subset (1-\epsilon, 1+\epsilon)} M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(X) = 1.$$

Moreover, for any $\mathscr{L} \in \mathit{Conf}_N(\mathbb{Z})$, we have:

$$\lim_{q \nearrow 1} M^{\alpha,\beta,\gamma,\delta}_{N;q,t}(q^{\mathscr{L}}) = M^{z,z_0,w,w_0}_{N,2\tau}(\mathscr{L}).$$

Proof. In this limit, the negative part of the q-lattice becomes negligible. However, within the framework of our approach, one cannot construct a (q,t)-version of discrete beta-ensembles solely on $\mathbb{L}_+$. The two-sided lattice $\mathbb{L}$ appears to be absolutely necessary. □

It would be interesting to find a representation-theoretic interpretation of our construction, at least for the case $q = t$. A natural suggestion would be to work with representations of the quantized algebras $\mathcal{U}_q(\mathfrak{gl}(N), \mathbb{C})$. However, reconciling this algebra with the two-sided lattice $\mathbb{L}$ remains unclear.

20.4 Big q-Jacobi symmetric functions

20.4.1 Big q-Jacobi symmetric polynomials

We focus on the degenerate series of parameters $(\alpha, \beta, \gamma, \delta)$:

$$\beta < 0 < \alpha, \quad \gamma = \overline{\delta} \in \mathbb{C} \setminus \mathbb{R},$$

and consider the truncated q-lattice

$$\mathbb{L}^{\alpha,\beta} := \mathbb{L}[\beta^{-1}q, \alpha^{-1}q] = \beta^{-1}q\mathbb{Z}_{\geq 1} \cup \alpha^{-1}q\mathbb{Z}_{\geq 1}.$$

Let $\mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta})$ denote the set of N-particle configurations of length $\leq N$, and let $P_{\lambda|N}(X; q, t)$ denote the N-variate Macdonald polynomial indexed by λ.

There exists a basis $\{\varphi_{\lambda|N}(X) : \lambda \in \mathbb{Y}(N)\}$ in $\mathrm{Sym}(N)$ such that

$$\varphi_{\lambda|N}(X) = P_{\lambda|N}(X) + \text{lower degree terms}$$

and

$$(\varphi_{\lambda|N}, \varphi_{\mu|N}) = 0, \quad \lambda \neq \mu.$$

The polynomials $\varphi_{\lambda|N}$ are called the N-variate symmetric big q-Jacobi polynomials. In the simplest case $N = 1$, these are the classic univariate big q-Jacobi polynomials discovered by Andrews and Askey [In: Lecture Notes in Math., vol. 1171, 1984].

As above, for $\tau > 1$ we additionally assume that the configurations are τ-sparse.

Let

$$\mathrm{Sym}(N) := \mathbb{R}[x_1, \dots, x_N]^{S_N}$$

denote the $\mathbb{R}$-algebra of symmetric polynomials in N variables. There is a natural embedding

$$\mathrm{Sym}(N) \xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}),$$

$$f \mapsto f(X), \quad X = (x_1, \dots, x_N) \in \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}).$$

Recall that for each N, we have defined a hypergeometric probability measure $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}$ on $\mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta})$. Using the embedding ι, we can realize $\mathrm{Sym}(N)$ as a dense subspace of the Hilbert space

$$\ell^2(\mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}), M^{\alpha,\beta,\gamma,\delta}_{N;q,t}).$$

Let $(\cdot,\cdot)$ denote the induced scalar product in $\mathrm{Sym}(N)$.

Theorem 20.10 (Stokman)**.** *Let λ range over the set $\mathbb{Y}(N)$ of partitions of length $\leq N$ and let $P_{\lambda|N}(X;q,t)$ denote the N-variate Macdonald polynomial indexed by λ. There exists a basis $\{\varphi_{\lambda|N}(X) : \lambda \in \mathbb{Y}(N)\}$ in Sym(N) such that*

$$\varphi_{\lambda|N}(X) = P_{\lambda|N}(X) + \textit{lower degree terms},$$

and

$$(\varphi_{\lambda|N}, \varphi_{\mu|N}) = 0, \quad \lambda \neq \mu.$$

The polynomials $\varphi_{\lambda|N}$ are called the **N-variate symmetric big q-Jacobi polynomials**. In the simplest case $N = 1$, these are the classic univariate big q-Jacobi polynomials discovered by Andrews and Askey in 1984.

20.4.2 "Almost-stable" expansion on Macdonald polynomials

Theorem 20.11. *The expansion of N-variate big q-Jacobi polynomials in the basis of Macdonald polynomials has the form*

$$\varphi_{\lambda|N} = \sum_{\mu:\mu\subseteq\lambda} \frac{(t^N;q,t)_\lambda}{(t^N;q,t)_\mu}\, \pi(\lambda,\mu;q,t;\alpha,\beta,\gamma,\delta)\, P_{\mu|N},$$

where

$$(t^N;q,t)_\lambda := \prod_{i=1}^{l(\lambda)} (t^{N+1-i};q)_{\lambda_i} = \prod_{(i,j)\in\lambda} \left(1 - q^{\lambda_i+j-1}t^{N+1-i}\right),$$

and

$$(t^N;q,t)_\mu := \prod_{i=1}^{l(\mu)} (t^{N+1-i};q)_{\mu_i} = \prod_{(i,j)\in\mu} \left(1 - q^{\mu_i+j-1}t^{N+1-i}\right)$$

are certain products of q-Pochhammer factors. The coefficients $\pi(\lambda,\mu;q,t;\alpha,\beta,\gamma,\delta)$ are specific terms that do not depend on N and have an explicit expression.

We call this expansion **almost stable**, as the dependence on N is localized in the fraction

$$\frac{(t^N;q,t)_\lambda}{(t^N;q,t)_\mu}.$$

The proof (Olshanski, Comm. Math. Phys., 2021) relies on results from Rains [Transf. Groups, 2005] and the theory of interpolation of Macdonald polynomials due to Okounkov, Knop, and Sahi.

20.4.3 Big q-Jacobi symmetric functions

Let

$$\mathbb{Y} = \bigcup_N \mathbb{Y}(N)$$

denote the set of all partitions (Young diagrams), and let

$$\mathrm{Sym} := \varprojlim \mathrm{Sym}(N)$$

denote the $\mathbb{R}$-algebra of symmetric functions.

Recall the embedding

$$\mathrm{Sym}(N) \xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta}),$$

where $f \mapsto f(X)$ with $X = (x_1, \dots, x_N) \in \mathrm{Conf}_N(\mathbb{L}^{\alpha,\beta})$.

Likewise, we have a natural embedding

$$\mathrm{Sym} \xrightarrow{\iota} \text{bounded functions on } \mathrm{Conf}_\infty(L_{\alpha,\beta}),$$

where $F \mapsto F(X)$ with $X = \{x_i\} \in \mathrm{Conf}_\infty(L_{\alpha,\beta})$, and where $\mathrm{Conf}_\infty(L_{\alpha,\beta})$ is the space of ∞-particle τ-sparse configurations on the truncated q-lattice $L_{\alpha,\beta}$.

We can regard Sym as an algebra of bounded functions on the (totally disconnected topological) space $\mathrm{Conf}_\infty(L_{\alpha,\beta})$.

Note that for fixed partitions $\lambda, \mu \in Y$,

$$\lim_{N\to\infty} (t^N; q, t)_\lambda = 1, \quad \lim_{N\to\infty} (t^N; q, t)_\mu = 1.$$

It follows that, as $N \to \infty$, the N-variate big q-Jacobi polynomials converge, in a natural sense, to certain symmetric functions

$$\Phi_\lambda = \Phi(-; q, t; \alpha, \beta, \gamma, \delta) = \sum_{\mu:\mu\subseteq\lambda} \pi(\lambda, \mu; q, t; \alpha, \beta, \gamma, \delta) P_\mu(-; q, t),$$

where the $P_\mu(-; q, t)$, $\mu \in Y$, are the Macdonald symmetric functions.

We refer to the functions $\Phi(-; q, t; \alpha, \beta, \gamma, \delta)$ as the **big q-Jacobi symmetric functions**. We regard them as bounded functions on $\mathrm{Conf}_\infty(L_{\alpha,\beta})$.

By Stokman's theorem, the measures $M^{\alpha,\beta,\gamma,\delta}_{N;q,t}$ are the orthogonality measures for the N-variate big q-orthogonal polynomials $\varphi^{|N}_\lambda$.

The next result is its analog in the context of symmetric functions.

Theorem 20.12. *The limit measure on infinite configurations in the truncated lattice,*

$$M^{\alpha,\beta,\gamma,\delta}_{\infty;q,t} = \lim_{N\to\infty} M^{\alpha,\beta,\gamma,\delta}_{N;q,t} \in \mathit{Prob}(\mathit{Conf}_\infty(L_{\alpha,\beta}))$$

is the orthogonality measure for the symmetric functions $\Phi(-;q,t;\alpha,\beta,\gamma,\delta)$.

That is, the big q-Jacobi symmetric functions form an orthogonal basis in the Hilbert space

$$L^2(Conf_\infty(\mathbb{L}_{\alpha,\beta}), M^{\alpha,\beta,\gamma,\delta}_{\infty;q,t}).$$

This theorem is related to the general idea of constructing analogs of various systems of classical orthogonal polynomials in the algebra Sym of symmetric functions.

Other results in this direction: Cuenca-Olshanski [Mosc. Math. J., 2020]. There, we show that part of the q-Askey scheme can be transferred into Sym.

In a different form, the idea of lifting N-variate analogs of orthogonal polynomials to the algebra Sym is present in earlier papers by Rains [Transf. Groups, 2005], Sergeev-Veselov [Adv. Math., 2009], and Desrosiers-Hallnäs [SIGMA, 2012].

20.5 Stochastic links connecting N-particle ensembles with varying $N = 1, 2, 3, ...$

20.5.1 Sketch of abstract formalism

A **stochastic link** $\Lambda : \Omega \dashrightarrow \Omega'$ between two spaces Ω and Ω' is defined as a Markov kernel $\Lambda(x, dy)$ on $\Omega \times \Omega'$. This means that for any fixed $x \in \Omega$, $\Lambda(x, -)$ is a probability measure on Ω'.

In particular, if both spaces are discrete, Λ is simply a stochastic matrix of size $\Omega \times \Omega'$, with entries $\Lambda(x, y)$ that are nonnegative, and the row sums of the matrix are all equal to 1.

A stochastic link $\Lambda : \Omega \dashrightarrow \Omega'$ can be viewed as a generalized map. The difference with conventional maps is that the image of a point is not a single point, but rather a probability distribution. Like ordinary maps, Λ induces a map on probability measures:

$$\mathrm{Prob}(\Omega) \to \mathrm{Prob}(\Omega'), \quad M \mapsto M\Lambda.$$

Specifically, for $M \in \mathrm{Prob}(\Omega)$, we have

$$(M\Lambda)(dy) = \int_{x\in\Omega} M(dx)\Lambda(x, dy).$$

In the case of discrete spaces, this operation corresponds to multiplying a row vector by a matrix.

Let us consider the category whose objects are sufficiently well-behaved spaces, and whose morphisms are stochastic links. Suppose we are given an infinite chain of spaces connected by stochastic links:

$$\Omega_N \overset{\Lambda_N^{N-1}}{\dashleftarrow} \Omega_{N-1} \overset{\Lambda_2^1}{\dashleftarrow} \Omega_2 \overset{\Lambda_3^2}{\dashleftarrow} \cdots \overset{\Lambda_{N-1}^{N-2}}{\dashleftarrow} \Omega_{N-1} \overset{\Lambda_N^{N-1}}{\dashleftarrow} \Omega_N \overset{\Lambda_{N+1}^{N}}{\dashleftarrow} \cdots$$

Under suitable conditions, one can prove the existence of a projective limit:

$$\Omega_\infty = \varprojlim(\Omega_N, \Lambda_N^{N-1}).$$

Definition 20.13. *Assume each Ω_N is equipped with a probability measure $M_N \in Prob(\Omega_N)$. We say that the family $\{M_N\}$ is **coherent** if*

$$M_N \Lambda_N^{N-1} = M_{N-1}, \quad \forall N \geq 2.$$

Theorem 20.14. *There exists a one-to-one correspondence between coherent families $\{M_N\}$ of measures and probability measures $M_\infty \in Prob(\Omega_\infty)$.*

Thus, any coherent family on a chain $\{\Omega_N, \Lambda_N^{N-1}\}$ gives rise to a probability measure $M_\infty \in \mathrm{Prob}(\Omega_\infty)$. In this way, $(\Omega_\infty, M_\infty)$ serves as the large-N limit of the probability spaces (Ω_N, M_N).

20.5.2 Stochastic links $\Gamma_{N-1}^N : \mathbf{Conf}_N(\mathbb{L}) \dashrightarrow \mathbf{Conf}_{N-1}(\mathbb{L})$

Returning to our setting, we take $\Omega_N = \mathrm{Conf}_N(\mathbb{L})$, the set of τ-sparse N-particle configurations on L, and define stochastic links between these sets.

Theorem 20.15. *For each $N \geq 2$, there exists a stochastic matrix $\Lambda_N^{N-1}(X, Y)$ of size $Conf_N(\mathbb{L}) \times Conf_{N-1}(\mathbb{L})$, which is consistent with the Macdonald polynomials in the sense that*

$$\sum_{Y \in Conf_{N-1}(\mathbb{L})} \Lambda_N^{N-1}(X, Y) \frac{P_{\lambda|N-1}(Y; q, t)}{(t^{N-1}; q, t)^\lambda} = \frac{P_{\lambda|N}(X; q, t)}{(t^N; q, t)^\lambda}$$

for any $X \in Conf_N(\mathbb{L})$ and any $\lambda \in Y(N-1)$.

Furthermore, the entries of $\Lambda_N^{N-1}(X, Y)$ are nonzero if and only if the configurations X and Y interlace in a certain sense. Such a matrix is unique.

20.5.3 Identification of the projective limit space

Theorem 20.16. *The projective limit of the sequence of sets*

$$Conf_1(\mathbb{L}) \dashleftarrow Conf_2(\mathbb{L}) \dashleftarrow Conf_3(\mathbb{L}) \dashleftarrow \cdots$$

connected by the stochastic links Λ_N^{N-1} can be naturally identified with the space $Conf_\infty(\mathbb{L})$ of infinite τ-sparse particle configurations on $\mathbb{L}$.

This theorem asserts that Conf$\infty(\mathbb{L})$ is the universal object with the property that there are stochastic links

$$\Lambda_\infty^N : \mathrm{Conf}\infty(\mathbb{L}) \longrightarrow \mathrm{Conf}_N(\mathbb{L}), \ \forall N \geq 1,$$

such that

$$\Lambda_\infty^N \Lambda_N^{N-1} = \Lambda_\infty^{N-1}, \quad \forall N \geq 2.$$

20.5.4 The coherency relation

Fix a quadruple $(\alpha, \beta, \gamma, \delta)$ from the principal or degenerate series and consider the corresponding hypergeometric measures $M^{N;q,t}_{\alpha,\beta,\gamma,\delta} \in \mathrm{Prob}(\mathrm{Conf}_N(\mathbb{L}))$.

Theorem 20.17. *These measures satisfy the coherency relation:*

$$M^{N;q,t}_{\alpha,\beta,\gamma,\delta} \Lambda^{N-1}_N = M^{N-1;q,t}_{\alpha,\beta,\gamma,\delta}, \quad N \geq 2.$$

Written explicitly, this becomes the following nontrivial combinatorial summation formula:

$$\sum_{X \in \mathrm{Conf}_N(\mathbb{L})} M^{N;q,t}_{\alpha,\beta,\gamma,\delta}(X) \Lambda^{N-1}_N(X, Y) = M^{N-1;q,t}_{\alpha,\beta,\gamma,\delta}(Y), \quad \text{for each } Y \in \mathrm{Conf}_{N-1}(\mathbb{L}).$$

This formula is first proven for the degenerate series, using the big q-Jacobi polynomials. The result is then extended to the principal series by analytic continuation.

This result, in conjunction with the abstract formalism, leads to the main theorem: the existence of the large-N limit measure $M^{\infty;q,t}_{\alpha,\beta,\gamma,\delta}$.

21 Da-jun Zhang: Elliptic Solitons Related To The Lamé Functions

Abstract

In this talk I will report recent progress on the elliptic solitons related to the Lamé functions. Apart from the classical solitons that are composed by usual exponential type plane wave factors, there exist "elliptic solitons" which are composed by the Lamé-type plane wave factors and expressed using Weierstrass functions. Recently, we found vertex operators to generate tau functions for such type of solitons. We also established an elliptic scheme of direct linearization approach.

Contents

21.1 Introduction

21.1.1 The Krichever-Novikov Equation

In an attempt to search for a geometric solution of the KP equation, Krichever discovered the following equation:

Definition 21.1 (Krichever, Novikov, 1981)**.** *The* ***Krichever-Novikov equation*** *is given by*

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x} + \frac{3}{8}v_x - 6\wp(2v)v_x^3.$$

Although not as famous as the KdV or KP equations, it is interesting because it is elliptic. There is also a discrete version of this equation, found by considering the Backlund transformation of the continuous one:

Definition 21.2 (Adler, 1998; Hietarinta, 2003)**.** *The* ***discrete Krichever-Novikov equation*** *is given by:*

$$p(u\tilde{u} + \widehat{u}\widehat{\tilde{u}}) - q(u\widehat{u} + \tilde{u}\widehat{\tilde{u}}) - r(u\widehat{\tilde{u}} + \tilde{u}\widehat{u}) + pqr(1 + u\tilde{u}\widehat{u}\widehat{\tilde{u}}) = 0$$

where

$$(p, P) = (\sqrt{k}\, sn(\alpha; k), sn'(\alpha; k)),$$
$$(q, R) = (\sqrt{k}\, sn(\beta; k), sn'(\beta; k)),$$

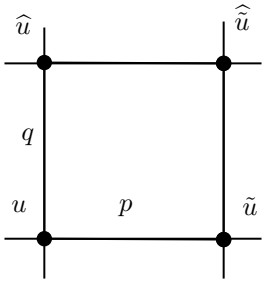
and

$$(r, R) = (\sqrt{k}\, sn(\gamma; k), sn'(\gamma; k)), \gamma = \alpha - \beta.$$

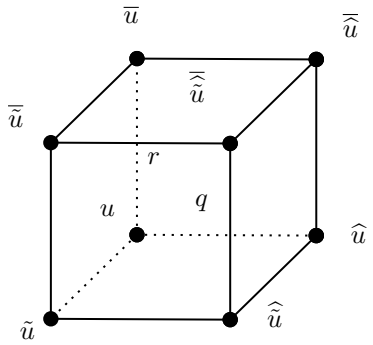
The points on the elliptic curve are defined by:

$$\Gamma = \{(x, X) : X^2 = x^4 + 1 - (k + 1/k)x^2\}$$

Letting $u \equiv u_{n,m}, \tilde{u} \equiv u_{n+1,m}, \widehat{u} \equiv u_{n,m+1}, \widehat{\tilde{u}} \equiv u_{n+1,m+1}$, we get:



One of the reasons why this is so interesting is if we put the same equation on six sides of the cube



we get the multidimensional consistency property, which in our case is the consistency condition around the cube:

$$Q(u, u_e, u_b, u_{be}; p, q) = 0.$$

Proposition 21.3 (Adler, Bobenko, Suris, 2003)**.** *Given*

- *Linearity with respect to each* $\{u, \tilde{u}, \widehat{u}, \widehat{\tilde{u}}\}$
- *Symmetry: Q invariant under D_4*
- *Tetrahedon condition:* $\overline{\widetilde{\widehat{u}}} = f(\tilde{u}, \widehat{u}, \overline{u}; p, q, r)$

we can classify all quad equations:

$$\begin{aligned}
&\textit{(Q4)}: && p(u\tilde{u} + \widehat{u}\widehat{\tilde{u}}) - q(u\widehat{u} + \tilde{u}\widehat{\tilde{u}}) - r(u\widehat{\tilde{u}} + \tilde{u}\widehat{u}) + pqr(1 + u\tilde{u}\widehat{u}\widehat{\tilde{u}}) = 0\\
&\textit{(Q3}(\delta)\textit{)}: && (q^2 - p^2)(u\widehat{\tilde{u}} + \tilde{u}\widehat{u}) + q(p^2 - 1)(u\tilde{u} + \widehat{u}\widehat{\tilde{u}}) - p(q^2 - 1)(u\widehat{u} + \tilde{u}\widehat{\tilde{u}}) - \delta^2(p^2 - q^2)(p^2 - 1)(q^2 - 1)/(4pq)\\
&\textit{(Q2)}: && p(u - \widehat{u})(\tilde{u} - \widehat{\tilde{u}}) - q(u - \tilde{u})(\widehat{u} - \widehat{\tilde{u}}) + pq(p - q)(u + \tilde{u} + \widehat{u} + \widehat{\tilde{u}}) - pq(p - q)(p^2 - pq + q^2) = 0\\
&\textit{(Q1}(\delta)\textit{)}: && p(u - \widehat{u})(\tilde{u} - \widehat{\tilde{u}}) - q(u - \tilde{u})(\widehat{u} - \widehat{\tilde{u}}) + \delta^2 pq(p - q) = 0\\
&\textit{(A2)}: && (q^2 - p^2)(u\tilde{u}\widehat{u}\widehat{\tilde{u}} + 1) + q(p^2 - 1)(u\widehat{u} + \tilde{u}\widehat{\tilde{u}}) - p(q^2 - 1)(u\tilde{u} + \widehat{u}\widehat{\tilde{u}}) = 0\\
&\textit{(A1}(\delta)\textit{)}: && p(u + \widehat{u})(\tilde{u} + \widehat{\tilde{u}}) - q(u + \tilde{u})(\widehat{u} + \widehat{\tilde{u}}) - \delta^2 pq(p - q) = 0\\
&\textit{(H3}(\delta)\textit{)}: && p(u\tilde{u} + \widehat{u}\widehat{\tilde{u}}) - q(u\widehat{u} + \tilde{u}\widehat{\tilde{u}}) + \delta(p^2 - q^2) = 0\\
&\textit{(H2)}: && (u - \widehat{\tilde{u}})(\tilde{u} - \widehat{u}) + (q - p)(u + \tilde{u} + \widehat{u} + \widehat{\tilde{u}}) + q^2 - p^2 = 0\\
&\textit{(H1)}: && (u - \widehat{\tilde{u}})(\tilde{u} - \widehat{u}) = p - q
\end{aligned}$$

Q4 is precisely the Krichever-Novikov equation, and the other ones have similar names. The Krichever-Novikov is the most important and interesting one in this list.

21.1.2 Elliptic Solitons

Some researchers found solutions for all equations except for Q4 - this is now called an elliptic soliton.

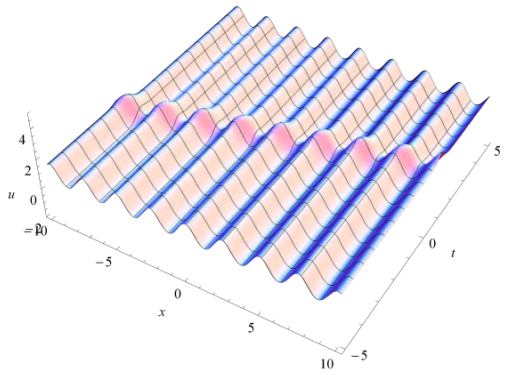
Definition 21.4. *The **KdV equation** (Korteweg-de Vries (KdV)) is given by*

$$u_t = 6uu_x + u_{xxx}$$

Definition 21.5. *The **1SS equation** (1-Soliton Solution) is given by*

$$u = 2(\ln f)_{xx}, \quad f = 1 + e^{kx+k^3t}$$

If we compare the usual soliton vs the elliptic soliton, the plane wave factor (PWF) is the difference. In the usual soliton, if we replace e^{kx+k^3t} with the **Lamé** function $\frac{\sigma(x+k)}{\sigma(x)\sigma(k)}e^{-\zeta(k)x-\wp'(k)t}$, we obtain an image of the elliptic 1-soliton of the KdV: where



$$u(x,t) = -2\wp(x+w_2) + 2\left(\ln\left(1+\tilde{\Psi}_x(k,k)e^{-4\wp'(k_1)t}\right)\right)_{xx},$$

and

$$\tilde{\Psi}_x(a,b) = \frac{\sigma(x+a+b+w_2)}{\sigma(x+w_2)\sigma(a+b)}e^{-(\zeta(a)+\zeta(b))x-\zeta(w_2)(a+b)}.$$

21.1.3 Lamé Function and KdV

Given the KdV equation

$$u_t = 6uu_x + u_{xxx}$$

we have an associated Lax pair given by

$$\varphi_{xx} = (\lambda - u)\varphi$$

$$\varphi_t = 4\varphi_{xxx} + 6u\varphi_x + 3u_x\varphi$$

The solution to the KdV is given by $u = -2\wp(x)$, and from the Lax pair we can solve for the **Lamé function**

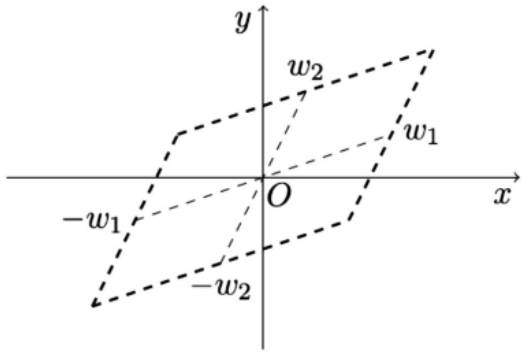
$$\varphi_{xx} = (\wp(k) + 2\wp(x))\varphi$$

which gives:

$$\varphi(x) = \frac{\sigma(x+k)}{\sigma(x)\sigma(k)}e^{-\zeta(k)x}$$

21.1.4 Weierstrass Functions

Consider the fundamental period parallelogram $\mathbb{D}$ with boundary Ω: Then we



obtain the Weierstrass functions: $\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \wp(z) = -\zeta'(z)$ where

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + O(z^6),$$

$$\zeta(z) = \frac{1}{z} - \frac{g_2}{60}z^3 - \frac{g_3}{140}z^5 + O(z^7),$$

and

$$\sigma(z) = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 + O(z^9).$$

Using this data, we can define an elliptic curve

$$y^2 = 4x^3 - g_2x - g_3.$$

There are many useful identities for the Weierstrass function. Here are a couple:

Proposition 21.6.

1.

$$\wp(z) - \wp(u) = -\frac{\sigma(z+u)\sigma(z-u)}{\sigma^2(z)\sigma^2(u)},$$

2.

$$\eta_u(z) = \zeta(z+u) - \zeta(z) - \zeta(u) = \frac{1}{2}\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)},$$

3.

$$\wp(z) + \wp(u) + \wp(z+u) = \eta_u^2(z)$$

4.

$$\chi_{u,v}(z) = \zeta(u) + \zeta(v) + \zeta(z) - \zeta(u+v+z) = \frac{\sigma(u+v)\sigma(u+z)\sigma(z+v)}{\sigma(u)\sigma(v)\sigma(z)\sigma(z+u+v)}.$$

5. *Frobenius-Stickelberger determinant (elliptic van der Monde):*

$$|1, \wp(k), \wp'(k), \wp''(k), \cdots, \wp^{(n-2)}(k)|$$
$$= (-1)^{\frac{(n-1)(n-2)}{2}} \left(\prod_{s=1}^{n-1} s! \right) \frac{\sigma(k_1 + \cdots + k_n) \prod_{i<j} \sigma(k_i - k_j)}{\sigma^n(k_1)\sigma^n(k_2), \cdots \sigma^n(k_n)},$$

where $f(k) = (f(k_1), f(k_2), \cdots, f(k_n))^T$.

6.

$$\prod_{j=1}^{n} \Phi_x(k_j) = \frac{(-1)^{n-1}}{(n-1)!} \Phi_x(k_1 + \cdots + k_n) \frac{|1, \wp(k), \wp'(k), \cdots, \wp^{(n-2)}(k)|}{|1, \eta_x(k), \wp(k), \wp'(k), \cdots, \wp^{(n-3)}(k)|},$$

where $\Phi_a(b) = \frac{\sigma(a+b)}{\sigma(a)\sigma(b)}$.

21.1.5 Hirota Bilinear Operator

Now, let's talk about the Hirota bilinear operator.

Definition 21.7. ***Hirota's bilinear operator*** D *is given by:*

$$D_t^m D_x^n f \cdot g = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n f(t, x) g(t', x')|_{t'=t, x'=x}$$

or equivalently,

$$e^{D_x + \kappa D_y} f(x, y) \cdot g(x, y) = f(x + \epsilon, y + \kappa) g(x - \epsilon, y - \kappa).$$

Proposition 21.8. *Let the plane wave factor (PWF) be* e^{η_j}*, where* $\eta_j = a_j x + b_j t + c_j$*. Then, the following properties hold:*

1. *The action of Hirota's operator on two plane wave factors is given by:*

$$D_x^n D_t^m e^{\eta_1} \cdot e^{\eta_2} = (a_1 - a_2)^n (b_1 - b_2)^m e^{\eta_1 + \eta_2}$$

2. *The gauge property for Hirota's bilinear operator is:*

$$D_x^n D_t^m (e^{\eta_1} f) \cdot (e^{\eta_1} g) = e^{2\eta_1} D_x^n D_t^m f \cdot g$$

Example 21.9. *The (KdV) equation can be expressed in Hirota's bilinear form as follows:*

$$u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}, \quad u = 2(\ln \tau)_{xx} \quad \textit{(KdV)}$$

The bilinear form of the KdV equation is:

$$(D_x^4 - D_x D_t) \tau \cdot \tau = 0$$

21.1.6 D vs Lamé-type PWF

All results from this section are by Li and Zhang in 2022.

Theorem 21.10. *The Lamé-type PWF of the KdV is given by:*

$$\rho_i(x,t) = \Phi_x(2k_i)e^{\xi_i}, \quad \xi_i = -2\zeta(k_i)x + \wp'(k_i)t + \xi_i^{(0)}$$

where $\Phi_x(y) = \frac{\sigma(x+y)}{\sigma(x)\sigma(y)}$. *The derivatives of* ρ_i *are:*

$$\rho_{i,x} = -\chi_{k_i,k_i}(x)\rho_i$$

$$\rho_{i,xx} = 2\eta_{k_i}(x)\rho_{i,x}$$

and

$$\rho_{i,xxx} = (6\wp(x) + 2\wp(x+k_i) + 4\wp(k_i))\rho_{i,x}$$

where

$$\eta_x(y) = \zeta(x+y) - \zeta(x) - \zeta(y)$$

and

$$\chi_{\delta,\epsilon}(\gamma) = \zeta(\delta) + \zeta(\epsilon) + \zeta(\gamma) - \zeta(\delta+\epsilon+\gamma)$$

Theorem 21.11. *Property 1 becomes*

$$D_x^2\rho_i \cdot \rho_i = 2(\wp(x) - \wp(x+2k_i))\rho_i^2$$

$$D_x^4\rho_i \cdot \rho_i = 12\wp(x)D_x^2\rho_i \cdot \rho_i$$

$$D_x^{2n}\varrho \cdot \varrho = \frac{\wp^{(2n-1)}(x)}{\wp'(x)}D_x^2\varrho \cdot \varrho$$

where $\varrho = \Phi_x(a)e^{bx+ct}$ *for* $a, b, c \in \mathbb{C}$.

Theorem 21.12. *A general formula for the Lamé-type PWF is:*

$$\varrho_i = \Phi_x(a_i)e^{b_i x + c_i t}, \quad a_i, b_i, c_i \in \mathbb{C}$$

Then, for two functions ϱ_1 *and* ϱ_2, *we have the following bilinear operator action:*

$$D_x^n D_t^m \varrho_1 \cdot \varrho_2 = (c_1 - c_2)^m \prod_{i=1}^{n}(G_1, G_2, \ldots, G_n)\varrho_1\varrho_2$$

where Y_n *are the Bell polynomials:*

$$Y_n(y_1, y_2, \ldots, y_n) = e^{-y}\partial_x^n e^y$$

with $y := y(x)$ *and* $y_i := \partial_x^i y(x)$. *The functions* $G_m(x)$ *are:*

$$G_m(x) = \frac{\partial_x^{m-1}\alpha_1(x) + (-1)^m\partial_x^{m-1}\alpha_2(x)}{}$$

and

$$\alpha_i(x) = \zeta(x+a_i) - \zeta(x) + b_i.$$

Theorem 21.13. *The following property 2 (quasi-gauge property) holds:*

$$D_x^n D_t^m(\varrho f)\cdot(\varrho g) = \varrho^2 D_x^n D_t^m f\cdot g + \sum_{l=1}^{\frac{n}{2}} \binom{n}{2l} (D_x^{2l}\varrho\cdot\varrho) D_x^{n-2l} D_t^m f\cdot g$$

21.1.7 KdV: τ Function and Vertex Operator

This section is due to Lepowsky, Wilson, 1978 and Date, Kashiwara, Miwa, 1981.

Definition 21.14. *The **bilinear KdV equation** can be expressed as:*

$$(4D_xD_t - D_x^4)\tau\cdot\tau = 0$$

where τ is a τ-function.

Theorem 21.15. *The general form of the τ-function τ_N is:*

$$\tau_N = \sum_{J\subset S}\left[\left(\prod_{i\in J} c_i\right)\left(\prod_{i,j\in J, i<j} A_{ij}\right)\exp\left(2\sum_{i\in J}\xi_i\right)\right]$$

where c_i are arbitrary constants, and A_{ij} is given by:

$$A_{ij} = \frac{(k_i - k_j)^2}{(k_i - k_j)^2}$$

and

$$\xi_i = k_i x + k_i^3 t$$

with $S = \{1, 2, \ldots, N\}$ and $J \subset S$.

Definition 21.16. *The **vertex operator** $X(k)$ is defined as:*

$$X(k) = e^{2\xi(t,k)} e^{-2\xi(\partial, ek^{-1})}$$

where $\xi(t,k)$ is given by:

$$\xi(t,k) = \sum_{j=0}^{1} k^{2j+1} t^{2j+1}$$

where $\boldsymbol{t} = (t_1 = x)$, $\tilde{\partial} = \left(\partial_1 \frac{\partial_3}{3}\right)$, and $\partial j = \partial_{t_j}$.

21.2 II Bilinear: τ-Functions, Vertex Operators

21.2.1 KdV and KP

Definition 21.17. *The pKdV equation is given by*

$$\overline{v}_t - \frac{3}{4}\overline{v}_x^2 - \frac{1}{4}\overline{v}_{xxx} = 0 \quad \textit{(pKdV)}$$

where the transformation is

$$\overline{v} = 2\zeta(x) + \frac{1}{4}g_2 t + 2(\ln\tau)_x$$

The bilinear form of KdV is:

$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2)\tau\cdot\tau = 0$$

Remark 21.18. *An alternative form is given by $\tau_0 = \sigma(x)\tau$, where:*

$$(D_x^4 - 4D_xD_t - g_2)\tau_0\cdot\tau_0 = 0$$

Proposition 21.19. *The 1-soliton solution (1SS) is given by:*

$$\tau_1 = 1 + \rho_1(x,t) = 1 + \Phi_x(2k_1)e^{\xi_1}$$

where $\Phi_x(y) = \frac{\sigma(x+y)}{\sigma(x)\sigma(y)}$, and ξ_1 is defined as:

$$\xi_1 = -2\zeta(k_1)x + \wp'(k_1)t + \xi_1^{(0)}$$

The 2-soliton solution (2SS) is:

$$\tau_2 = 1 + \rho_1(x,t) + \rho_2(x,t) + f^{(2)}(x,t)$$

where

$$f^{(2)}(x,t) = \frac{A_{12}\sigma(x+2k_1+2k_2)}{\sigma(x)\sigma(2k_1)\sigma(2k_2)}e^{\xi_1+\xi_2}$$

and

$$A_{12} = \frac{\sigma^2(k_1-k_2)}{\sigma^2(k_1+k_2)}$$

Here, the individual components $\rho_i(x,t)$ are:

$$\rho_i(x,t) = \Phi_x(2k_i)e^{\xi_i}, \quad \xi_i = -2\zeta(k_i)x + \wp'(k_i)t + \xi_i^{(0)}$$

where $i = 1, 2$.

Definition 21.20. *The **Wronskian** is defined as:*

$$W = \left|\varphi, \frac{\partial\varphi}{\partial x}, \frac{\partial^2\varphi}{\partial x^2}, \cdots, \frac{\partial^{N-1}\varphi}{\partial x^{N-1}}\right| = |0, 1, 2, \cdots, N-1| = |N-1|$$

The NSS to the bilinear KdV is given by $\tau = |\widehat{N-1}|$

$$\varphi_{j,xx} = (\wp(k_j) + 2\wp(x))\varphi_j$$

$$\varphi_{j,t} = \varphi_{j,xxx} - 3\wp(x)\varphi_{j,x} - \frac{3}{2}\wp'(x)\varphi_j$$

for each $j = 1, 2, \cdots, N$ and $k_j \in \mathbb{C}$.

A general solution for φ_j is given by:

$$\varphi_j = a_j^+ \varphi_j^+ + a_j^- \varphi_j^-$$

where $\varphi_j^\pm$ are Lamé functions:

$$\varphi_j^\pm = \Phi_x(\pm k_j) e^{\mp \gamma_j}$$

and γ_j is defined as:

$$\gamma_j = \zeta(k_j)x - \frac{1}{2}\wp'(k_j)t + \gamma_j^{(0)}$$

Proposition 21.21. *Hirota's form for f is:*

$$f = \sum_{\mu=0,1} \frac{\sigma\left(x + 2\sum_{i=1}^N \mu_i k_i\right)}{\sigma(x)\prod_{j=1}^N \sigma^{\mu_j}(2k_j)} \exp\left(\sum_{j=1}^N \mu_j \theta_j + \sum_{1\le i<j\le N} \mu_i \mu_j a_{ij}\right)$$

where

$$\theta_j = -2\zeta(k_j)x + \wp'(k_j)t + \theta_j^{(0)}$$

and the matrix element $e^{a_{ij}}$ is given by:

$$e^{a_{ij}} = A_{ij} = \left(\frac{\sigma(k_i - k_j)}{\sigma(k_i + k_j)}\right)^2$$

The bilinear KdV equation and its Hirota form are connected with τ-functions as follows:

$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2)\tau \cdot \tau = 0$$
$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2) f \cdot f = 0$$

where $f = \frac{\tilde{\tau}}{\tilde{g}}$, $\tilde{g} = g\left(x + \sum_{i=1}^N k_i\right)$, and $\tilde{\tau} = \tau\left(x + \sum_{i=1}^N k_i\right)$, where the function g is given by:

$$g = (-1)^{\frac{N(N-1)}{2}} \frac{\sigma(x - \sum_{i=1}^N k_i)}{\sigma(x)} \frac{\prod_{1\le i<j\le N}\sigma(k_i - k_j)}{\sigma(k_1)\cdots\sigma(k_N)} \exp\left(\sum_{i=1}^N \gamma_i\right)$$

Proposition 21.22. *The vertex operator for τ in Hirota's form is given by:*

$$f = \tau_N(\bar{\mathbf{t}})$$

where:

$$\tau_N(\overline{\mathbf{t}}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i,j\in J, i<j} A_{ij}\right)\frac{\sigma\left(t_1 + 2\sum_{i\in J} k_i\right)}{\sigma(t_1)}\prod_{i\in J}\sigma(2k_i)\exp\left(\sum_{i\in J}\theta_{[e]}(\overline{\mathbf{t}}, k_i)\right)$$

where c_i are arbitrary constants, $S = \{1, 2, \cdots, N\}$, and $J \subset S$.

Theorem 21.23 (Date, Kashiwara, Miwa, 1981)**.** *The vertex operator $X(k)$ is defined as:*

$$X(k) = \Phi_{t_1}(2k)e^{\theta_{[e]}(\overline{\mathbf{t}},k)}e^{\theta(\overline{\tilde{\partial}},k)}$$

Then we can use the vertex operator to generate τ:

$$\tau_N(\overline{\mathbf{t}}) = e^{c_N X(k_N)}\tau_{N-1}(\overline{\mathbf{t}}), \quad \tau_0(\overline{\mathbf{t}}) = 1$$

where the time variables and differential operators are defined as:

$$\overline{\mathbf{t}} = (t_1 = x, t_3, \cdots, t_{2n+1}, \cdots),$$

$$\overline{\tilde{\partial}} = \left(\frac{\partial}{\partial t_1}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots, \frac{1}{2n+1}\frac{\partial}{\partial t_{2n+1}}, \cdots\right),$$

and the functions $\theta(t,k)$ and $\theta_{[e]}(t,k)$ are given by:

$$\theta(\overline{\mathbf{t}}, k) = 2\sum_{n=0}^{\infty} k^{2n+1}t_{2n+1},$$

$$\theta_{[e]}(\overline{\mathbf{t}}, k) = -2\sum_{n=0}^{\infty}\frac{\zeta^{(2n)}(k)}{(2n)!}t_{2n+1}$$

where $\zeta^{(2n)}(k)$ are the values of the zeta function evaluated at even integers.

Proposition 21.24 (Bilinear Identity)**.** *The bilinear identity is given by:*

$$\oint_\Omega \frac{dq}{2\pi i}h(\overline{\mathbf{t}}, q)h(\overline{\mathbf{t}}', -q) = 0$$

where $h(\overline{\mathbf{t}}, q)$ is defined as:

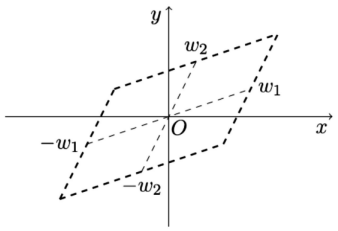
$$h(\overline{\mathbf{t}}, q) = X(\overline{\mathbf{t}}, q)\tau(\overline{\mathbf{t}})$$

and

$$X(\overline{\mathbf{t}}, q) = \frac{\sigma(t_1 + q)}{\sigma(q)}e^{\frac{1}{2}\theta_{[e]}(\overline{\mathbf{t}},q)}e^{\frac{1}{2}\theta(\overline{\tilde{\partial}},q)}$$

Note that $h(\overline{\mathbf{t}}, q)$ is doubly periodic with respect to q; here is the fundamental period parallelogram $\mathbb{D}$ with boundary Ω.

The bilinear identity can also be written in residue form:



Proposition 21.25 (Bilinear Identity in Residue Form)**.**

$$Res_{q=0}\left[h(\overline{\mathbf{t}},q)h(\overline{\mathbf{t}}',-q)\right]=0$$

Redefine $\tau'\left(\overline{\mathbf{t}}\right)$ as:

$$\tau'\left(\overline{\mathbf{t}}\right)=\sigma(t_1)\tau(t)$$

The bilinear identity becomes:

$$\oint_{\Omega}\frac{dq}{2\pi i}\frac{1}{\sigma^2(q)}e^{\frac{1}{2}\theta_{[e]}(t-\overline{\mathbf{t}}',q)}\tau'(\overline{\mathbf{t}}+\overline{\epsilon}(q))\tau'(\overline{\mathbf{t}}'-\overline{\epsilon}(q))=0$$

Now, introduce $\overline{\mathbf{t}}=\overline{\mathbf{x}}+\overline{\mathbf{y}}$ and $\overline{\mathbf{t}}'=\overline{\mathbf{x}}-\overline{\mathbf{y}}$, where $\overline{\mathbf{x}}=(x_1,x_3,\ldots)$ and $\overline{\mathbf{y}}=(y_1,y_3,\ldots)$. The identity becomes:

$$\oint_{\Omega}\frac{dq}{2\pi i}\frac{1}{\sigma^2(q)}e^{\theta_{[e]}(\overline{\mathbf{y}},q)}e^{(\overline{\mathbf{y}}+\overline{\epsilon}(q))\cdot\mathbf{D}_{\overline{\mathbf{x}}}}\tau'(x)\cdot\tau'(x)=0$$

Proposition 21.26 (The Residue Form)**.** *Taking the residue at $q=0$, we have:*

$$Res_{q=0}\left[\frac{1}{\sigma^2(q)}e^{\theta_{[e]}(\overline{\mathbf{y}},q)}e^{(\overline{\mathbf{y}}+\overline{\epsilon}(q))\cdot D_x}\tau_0(\overline{\mathbf{x}})\cdot\tau_0(\overline{\mathbf{x}})\right]=0$$

where $\mathbf{D}_{\overline{\mathbf{x}}}=(D_{x_1},D_{x_3},D_{x_5},\ldots)$ *and* $\overline{\epsilon}(q)=(q,\frac{q^3}{3},\ldots,\frac{q^{2n+1}}{2n+1},\ldots)$.

The difficulty arises from the form of the $\theta_{[e]}(t,k)$ function:

$$\theta_{[e]}(\overline{\mathbf{t}},k)=-2\sum_{n=0}^{\infty}\frac{\zeta^{(2n)}(k)}{(2n)!}t_{2n+1}=\sum_{j=-\infty}^{\infty}s_j(\overline{\mathbf{t}})k^j.$$

Proposition 21.27 (Li, Zhang, 2022)**.** *The algorithm is given by:*

$$Res_{q=0}\left[\left(\overline{\mathbf{B}}+\overline{\mathbf{D}}_{\overline{\mathbf{x}}}\right)^{\overline{\beta}}\Big|_{\leq 1}\left(\sum_{n=0}^{\infty}X_n\sum_{j=0}^{n}p_j(\tilde{\mathbf{D}}_{\overline{\mathbf{x}}})\mu_{n-j}q^{n-2}\right)\tau'(\overline{\mathbf{x}})\cdot\tau'(\overline{\mathbf{x}})\right]=0$$

where:

- $\overline{\beta} = (\beta_1, \beta_3, \ldots, \beta_{2j+1}, \ldots)$ *and* $\beta_j \geq 0$,
- $|\overline{\beta}| = \sum_{j=0}^{\infty} \beta_{2j+1}$,
- $||\overline{\beta}|| = \sum_{j=0}^{n} (2j+1)\beta_{2j+1}$,
- $e^{\xi(\mathbf{t},k)} = \sum_{n=0}^{\infty} p_n(\mathbf{t})k^n$,
- $\frac{1}{\sigma^2(q)} = \sum_{j=0}^{\infty} \mu_j q^{j-2}$,
- $\overline{B} = -2\left(\zeta(q), \frac{\zeta''(q)}{2!}, \ldots, \frac{\zeta^{(2n)}(q)}{(2n)!}, \ldots\right)$,
- $\tilde{\mathbf{D}}_{\overline{\mathbf{x}}} = (D_{x_1}, 0, \frac{1}{3}D_{x_3}, 0, \frac{1}{5}D_{x_5}, \ldots)$.

Example 21.28. *Consider* $\beta = (3, 0, 0, \ldots)$. *The bilinear identity becomes:*

$$(D_{x_1}^4 - 4D_{x_1}D_{x_3} - g_2)\tau' \cdot \tau' = 0$$

Example 21.29. *Consider* $\beta = (2, 1, 0, \ldots)$. *The bilinear identity becomes:*

$$(D_{x_1}^6 + 4D_{x_1}^3 D_{x_3} - 32D_{x_3}^2 + 3g_2 D_{x_1}^2 - 24g_3)\tau' \cdot \tau' = 0$$

Example 21.30. *Consider* $\beta = (5, 0, 0, \ldots)$. *The bilinear identity becomes:*

$$(D_{x_1}^6 + 40D_{x_1}^3 D_{x_3} + 40D_{x_3}^2 - 216D_{x_1}D_{x_5} + 3g_2 D_{x_1}^2 - 24g_3)\tau' \cdot \tau' = 0$$

Let's move onto discussing KP.

Proposition 21.31. *The bilinear equation for the pKP is given by:*

$$4v_t - v_{xxx} - 3(v_x)^2 - 3\partial^{-1}v_{yy} = 0, \quad (pKP)$$

where v *is expressed as:*

$$v = 2\zeta(x) + \frac{g_2}{4}t + 2(\ln\tau)_x$$

Proposition 21.32. *The bilinear KP equation is:*

$$(D_x^4 - 4D_xD_t - 12\wp(x)D_x^2 + 3D_y^2)\tau \cdot \tau = 0$$

or equivalently:

$$(D_x^4 - 4D_xD_t + 3D_y^2 - g_2)\tau' \cdot \tau' = 0$$

where $\tau' = \sigma(x)\tau$.

Proposition 21.33. *To find the* τ *function, let* $\tau = \left|\widehat{N-1}\right|$ *with* $f = \tilde{\tau}\tilde{g}$. *The* τ*-function for* N *solitons is given by:*

$$\tau_N(\mathbf{t}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i,j\in J, i<j} A_{ij}\right)\frac{\sigma\left(t_1 + \sum_{i\in J}(k_i - l_i)\right)}{\sigma(t_1)\prod_{i\in J}\sigma(k_i - l_i)} e^{\sum_{i\in J}(\xi_{[e]}(\mathbf{t},k_i) - \xi_{[e]}(\mathbf{t},l_i))}$$

where A_{ij} is defined as:

$$A_{ij} = \frac{\sigma(k_i - k_j)\sigma(l_i - l_j)}{\sigma(k_i - l_j)\sigma(l_i - k_j)}.$$

Proposition 21.34. *The τ-function for N solitons is:*

$$\tau_N(t) = \sum_{J \subset S} \prod_{i \in J} c_i \left(\prod_{i<j \in J} A_{ij} \right) \frac{\sigma\left(t_1 + \sum_{i \in J}(k_i - l_i)\right)}{\sigma(t_1)} \prod_{i \in J} \sigma(k_i - l_i) e^{\sum_{i \in J}(\xi[e](t,k_i) - \xi[e](t,l_i))}$$

where A_{ij} is given by:

$$A_{ij} = \frac{\sigma(k_i - k_j)\sigma(l_i - l_j)}{\sigma(k_i - l_j)\sigma(l_i - k_j)}$$

Proposition 21.35. *The vertex operator to generate τ is:*

$$X(k,l) = \Phi_{t_1}(k-l) e^{\xi[e](t,k) - \xi[e](t,l)} e^{\xi(\partial, ek) - \xi(\partial, el)}$$

Then the τ-function is generated as:

$$\tau_N(t) = e^{c_N} X(k_N, l_N) \tau_{N-1}(t), \quad \tau'\left(\overline{\mathbf{t}}\right) = 1$$

We use the following notations:

$$\mathbf{t} = (t_1 = x, t_2, \dots, t_n, \dots)$$

$$\tilde{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \dots, \frac{1}{n}\frac{\partial}{\partial t_n}, \dots \right)$$

$$\xi(\mathbf{t}, k) = \sum_{n=1}^{\infty} k^n t_n, \quad \xi_{[e]}(\mathbf{t}, k) = \sum_{n=1}^{\infty} (-1)^n \zeta^{(n-1)}(k) \frac{t_n}{(n-1)!}, \quad \zeta^{(i)}(k) = \partial_k^i \zeta(k).$$

Proposition 21.36 (Li, Zhang, 2022)**.** *The bilinear identity is:*

$$\oint_{\Omega} \frac{dq}{2\pi i} h(\mathbf{t}, q) h^*(\mathbf{t}', q) = 0$$

where $h(\mathbf{t}, q)$ and $h^*(\mathbf{t}, q)$ are defined as:

$$h(\mathbf{t}, q) = X(\mathbf{t}, q)\tau(\mathbf{t}), \quad h^*(\mathbf{t}, q) = X^*(\mathbf{t}, q)\tau(\mathbf{t})$$

with:

$$X(\mathbf{t}, q) = \frac{\sigma(t_1 + q)}{\sigma(q)} e^{\xi[e](\mathbf{t}, q)} e^{\xi(\tilde{\partial}, q)}, \quad X^*(\mathbf{t}, q) = \frac{\sigma(t_1 - q)}{\sigma(-q)} e^{-\xi[e](\mathbf{t}, q)} e^{-\xi(\tilde{\partial}, q)}$$

Proposition 21.37.

$$Res_{q=0}\left[(B+D_x)^{\beta}\left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}p_j(D_xe)^{\mu_{n-j}}q^{n-2}\right)\right)\tau_0(x)\cdot\tau_0(x)\right]=0$$

.

Example 21.38.

$$(D_{x_1}^4+3D_{x_2}^2-4D_{x_1}D_{x_3}-g_2)\tau'\cdot\tau'=0$$

$$(D_{x_1}^3D_{x_2}+2D_{x_2}D_{x_3}-3D_{x_1}D_{x_4})\tau'\cdot\tau'=0$$

$$(D_{x_1}^6+45D_{x_1}^2D_{x_2}^2+20D_{x_1}^3D_{x_3}+40D_{x_3}^2+90D_{x_2}D_{x_4}-216D_{x_1}D_{x_5}+3g_2D_{x_1}^2-24g_3)\tau'\cdot\tau'=0$$

21.2.2 Elliptic N-th Roots of Unity

Following Nijhoff, Sun, Zhang, 2023.

Definition 21.39. *There exist distinct $\omega_j(\delta)$, for $j=0,1,2,\ldots,N-1$, up to the periodicity of the periodic lattice, such that the following equation holds:*

$$\prod_{j=0}^{N-1}\Phi_\kappa(\omega_j(\delta))=\frac{1}{(N-1)!}\left(\wp^{(N-2)}(-\kappa)-\wp^{(N-2)}(\delta)\right)$$

where $\omega_0(\delta)=\delta$ and all $\{\omega_j(\delta)\}$ are independent of κ. These values $\{\omega_j(\delta)\}_{j=0}^{N-1}$ are called the ***elliptic N-th roots of unity***

.

Proposition 21.40. *These roots also satisfy the following identities:*

$$\sum_{j=0}^{N-1}\omega_j(\delta)=0$$

and

$$\sum_{j=0}^{N-1}\zeta^{(l)}(\omega_j(\delta))=0,\quad (l=0,1,\ldots,N-2)$$

The discrete PWF/dispersion relation for usual solitons is given by

$$G_N(p,k):=\prod_{j=1}^{N}\alpha_j(p_j-k_j)=\prod_j(p-\omega_j(k)),\quad \alpha_N\equiv 1$$

with PWF

$$\rho = \left(\frac{p+k}{p+\omega_j(k)}\right)^n \left(\frac{q+k}{q+\omega_j(k)}\right)^m$$

We consider various elliptic N-th roots of unity:

- For $N = 2$:

$$\Phi_\kappa(\delta)\Phi_\kappa(-\delta) = \wp(\kappa) - \wp(\delta)$$

- For $N = 3$:

$$\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta)) = -\frac{1}{2}\left(\wp'(\kappa) + \wp'(\delta)\right)$$

- For $N = 4$:

$$\Phi_\kappa(\delta)\Phi_\kappa(\omega_1(\delta))\Phi_\kappa(\omega_2(\delta))\Phi_\kappa(\omega_3(\delta)) = \frac{1}{6}\left(\wp''(\kappa) - \wp''(\delta)\right)$$

Then the elliptic PWF is given by:

$$\rho = \prod_j \left(\frac{\Phi_p(\omega_j(k))}{\Phi_p(\omega_0(k))}\right)^n \left(\frac{\Phi_q(\omega_j(k))}{\Phi_q(\omega_0(k))}\right)^m$$

21.2.3 Reduction by Dispersion Relation

Now we discuss reduction by dispersion relation. The τ function of KP is given by

$$\tau_N(\mathbf{t}) = \sum_{J\subset S}\left(\prod_{i\in J} c_i\right)\left(\prod_{i<j\in J} A_{ij}\right)\frac{\sigma\left(t_1 + \sum_{i\in J}(k_i - l_i)\right)}{\sigma(t_1)}\prod_{i\in J}\frac{\sigma(k_i - l_i)}{\sigma(2k_i)}e^{\sum_{i\in J}\left(\xi_{[e]}(t,k_i) - \xi_{[e]}(t,l_i)\right)}$$

where

$$\xi^{[e]}(t,k) = \sum_{n=1}^{\infty}(-1)^n\zeta^{(n-1)}(k)\frac{t^n}{n!}$$

It is straightforward to reduce to KdV Hierarchy by setting $l_i = -k_i$.

It is much harder to reduce to Boussinesq. It is okay to reduce from the KP equation to the Boussinesq equation, but we fail by trying to reduce the KP hierarchy to the Boussinesq hierarchy (because the elliptic cube root of unity is not the 6th root of unity). However, this can be resolved by redefining $t_2 \to t_2 + \frac{1}{2}g_2 t_6$. This gives

$$\wp^{(4)}(\omega_1(\delta)) - \wp^{(4)}(\delta) = 30\left(\wp^{(0)}(\omega_1(\delta)) - \wp^{(0)}(\delta)\right)\left(\wp^{(0)}(\omega_1(\delta)) + \wp^{(0)}(\delta)\right) + 12g_2\left(\wp(\omega_1(\delta)) - \wp(\delta)\right)$$

21.2.4 Degeneration by Period

The elliptic curve is given by:

$$y^2 = R(x) = 4x^3 - g_2 x - g_3$$

The condition for degeneration is:

$$\Delta = g_2^3 - 27g_3^2 = 0$$

For the trigonometric and hyperbolic degeneration, the relations are:

$$g_2 = \frac{4}{3}\alpha^4, \quad g_3 = \frac{8}{27}\alpha^6, \quad \alpha = \frac{\pi}{2w}$$

The elliptic functions in this case are given by:

$$\sigma(q) = \frac{1}{\alpha} e^{\frac{1}{6}\alpha q^2} \sin(\alpha q), \quad \zeta(q) \quad = \frac{1}{3}\alpha^2 q + \alpha \cot(\alpha q) \wp(q) = -\frac{1}{3}\alpha^2 + \alpha^2 \csc^2(\alpha q)$$

For the rational case, where $g_2 = g_3 = 0$, we have:

$$\sigma(q) = q, \quad \zeta(q) = \frac{1}{q}, \quad \wp(q) = \frac{1}{q^2}$$

More generally, the residue calculation for the bilinear KP equation becomes:

$$\mathrm{Res}_{q=0} \left[\left(\mathbf{B} + \mathbf{D}_{\mathbf{x}} \right)^{\beta} \Big|_{\leq 1} \left(\sum_{n=0}^{\|\beta\|-1} \sum_{j=0}^{n} p_j \left(\tilde{\mathbf{D}}_{\mathbf{x}} \mu_{n-j} q^{n-2} \right) \right) \tau'(\mathbf{x}) \cdot \tau(\mathbf{x}) \right] = 0$$

- For the trigonometric/hyperbolic case, we express the τ' function as:

$$\tau' = e^{\frac{1}{6}(\alpha x_1)^2} \sin(\alpha x_1) \tau_N(\mathbf{x})$$

  The $\tau_N(\mathbf{x})$ is given by:

$$\tau_N(\mathbf{x}) = \sum_{J \subset S} \left(\prod_{i \in J} c_i \right) \left(\prod_{i<j \in J} A_{ij} \right) \frac{\sin(\alpha(x_1 + \sum_{i \in J}(k_i - l_i)))}{\sin(\alpha x_1) \prod_{i \in J} \sin(\alpha(k_i - l_i))} \times \exp\left(\sum_{i \in J} \left(\xi^{[t]}(\mathbf{x}, k_i) - \xi^{[t]}(\mathbf{x}, l_i) \right) \right)$$

  where

$$\xi^{[t]}(\mathbf{x}, k) = \alpha \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{(n-1)!} \partial_k^{n-1} \cot(\alpha k)$$

  The term A'_{ij} is defined as:

$$A'_{ij} = \frac{\sin(\alpha(k_i - k_j)) \sin(\alpha(l_i - l_j))}{\sin(\alpha(k_i - l_j)) \sin(\alpha(l_i - k_j))}$$

- For the rational degeneration, we have:

$$\tau' = x_1 \tau_N(\mathbf{x})$$

The expression for $\tau_N(x)$ is:

$$\tau_N(\mathbf{x}) = \sum_{J \subset S} \left(\prod_{i \in J} c_i \right) \left(\prod_{i<j \in J} A_{ij} \right) \frac{x_1 + \sum_{i \in J}(k_i - l_i)}{x_1 \prod_{i \in J}(k_i - l_i)} \exp\left(\sum_{i \in J} (\xi_{[r]}(\mathbf{x}, k_i) - \xi_{[r]}(\mathbf{x}, l_i)) \right)$$

where:

$$\xi^{[r]}(\mathbf{x}, k) = -\sum_{n=1}^{\infty} \frac{1}{k^n} x^n, \quad A_{ij} = \frac{(k_i - k_j)(l_i - l_j)}{(k_i - l_j)(l_i - k_j)}$$

Following Li and Zhang, 2023, we discuss the discrete KdV and KP equations. For the bilinearization, the bilinear form is given by:

$$(u - \widehat{\tilde{u}})(\tilde{u} - \widehat{b}) = p^2 - q^2$$

where:

$$p^2 = \wp(\delta) - e_0, \quad q^2 = \wp(\epsilon) - e_0$$

$$u = \zeta(\xi + N\gamma) - N\zeta(\gamma) - n\zeta(\delta) - m\zeta(\epsilon) - h\zeta(\gamma) - \zeta(\xi_0) + \frac{g}{f}$$

$$\xi = n\delta + m\epsilon + h\gamma.$$

Then

$$\mathcal{H}_1 \equiv \chi_{\delta,-\epsilon}(\widehat{\xi} + N\gamma)\tilde{f}\widehat{f} + \tilde{f}\widehat{g} - \tilde{g}\widehat{f} - \Phi_\delta(-\epsilon) f \widehat{\tilde{f}} = 0$$

and

$$\mathcal{H}_2 \equiv \chi_{\delta,\epsilon}(\xi + N\gamma) f \widehat{\tilde{f}} + \widehat{\tilde{f}} g - \tilde{g} f - \Phi_\delta(\epsilon) \tilde{f} \widehat{f} = 0$$

where the function $\chi_{u,v}(z)$ is given by:

$$\chi_{u,v}(z) = \zeta(u) + \zeta(v) + \zeta(z) - \zeta(u + v + z)$$

The NSS is expressed as follows:

$$f = \sigma(\xi)|\widehat{N-1}|, \quad g = \sigma(\xi)|\widehat{N-2}, N|$$

where the vector ϕ is given by:

$$\phi = (\phi_1, \cdots, \phi_N)^T, \quad \phi_i = \rho^-_{n,m,h}(k_i)\Phi_\xi(k_i) + \rho^-_{n,m,h}(l_i)\Phi_\xi(l_i)$$

where $\rho^{\pm}_{n,m,h}(z)$ is defined as:

$$\rho^{\pm}_{n,m,h}(z) = \left(\frac{\sigma(\delta \pm z)}{\sigma(\delta)\sigma(\pm z)} \right)^n \left(\frac{\sigma(\epsilon \pm z)}{\sigma(\epsilon)\sigma(\pm z)} \right)^m \left(\frac{\sigma(\gamma \pm z)}{\sigma(\gamma)\sigma(\pm z)} \right)^h \rho^{\pm}_{0,0,0}$$

Now we move on to discuss the lpKdV τ function and vertex operator. The τ function is given by:

$$\tau_N = \sum_{J\subset S} \frac{\sigma\left(\xi + 2\sum_{i\in J} k_i\right)}{\sigma(\xi)\prod_{i\in J}\sigma(2k_i)} \left(\prod_{i,j\in J, i<j} A_{ij}\right) \prod_{i\in J} \rho_{n,m,h}(k_i)$$

where A_{ij} is defined as:

$$A_{ij} = \left(\frac{\sigma(k_i - k_j)}{\sigma(k_i + k_j)}\right)^2$$

and $\rho_{n,m,h}(k_i)$ is defined as:

$$\rho_{n,m,h}(k_i) = \left(\frac{\sigma(k_i-\delta)}{\sigma(k_i+\delta)}\right)^n \left(\frac{\sigma(k_i-\epsilon)}{\sigma(k_i+\epsilon)}\right)^m \left(\frac{\sigma(k_i-\gamma)}{\sigma(k_i+\gamma)}\right)^h \rho_{0,0,0}(k_i)$$

In this case, the vertex operator is the same as for the continuous KdV case. After redefining the coefficients c_i

$$c_i = \left(\frac{\sigma(k_i-\gamma)}{\sigma(k_i+\gamma)}\right)^h \rho_{0,0,0}(k_i)$$

and introducing Miwa's coordinates

$$t_{2j+1} = \frac{\delta^{2j+1}n + \epsilon^{2j+1}m}{2j+1}$$

we obtain

$$\rho = e^{\theta_{[e]}(\bar{\mathbf{t}},k)}$$

where $\theta_{[e]}(\bar{\mathbf{t}},k)$ is given by:

$$\theta_{[e]}(\bar{\mathbf{t}},k) = -2\sum_{n=0}^{\infty} \frac{\zeta^{(2n)}(k)}{(2n)!} t_{2n+1}.$$

The equation for lpKP is given by:

$$(\widehat{\widetilde{w}} - \widehat{\tilde{w}})(\overline{w} - \widehat{w}) = (\tilde{\widetilde{w}} - \widehat{\tilde{w}})(\overline{w} - \tilde{w})$$

where

$$w = \zeta(\xi + N\gamma) - N\zeta(\gamma) - n\zeta(\delta) - m\zeta(\epsilon) - h\zeta(\gamma) - \zeta(\xi_0) + \frac{g}{f}.$$

The following equations hold for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H}_1 \equiv \chi_{\epsilon,-\gamma}(\xi + (N+1)\gamma)\overline{f}\widehat{f} + \overline{g}\widehat{f} - \widehat{g}\overline{f} - \Phi_\epsilon(-\gamma) f\widehat{\overline{f}} = 0$$

$$\mathcal{H}_2 \equiv \chi_{\delta,-\gamma}(\xi + (N+1)\gamma)\tilde{f}\overline{f} + \overline{g}\tilde{f} - \tilde{g}\overline{f} - \Phi_\delta(-\gamma) f\tilde{\overline{f}} = 0$$

There is also more research on what is known as the Casoratian solution, the τ function, the vertex operator, and the discrete AKP reduction (due to Wang, Zhang, Maruno, 2024), but we will not discuss them here.

21.3 Discrete Linearisation (DL) Approach

We will now move on to discuss the elliptic DL scheme and the Marchenko equation.

21.3.1 Elliptic Scheme of DL Approach

Proposition 21.41 (Fokas, Ablowitz, 1981)**.** *The DL for the KdV equation is given by*

$$\varphi(x,t;k) + ie^{i(kx+k^3t)} \int_L \frac{\varphi(x,t;l)}{l+k} d\lambda(l) = e^{i(kx+k^3t)},$$

where $u = -\partial_x \int_L \varphi(x,t;l) d\lambda(l)$.

In this case, the Lax pair is needed.

Proposition 21.42 (Nijhoff, Qiuspel, Caple, et al, 1980s)**.** *The DL+ is an infinite matrix. Given*

$$\mathbf{u}_k + \rho_k \int\int_D d\lambda(l,l')\mathbf{u}_l \sigma_l^0 \Omega_{k,l'} = \rho_k \mathbf{c}_k,$$

where $\mathbf{c}_k = (\cdots, k^{-1}, 1, k, \cdots)^T$ *and* $\Omega_{k,l'} = \frac{1}{k+l'}$*, the matrix* $\mathbf{U}$ *is defined as:*

$$\mathbf{U} = \int\int_D d\lambda(k,k')\mathbf{u}_k^t \mathbf{c}\sigma_{k'}.$$

The Discrete PWF is given by

$$\rho_k = \prod_{j=1}^N (p_j + k)^{n_j}, \quad \sigma_k^0 = \prod_{j=1}^N (p_j - k')^{-n_j}.$$

and the continuous PWF is given by

$$\rho_k = e^{\sum_{j=1}^\infty k_j t_j}, \quad \sigma_{k'} = e^{-\sum_{j=1}^\infty k'^j t_j}.$$

In this case, there is no need to check the Lax pair!

Now we move onto some major results on the DLA case.

Proposition 21.43 (Nijhoff, Sun, Zhang, 2023)**.** *In the DLA case, the scheme is given by*

$$\mathbf{u}_\kappa + \rho_\kappa \int\int_D d\mu(\ell,\ell')\sigma_{\ell'}\mathbf{u}_\ell \Phi_\xi(\kappa+\ell') = \rho_\kappa \Phi_\xi(\mathbf{\Lambda})\mathbf{c}_\kappa,$$

where $\mathbf{U}_\xi := \int\int_D d\mu(\ell,\ell')\mathbf{u}_\ell(\xi)^t \mathbf{c}_{\ell'}\sigma_{\ell'}\Phi_\xi(t\mathbf{\Lambda})$*, and* $\Phi_\xi(x) := \frac{\sigma(x+\xi)}{\sigma(x)\sigma(\xi)}$*,* $\mathbf{\Lambda}\mathbf{c}_\kappa, \xi = \xi_0 - n\delta - m\epsilon - l\nu$*. Additionally,*

$$\rho_\kappa(n,m,l) = (\Phi_\delta(\kappa))^n (\Phi_\epsilon(\kappa))^m (\Phi_\nu(\kappa))^l \rho_\kappa(0,0,0),$$

and

$$\sigma_{\kappa'}(n,m,l) = (\Phi_\delta(-\kappa'))^{-n} (\Phi_\epsilon(-\kappa'))^{-m} (\Phi_\nu(-\kappa'))^{-l} \sigma_{\kappa 0}(0,0,0).$$

Proposition 21.44 (Nijhoff, Sun, Zhang, 2023)**.** *In the DLA case, the lattice KP is given by*

$$\begin{aligned}&[\zeta(\delta)-\zeta(\epsilon)+\zeta(\xi-\delta)-\zeta(\xi-\epsilon)]\,\widehat{\widetilde{u}}(\xi-\delta-\epsilon)\\&+[\zeta(\nu)-\zeta(\delta)+\zeta(\xi-\epsilon-\nu)-\zeta(\xi-\delta-\epsilon)]\,\widehat{u}(\xi-\epsilon)\\&+\left[\widehat{\widetilde{u}}(\xi-\delta-\epsilon)-\widehat{\widetilde{u}}(\xi-\epsilon-\nu)\right]\widehat{u}(\xi-\epsilon)+\mathit{cyclical}=0\end{aligned}$$

where $u(\xi):=(\mathbf{U}_\xi)_{0,0}$

Define the variables

$$v_\alpha(\xi)=1-\left([\zeta(\xi)+\zeta(\alpha)+\zeta(\mathbf{\Lambda})-\zeta(\xi+\alpha+\mathbf{\Lambda})]^{-1}\,\mathbf{U}_\xi\right)_{0,0},$$

$$w_\alpha(\xi)=1-\left(\left[\mathbf{U}_\xi\left(\zeta(\xi)+\zeta(\alpha)+\zeta({}^t\mathbf{\Lambda})-\zeta(\xi+\alpha+{}^t\mathbf{\Lambda})\right)\right]^{-1}\right)_{0,0},$$

$$\begin{aligned}s_{\alpha,\beta}(\xi)=\Big([\zeta(\xi)+\zeta(\alpha)+\zeta(\mathbf{\Lambda})-\zeta(\xi+\alpha+\mathbf{\Lambda})]^{-1}\cdot\mathbf{U}_\xi\\\cdot\left[\zeta(\xi)+\zeta(\beta)+\zeta(\mathbf{\Lambda}^t)-\zeta(\xi+\beta+\mathbf{\Lambda}^t)\right]^{-1}\Big)_{0,0}.\end{aligned}$$

Proposition 21.45 (Nijhoff, Sun, Zhang, 2023)**.** *For lattice mKP, the equation is given by:*

$$\begin{aligned}&[\zeta(\delta)-\zeta(\xi-\varepsilon)-\zeta(\alpha)+\zeta(\xi+\alpha-\delta-\varepsilon)]\hat{v}_\alpha(\xi-\varepsilon)\hat{\tilde{v}}_\alpha(\xi-\delta-\varepsilon)\\&-[\zeta(\varepsilon)-\zeta(\xi-\delta)-\zeta(\alpha)+\zeta(\xi+\alpha-\delta-\varepsilon)]\tilde{v}_\alpha(\xi-\delta)\hat{\tilde{v}}_\alpha(\xi-\delta-\varepsilon)+\mathit{cycl.}=0\end{aligned}$$

The lattice SKP equation is:

$$\begin{aligned}&\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\nu)s_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\nu)\tilde{s}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\nu)s_{\alpha,\beta}(\xi-\nu)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\varepsilon-\nu)\hat{s}_{\alpha,\beta}(\xi-\varepsilon-\nu)}\\=&\frac{1-\chi^{(1)}_{\alpha,-\delta}(\xi-\varepsilon)\hat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\delta}(\xi-\delta-\varepsilon)\hat{\tilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)}{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\varepsilon)\hat{s}_{\alpha,\beta}(\xi-\varepsilon)-\chi^{(1)}_{\beta,\nu}(\xi-\varepsilon-\nu)\hat{s}_{\alpha,\beta}(\xi-\varepsilon-\nu)}\\\times&\frac{1-\chi^{(1)}_{\alpha,-\nu}(\xi-\delta)\tilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\nu}(\xi-\delta-\nu)\tilde{s}_{\alpha,\beta}(\xi-\delta-\nu)}{1-\chi^{(1)}_{\alpha,-\varepsilon}(\xi-\delta)\tilde{s}_{\alpha,\beta}(\xi-\delta)-\chi^{(1)}_{\beta,\varepsilon}(\xi-\delta-\varepsilon)\hat{\tilde{s}}_{\alpha,\beta}(\xi-\delta-\varepsilon)}\end{aligned}$$

where:

$$\chi^{(1)}_{\delta,\varepsilon}(\gamma)=\zeta(\delta)+\zeta(\varepsilon)+\zeta(\gamma)-\zeta(\delta+\varepsilon+\gamma)$$

Now we move onto the DBSQ.

Definition 21.46. *The DBSQ is given by*

$$p_\xi\tilde{u}_{0,0}+\tilde{u}_{0,1}=p_\xi u_{0,0}-u_{1,0}-\tilde{u}_{0,0}u_{0,0},$$

$$q_\xi \hat{u}_{0,0} + \hat{u}_{0,1} = q_\xi u_{0,0} - u_{1,0} - \hat{u}_{0,0} u_{0,0},$$

$$\frac{1}{2\wp'(\delta) - \wp'(\varepsilon)}(p_\xi - q_\xi + \hat{u}_{0,0} - \tilde{u}_{0,0}) = \frac{1}{2\wp'(\delta) - \wp'(\varepsilon)}(p_\xi - q_\xi) + \hat{\tilde{u}}_{1,0} + u_{0,1} + u_{0,0}\hat{\tilde{u}}_{0,0} + (p_\xi + q_\xi)(\hat{\tilde{u}}_{0,0} - u_{0,0}),$$

where

$$u_{0,0} := (\mathbf{U}_\xi)_{0,0}, \quad u_{1,0} := (\mathbf{\Lambda}_\xi \mathbf{U}_\xi)_{0,0}, \quad u_{0,1} := (U_\xi{}^t \mathbf{\Lambda}_\xi)_{0,0}$$

Proposition 21.47. *The deformation of DBSQ is given by*

$$\tilde{w} - u\tilde{u} + v = 0,$$

$$\hat{w} - u\hat{u} + v = 0,$$

$$\frac{1}{2\wp'(\delta) - \wp'(\varepsilon)}(\hat{u} - \tilde{u}) = w - u\hat{\tilde{u}} + \hat{\tilde{v}}$$

Proposition 21.48. *Given the transformation*

$$u_{0,0} = x_0 - u, \quad u_{1,0} = y_0 - v - x_0 u_{0,0}, \quad u_{0,1} = z_0 - w - x_0 u_{0,0},$$

where

$$x_0 = \zeta(\xi) + n\zeta(\delta) + m\zeta(\varepsilon) - \zeta(\xi_0), \quad \xi = \xi_0 - n\delta - m\varepsilon,$$

$$y_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)),$$

$$z_0 = \frac{1}{2}x_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)),$$

we obtain the 1-component form (9-point)

$$\frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))(\hat{\tilde{u}} - \tilde{\tilde{u}}) - \frac{1}{2}(\wp'(\delta) - \wp'(\varepsilon))(\hat{\tilde{u}} - \hat{\tilde{u}}) = (\hat{\tilde{u}} - \hat{\tilde{\tilde{u}}})(\hat{u} - \hat{\tilde{\tilde{u}}}) - (\hat{u} - \tilde{u})(u - \hat{\tilde{\tilde{u}}})$$

There are other DBSQs, but we won't discuss them.

21.3.2 Marchenko Equation

Proposition 21.49 (Fokas, Ablowitz, 1981)**.** *The DLA for the KdV is given by*

$$\phi(x,t;k) + ie^{i(kx+k^3t)} \int_L \frac{\phi(x,t;l)}{l+k} d\lambda(l) = e^{i(kx+k^3t)},$$

where

$$u = -\partial_x \int_L \phi(x,t;l) d\lambda(l), \text{ Lax pair is needed.}$$

In this case, the Lax pair is needed.

To transform from the DLA to GLM, we can substitute

$$\psi(x,t;k) = \phi(x,t;k)e^{-i(kx+k^3t)},$$
$$K(x,y,t) = -\frac{1}{2}\int_L \psi(x,t;k)e^{i(ky+k^3t)}d\lambda(k),$$
$$F(x,t) = \int_L e^{i(ky+k^3t)}d\lambda(k),$$
$$K(x,y,t) + F(x+y,t) + \int_{+\infty}^{x} K(x,\xi;t)F(y+\xi,t)d\xi = 0,$$
$$u = 2\partial_x K(x,x;t).$$

Let's move on to the Fokas-Ablowitz's elliptic DLA. Recall that the KP equation is given by

$$u_t + u_{xxx} + 6uu_x + 3\partial_x^{-1}u_{yy} = 0.$$

Proposition 21.50. *The elliptic DLA for the KP is given by*

$$\psi(x,y,t;k) + \rho_k(y,t)\int\int_D \psi(x,y,t;l)\sigma_{l'}(y,t)\Psi_x(k,l')d\lambda(l,l') = \Psi_x(k)\rho_k(y,t),$$

where

$$u(x,y,t) = -2\wp(x) - 2\partial_x \int\int_D \psi(x,y,t;l)\sigma_{l'}(x,y,t)\Phi_x(l')d\lambda(l,l').$$

If we set

$$\rho_k(x,y,t) = \exp(\wp(k)y - 2\wp'(k)t + \rho^{(0)}(k)),$$
$$\Psi_x(k) = \frac{\sigma(x+k)}{\sigma(x)\sigma(k)}e^{-\zeta(k)x} = \Phi_x(k)e^{-\zeta(k)x},$$

then the Lax pair becomes

$$P\psi(x,y,t) = 0, \quad P = \partial_y - \partial_x^2 - u(x,y,t),$$
$$M\psi(x,y,t) = 0, \quad M = \partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x + 3\partial_x^{-1}u_y.$$

Proposition 21.51. *In the real-valued case where $w_1 > 0$, w_2 purely imaginary, we have*

$$\psi(x,y,t;k) + \rho_k(y,t)\int\int_D \psi(x,y,t;l)\sigma_{l'}(y,t)\tilde{\Psi}_x(k,l')d\lambda(l,l') = \tilde{\Psi}_x(k)\rho_k(y,t),$$

where

$$\tilde{\Psi}_x(a) = \frac{\sigma(x+a+w_2)}{\sigma(x+w_2)\sigma(a)}e^{-\zeta(a)x-\zeta(w_2)a},$$

and

$$\tilde{\Psi}_x(a,b) = \frac{\sigma(x+a+b+w_2)}{\sigma(x+w_2)\sigma(a+b)}e^{-(\zeta(a)+\zeta(b))x-\zeta(w_2)(a+b)},$$

Proposition 21.52. *In the non-singular case, $\tilde{\Psi}_x(k,l)$ exponentially decays when $x \to +\infty$:*

$$\int_{+\infty}^{x} \tilde{\Psi}_\xi(k)\tilde{\Psi}_\xi(l')d\xi = \tilde{\Psi}_x(k,l).$$

Now we finally reach the Marchenko equation.

Proposition 21.53. *The Marchenko equation is given by*

$$K(x,s,y,t) + F(x,s,y,t) + \int_{+\infty}^{x} K(x,\xi,y,t)F(\xi,s,y,t)d\xi = 0,$$

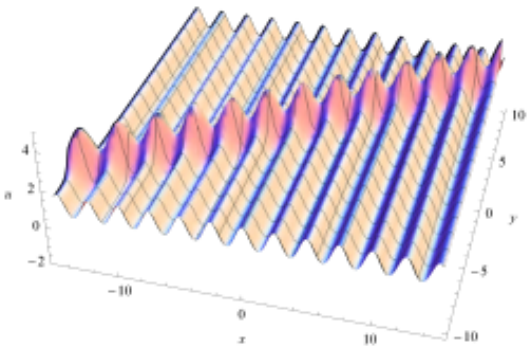
$$K(x,s,y,t) = -\int\int_D \psi(x,y,t;k)\tilde{\Psi}_s(k')\sigma_{k'}(y,t)d\lambda(k,k'),$$

$$F(x,s,y,t) = \int\int_D \tilde{\Psi}_x(k)\rho_k(y,t)\tilde{\Psi}_s(k')\sigma_{k'}(y,t)d\lambda(k,k'),$$

and it has the solution

$$u(x,y,t) = -2\wp(x+w_2) + 2\frac{\partial}{\partial x}K(x,x,y,t).$$

We conclude this section with an image of the elliptic 1-soliton of KP.



21.4 Related Problems

Consider the discrete Krichever-Novikov equation

$$p(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\hat{\tilde{u}}) - r(u\hat{\tilde{u}} + \tilde{u}\hat{u}) + pqr(1 + u\tilde{u}\hat{u}\hat{\tilde{u}}) = 0$$

where

$$(p,P) = (\sqrt{k}\,\mathrm{sn}(\alpha;k), \mathrm{sn}'(\alpha;k)), \quad (q,R) = (\sqrt{k}\,\mathrm{sn}(\beta;k), \mathrm{sn}'(\beta;k)),$$

$$(r,R) = (\sqrt{k}\,\mathrm{sn}(\gamma;k), \mathrm{sn}'(\gamma;k)), \quad \gamma = \alpha - \beta.$$

Proposition 21.54. *What are the algebras associated with the vertex operators given by:*

$$X(k) = \Phi_{t_1}(2k) e^{\theta_{[e]}(t,k)} e^{\theta(\tilde{\partial},k)}$$

and

$$\tau_N(t) = e^{c_N} X(k_N) \tau_{N-1}(t), \quad \tau_0(t) = 1?$$

Proposition 21.55. *Is there a higher order analogue of the Boussinesq equation*

$$\wp^{(4)}(\omega_1(\delta)) - \wp^{(4)}(\delta) = 30(\wp'(\omega_1(\delta)) - \wp'(\delta))(\wp'(\omega_1(\delta)) + \wp'(\delta)) + 12 g_2(\wp(\omega_1(\delta)) - \wp(\delta))?$$

Proposition 21.56. *Does there exist a new formulation of KP?*

Solution. In [Kakei, 2023] the following was discovered:

$$L_0 = V_0 \partial_x^{-1} V_0^{-1},$$

$$V_0 = 1 + \sum_{n=0}^{\infty} v_{0,n}(x) \partial_x^{-1},$$

$$v_{0,1}(x) = -\zeta(x), \quad v_{0,2}(x) = \frac{1}{2}[\zeta^2(x) - \wp(x)], \ldots$$

$$L_0 = \partial_x - \wp(x)\partial_x^{-1} + \frac{1}{2}\wp'(x)\partial_x^{-2} + \ldots,$$

$$L_0^2 = \partial_x^2 - 2\wp(x) - \frac{g_2}{10}\partial_x^{-2} - 6\wp(x)\wp'(x)\partial_x^{-3} + \ldots,$$

But there is still plenty more to explore. □

This is a good place to stop.

22 Anton Dzhamay: Geometry and Symmetry of Painlevé Equations

Abstract

We begin by an overview of how geometric ideas entered the theory of differential Painlevé equations in the work of of K.Okamoto, which led to the better understanding of their symmetries (Backlünd transformations) in terms of affine Weyl groups. These ideas were then extended by H.Sakai to the discrete (elliptic, multiplicative, and additive) Painlevé equations and resulted in the beautiful Sakai classification scheme for both differential and discrete Painlevé equations. In the latter case, it is the symmetry group that is the source of a discrete dynamics. In the second part of the talk we discuss the notion of an abstract discrete Painlevé equation and its various concrete realizations. This leads to the study of a refined identification problem, which is a classification of different orbits for the same abstract discrete Painlevé dynamic, and results in the appearance of special symmetry groups that are not a part of the general (i.e., generic) Sakai classification scheme. We illustrate this by an example of a discrete Painlevé-II equation and its symmetry group. This is based on a joint work with Yang Shi, Alex Stokes, and Ralph Willox.

Contents

22.1 Introduction

Instead of focusing on formulas and equations, we will see the general picture of the Painlevé equations from a geometric viewpoint. The plan of the talk is as follows:

- What are the Painlevé equations and why are people interested in them?
- How does geometry appear through the Okamoto space of initial conditions (a family of rational algebraic surfaces called the generalized Halphen surfaces)?

  The big idea is as follows: given some equation, we get a geometric object. There are symmetries associated with this geometric object called the Cremona isometries. These Cremona isometries are associated with surfaces and are linked to the group structure. In this case, the group structure of the Cremona isometries corresponds to an (affine and extended) Weyl group $\tilde{W}(R = D)$.
- By examining the geometry of this Weyl group, it becomes much more clear how to compute the symmetries as the symmetries create transformations in the equations, called Bäcklund transformations.
- Since the Weyl group is affine, we have translations, which we can think of as defining a discrete dynamical system, where we move along a fixed vector in affine space. Associated with this fixed vector is a nonlinear equation, which is commonly referred to as the discrete Painlevé equation. These discrete Painlevé equations are so named because we can take a continuous limit and return to a standard Painlevé equation.

  A typical process might involve starting with a Painlevé equation, like Painlevé VI, and finding its associated geometric object. You then examine the symmetries in this Painlevé VI case, which would correspond to an affine D_4 symmetry, observe the translations, and take the continuous limit. This process would yield a discrete Painlevé V equation.

Let's start with something really simple: $y' = y$. From this ODE, we get a (special) function $y(t) = Ce^t = C\sum \frac{t^n}{n!}$ for $t \in \mathbb{C}$.

So ODEs are interested because they often lead to new functions. For instance, if we go a step beyond simple equations, we encounter something like the Gauss hypergeometric equation:

$$E(\alpha, \beta, \gamma) : \frac{d^2W}{dt^2} + \left(\frac{\gamma - (\alpha + \beta + 1)t}{t(1-t)}\right)\frac{dW}{dt} - \frac{\alpha\beta}{t(1-t)}W = 0$$

This is one of the fundamental examples of differential equations that give rise to special functions through various degenerations. When we allow $t \in \mathbb{CP}^1$, this equation has three singular points: one at $t = 0$, another at $t = 1$, and the

third at ∞. These singular points arise because the coefficients of the equation become singular at these values.

Singularities can occur in two different contexts: within the equations themselves or within their solutions. The question is: how are these two types of singularities connected? If an equation has a singularity, does the solution necessarily have one as well?

For linear ordinary differential equations, the situation is more straightforward. If the coefficients of the equation stop being analytic at a certain point, then the solution may also develop a singularity at that point. However, it's not guaranteed. For example, in the case of the generalized hypergeometric equation, with parameters α, β, and γ, the solutions can exhibit different behaviors: we have

$$\begin{bmatrix} t=0 & t=1 & t=v_0 \\ 0 & 0 & \alpha \\ z-\gamma & 1-\gamma-\beta & \beta \end{bmatrix}$$

where a fundamental system of solutions might include one solution that remains regular, while another may exhibit power-law behavior near the singular point. So, in the linear case, the singularities of the solutions are entirely controlled by the equation itself.

However, when we transition from linear to nonlinear differential equations, things become more complicated. Many of the nice properties of linear equations disappear, making the study of singularities more challenging. If we're using differential equations to describe special functions, it makes sense to try and construct nonlinear special functions from nonlinear equations. But this introduces new difficulties.

Let's consider a very simple example: the equation $\frac{dw}{dt} = w^2$ with $w(t_0) = w_0$. This is a first-order separable equation that can be easily solved be rewritting it as

$$-\frac{dw}{w^2} = dt$$

and then substituting in the initial condition gives

$$w(t) = \frac{w_0}{1 - w_0(t - t_0)}$$

This solution has a pole because the denominator vanishes when $t = \frac{1}{w_0} + t_0$. The location of this pole is a movable singularity because it depends on the initial condition w_0. In this case, it's not the equation itself that determines where the singularity occurs, but rather the initial conditions.

While poles are manageable, if we look at a more complex equation like $3w^2 \frac{dw}{dt} = 1, w(0) = w_0$, the solution involves branch points and branch cuts. The equation might have a solution like:

$$w^3 = t + w_0^3 \implies w = (t + w_0^3)^{\frac{1}{3}}$$

so we have singularities at $-w_0^3$ and ∞, and we might want to introduce a branch cut. The location of the branch cut depends on the initial condition, which complicates things further. If we want to construct the Riemann surface for this solution, we would need to account for these branch points, making the solution much more difficult to handle.

So, while movable poles can be tolerated, movable branch points pose more significant issues. Mathematically speaking, equations with only poles are said to have the Painlevé property, while those with branch points do not. The Painlevé property refers to differential equations whose general solutions have no movable singularities other than poles. Specifically, the Painlevé property ensures that the general solution has no movable branch points.

22.2 The Painlevé Equations

Painlevé, along with his student Gambier and Fuchs, classified second-order equations that satisfy this property. These are now known as the Painlevé equations, which take the form:

1. **Painlevé I:**
$$\frac{d^2w}{dt^2} = 6w^2 + t$$

2. **Painlevé II:**
$$\frac{d^2w}{dt^2} = 2w^3 + tw + \alpha$$

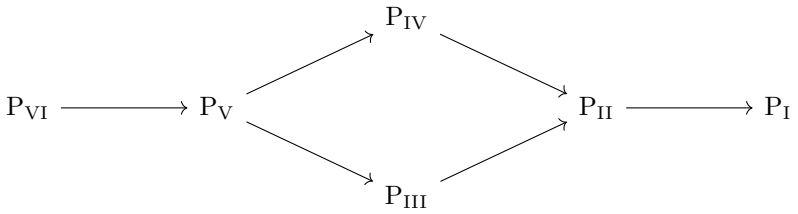
3. **Painlevé III:**
$$\frac{d^2w}{dt^2} = \frac{1}{w}\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{\alpha w^2 + \beta}{t} + \gamma w^3 + \frac{\delta}{w}$$

4. **Painlevé IV:**
$$\frac{d^2w}{dt^2} = \frac{1}{2w}\left(\frac{dw}{dt}\right)^2 + \frac{3}{2}w^3 + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}$$

5. **Painlevé V:**
$$\frac{d^2w}{dt^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{t} + \delta\frac{w(w+1)}{w-1}$$

6. **Painlevé VI:**
$$\frac{d^2w}{dt^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)\frac{dw}{dt}$$
$$+ \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{w^2} + \gamma\frac{t-1}{(w-1)^2} + \delta\frac{t(t-1)}{(w-t)^2}\right)$$

The Painlevé equations form a cascade, meaning that we can start with a higher-level equation like Painlevé V1 and degenerate it to lower levels, such as Painlevé V, Painlevé IV, Painlevé III, and so on down to Painlevé I. Each level has specific parameters, often denoted as α and β, which can yield special solutions. For instance, certain values of these parameters can produce hypergeometric functions as solutions to Painlevé VI.



P_V was studied by Kummer, P_{III} by Bessel, P_{IV} by Hermite, and P_{II} by Airy. It's important that there are some values of parameters that are special. But in general, it is very difficult to keep track of solutions of Painlevé equations: if the method of Frobenius doesn't work, how do we represent a solution? How do we study its properties? There are a large number of techniques. For example, we can use the Riemann Hilbert analysis. One approach to describe solutions is due to Okamoto.

22.3 Okamoto's Approach

Okamoto decided to understand the solutions by parameterizing them using initial conditions, which a very natural approach: By Cauchy's theorem, wer initial condition uniquely determines wer solution in some time interval.

For example, if we start with P_{IV}, it's better to rewrite it as a Hamiltonian system,

$$\mathrm{H}_{IV}(f,g;t) = 2fg^2 - (f^2 + tf + \kappa_0)g + \theta_\infty f.$$

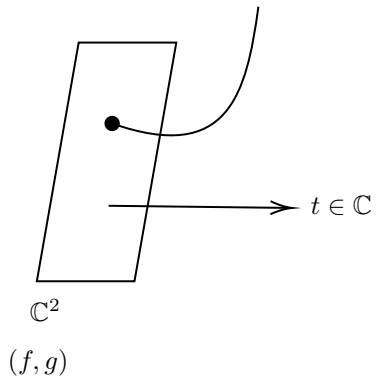
Notice that even though this function is called a Hamiltonian, it is actually time-dependent. So, it is not a constant of motion; it evolves with time. Nevertheless, we can still use the symplectic form. Taking the standard symplectic form $\omega = dg \wedge df$, we generate the equations:

$$\begin{aligned}\dot{F} &= Hg = 4fg - (f^2 + tf + \kappa_0)\\ \dot{g} &= -Hf = -2g^2 + 2fg + tg - \theta_\infty\end{aligned}$$

Notice that F is not just a coordinate and G is not just a momentum - it's a little more complicated. It turns out that this system, rewritten for $F(t)$, is equivalent to P_{IV} for $\omega(t) = f(t)$. We use parameters $\alpha = 1 + 2\theta_\infty - \kappa_0$ and $\beta = -\kappa_0^2$. The reason why we are showing these parameters is because when

we have equations coming from a particular applied problem, we need to use geometry to match the parameters of the problem with the standard parameters.

We can show that this system is equivalent to P_{IV}, and then we can start thinking about the space of initial conditions:



If we pick a point, we will have a solution. Remember that the type of singularities we allow for Painlevé equations are poles, so we can let the solution go to ∞ at some value of t. But on the other hand, some solution can go to ∞ for all values of t. To handle these infinities, we need to projectivize $\mathbb{C}^2$. There are two ways to do this:

- $\mathbb{C}^2 \to \mathbb{P}^2$. This is the better one for Painlevé I (which we are not focusing on).
- $\mathbb{C}^2 \to \mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. This is what we will do.

Anyone with knowledge of algebraic geometry will recognize that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are different, but birationally equivalent. This gives us a compact, complex surface (a two-dimensional complex algebraic variety). Since it's compact, we can discuss its homology, cohomology, etc.

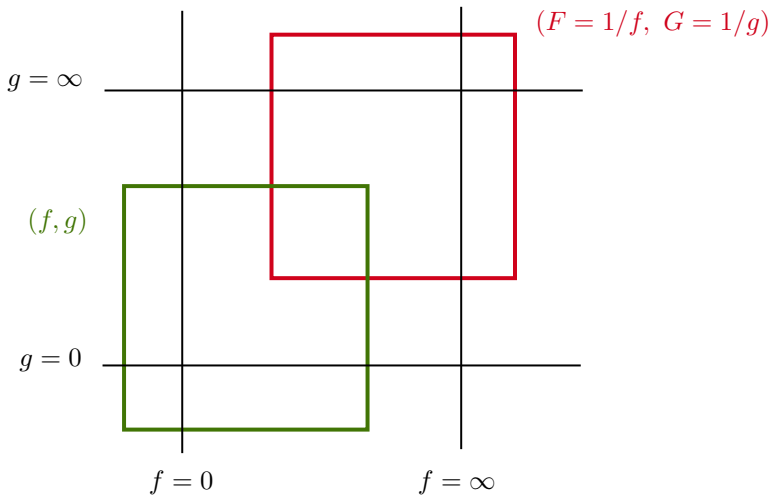
For example, by Poincaré duality,

$$H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) \simeq H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) \simeq H^1(-, \mathcal{O}^*) \simeq \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathrm{Div}/\sim$$

The divisor classes of curves are irreducible, co-dimension-one submanifolds and satisfy $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_f, \mathcal{H}_g\}$.

Now, let's think topologically. $\mathbb{P}^1 \times \mathbb{P}^1$ is similar to $S^2 \times S^2$, similar to how a torus is a product of two circles. In that case, each sphere is a generator in homology, and we have something similar here.

We can visualize this as follows:



The group $\mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_f, \mathcal{H}_g\}$ is equipped with an intersection product:

$$\mathcal{H}_f \cdot \mathcal{H}_f = \mathcal{H}_g \cdot \mathcal{H}_g = 0$$

$$\mathcal{H}_f \cdot \mathcal{H}_f = 1.$$

The second equation can be interpreted as: if we take a line $f = a$ and $g = b$, they intersect at one point. Of course, $\mathcal{H}_f$ and $\mathcal{H}_g$ are classes, but we can just allow deformations to make things work. To do this, take two representatives and use position or transversality. Take two representatives, where one is a shift of the other, and they do not intersect. The "right" way to think about this is that we can shift by pushing the zero section along the normal bundle of the embedding of this line in the projective plane, and we can do that in this case.

Recall that our symplectic form

$$\omega = dg \wedge df$$

is completely regular in the image, but if we extend it to the whole of $\mathbb{P}^1 \times \mathbb{P}^1$, it starts picking up poles. So, $\omega = dg \wedge df$ in the chart in the image, but it becomes $\omega = \frac{dG \wedge dF}{F^2 G^2}$ in the other chart, leaving us with poles of order 2.

Thus, the canonical class is

$$K_{\mathbb{P}^1 \times \mathbb{P}^1}[(\omega)] = -2\mathcal{H}_f - 2\mathcal{H}_g.$$

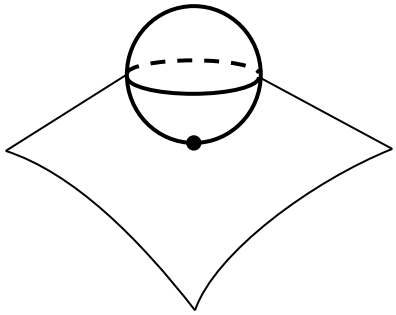
In Painlevé theory, this class plays a crucial role, turning into

$$-K = 2\mathcal{H}_f + 2\mathcal{H}_g$$

This is how geometry appears. However, we are not done because we can have more than one solution going to infinity.

If we have two solutions going to ∞, given a point, we would have one curve arrive at the point, followed by another curve, but they have different rates of growth. To fix this, we can perofrm the blow-up on the point, which introduces an additional direction of slopes, and then we lift the curve according to the slope. This separates them so that they don't intersect anymore.

All possible slopes (up to equivalence) form the projective line $\mathbb{P}^1$. But every time we do the blow-up, we must add the exceptional divisor, E, that corresponds to this line of slopes $E_i \simeq S^1 = \mathbb{P}^1$. Thus, Pic gets another class $\mathcal{E}_i = [E_i]$ which is the class of this sphere.



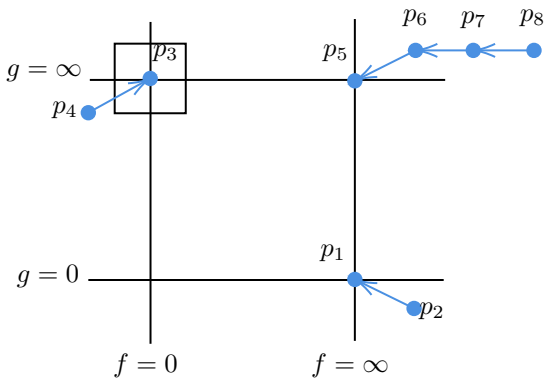
This is essentially a kind of surgery where we take a point, remove it, and then blow up a bubble at that point or $\#\overline{\mathbb{P}}^1$.

After performing this blow-up, the exceptional divisor E_I has the interesting property that it is what is called a minus-one curve:

$$\mathcal{E}_i^2 = -1 = c_1(N_{E \text{ in } X})$$

where X is the surface we get after the blow-up. Thus, we must adjust the geometry, as it is no longer just the projective geometry.

Consider Painlevé-IV. If we start with $\mathbb{C}^2$, add lines at infinity, and look at points where new trajectories can intersect. We start with 4 initial points. From the initial bad point, we do the surgery, create a bubble, extend the equation to that bubble, picked up another bad point, and continue this until we have created a ascade of infinitely close points, or we can think of it as curves touching each other. Eventually, we get something like this:



This is similar to a gauge transformation: we can always arrange for the points to be there, which normalizes the way the system looks. Before we proceed, we need to introduce coordinates to describe these points. Local coordinates around p_3 are given by $\left(f, G = \frac{1}{g}\right)$. Additionally, let

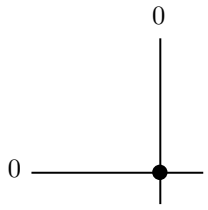
$$f = u_3 = U_3 V_3$$

$$G = u_3 v_3 = V_3$$

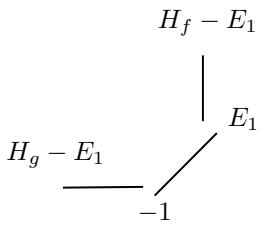
so that $v_3 = \frac{G}{f}$ is the slope coordinate. Now, we have two charts: one at the bottom of $\mathbb{P}^1$ and another at the top of $\mathbb{P}^1$. We can use either of these charts to specify the coordinates. If we do the computation, we get $p_4 = (u_3 = \kappa_0 m v_3 = 0)$, for example. The important point is that the coordinates of these points depend on the parameters of the equation, such as α, β, γ, or κ.

Now, we perform the blow-ups at these points and keep track of the intersection index of your curve. Let's start with point p_1.

Consider the intersection of the zero lines:

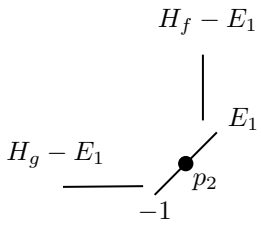


After blowing up the point, we get

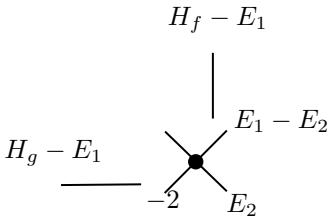


The intersection index is -1 for all of the lines.

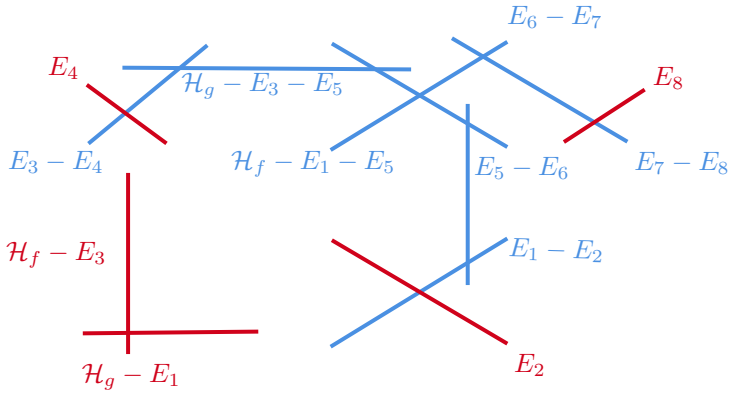
Then, more points appear:



If we blow up p_2, we get



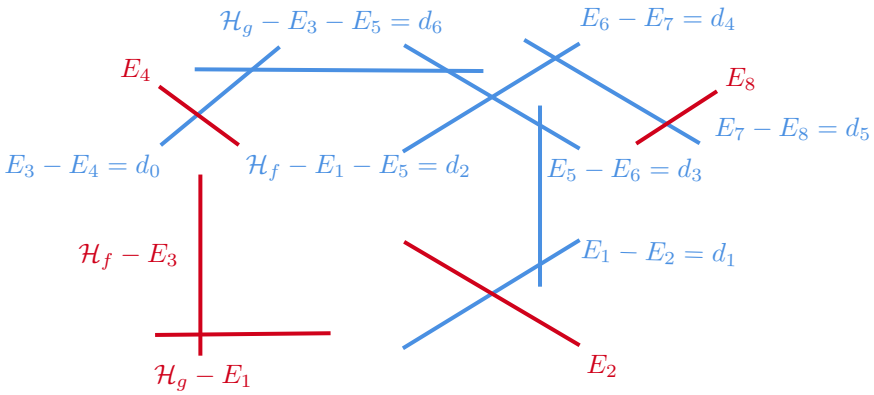
where the intersection index of $E_1 - E_2$ is -2. If we continue this, we get:



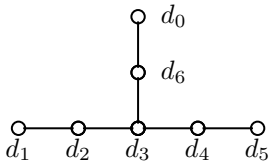
where the red curves are -1 and the blue curves are -2.

22.4 Towards Affine Dynkin Diagrams

This is where the affine Dynkin diagrams begin to appear. Let's look at the configuration of these blue curves. Labelling the blue curves with d_i:



we obtain the affine Dynkin diagram $E_6^{(1)}$:



This affine Dynkin diagram comes with a generalized Cartan matrix

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

This matrix is degenerate, having a null vector, which is exactly what the affine part of Dynkin diagram comes from. In fact, we can compute the coordinates of this vector directly from the diagram:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \delta = 0.$$

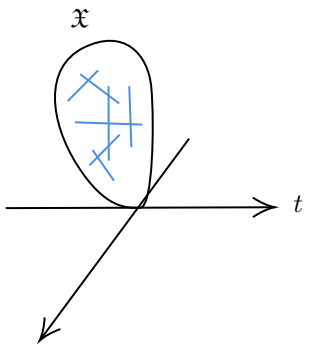
If we call the vector δ, it turns out that it is equal to the canonical divisor class, and then we have a decomposition of the anticanonical divisor classes:

$$\delta = -\kappa_X = \delta_0 + \delta_1 + 2\delta_2 + 3\delta_3 + 2\delta_4 + \delta_5 + 2\delta_6.$$

This decomposition corresponds to the geometry because each of these summands is the equivalence class of some -2 curve that we created.

When we construct a space of initial conditions, we need to add infinities to allow for points of intersection. When we add infinities, you get points where some trajectories merge. We separate them by blowing up, which creates a complicated algebraic surface. This surface encodes the space you created, and it is known as the $E_6^{(1)}$ surface.

From this surface, we can read off the symmetry



where the parameter space has parameters α, β or κ_0, θ_∞. Here,

$$\mathfrak{X} = \mathrm{Bl}_{p_1,\ldots,p_8}(\mathbb{P}^1 \times \mathbb{P}^1)$$

is a generalized Halphen surface, meaning that there is a unique anticanonical divisor class of canonical type:

$$(-\kappa_X) \in \mathrm{Pic}(\mathfrak{X}) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_f, \mathcal{H}_g, \mathcal{E}_1, ..., \mathcal{E}_8\}.$$

We can also look at $(-\kappa_X)^\perp$, which has two sublattices

$$\mathrm{Span}_{\mathbb{Z}}\{\delta_i\}$$

and

$$\mathrm{Span}_{\mathbb{Z}}\{\alpha_j\}$$

where $\alpha_j \cdot \delta_i = 0$. The first sublattice gives the surface, whereas the second gives the symmetry?

What do we mean by symmetry? First of all, the α's are generated by the condition that

$$\alpha_j \Delta_i = 0$$

These vectors are perpendicular to the Δ-vectors. You can compute what these α's are:

$$\alpha_0 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8.$$

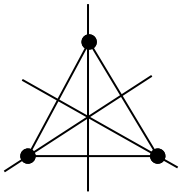
$$\alpha_1 = \mathcal{H}_f - \mathcal{E}_3 - \mathcal{E}_4$$
$$\alpha_2 = \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2$$

These form the Dynkin diagram $A_2^{(1)}$:

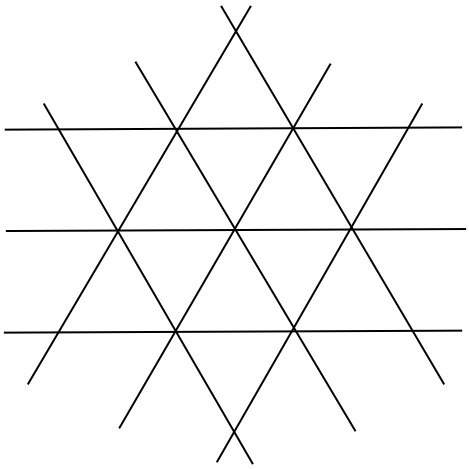
which is the symmetry group. We call it a symmetry group because it's an abstractified Weyl group with generators and relations:

$$W(A_2^{(1)}) = \langle w_0, ..., w_2 | w_i^2 = \mathrm{id}, w_i w_j w_i = w_j w_i w_j \rangle.$$

This group can be visualized as follows. If we consider the usual dihedral group D_3



we can forget about the triangle and take parallel shifts of the lines, we get a infinite triangular lattice which behaves like a kaleidoscope:



Taking a point and mirroring over the edges, we can create translations. Each translation is discrete Painlevé, so each element of the group induces a Backlünd transformation and their particular compositions give translational dynamics, which when written in coordinates, gives discrete Painlevé.

How does one get the elements? We have the representations of $W(A_2^{(1)})$ on $\mathrm{Pic}(\mathfrak{X})$. We can extend this to a vector space by tensoring $\mathrm{Pic}(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$, which

is done by reflections in the roots α_i:

$$w_i = r_{\alpha_i} : c \to c - 2\frac{c - \alpha_i}{\alpha_i \cdot \alpha_i}.$$

Finally, one can show

$$\begin{aligned} w_1 : f &\mapsto f \\ g &\mapsto g - \frac{\kappa_0}{f} \end{aligned}$$

This is an example of a symmetry of your system of equations, specifically a Backlünd transformation. We no longer fix the surface but move in a family of surfaces, giving us a symmetry that takes the solution of one Painlevé equation to the solution of another Painlevé equation for a different value of parameters. Iterating this with the limiting procedure, we get another Painlevé equation, and so on.

This is a good place to stop.