# Painlevé Equations and Symmetries

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These are my notes for Painlevé Equations and Symmetries, taught by Anton Dzhamay in Spring 2024.

## Contents

# 1 Motivation: New Functions

The motivation is creating new functions, such as elementary transcendental functions. What are these? With pure algebra on $x$, we can construct polynomials (e.g., $5x^7 - 3x^2 + 4$) and rational functions (e.g., $\frac{4x^2-3x+1}{x^2+x-7}$), but not much more. If we allow inverse functions, we can also take roots: $y = x^n \implies x = y^{1/n}$. What about "other" functions?

One way we can get new functions is through ODEs, allowing our solutions to be new functions. For example, if we take the ODE $y' = y$, with $y(x_0) = y_0$, then the solution is $y(x) = e^{(x-x_0)}y_0$, and now we have exponential functions.

Similarly, for the ODE $y' = y$ with $y(0) = 1$, we can assume the existence of an analytic solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We need convergence, etc., but for now, we assume everything we want exists. We can differentiate term-by-term:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \\ y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}. \end{aligned}$$

Setting the equations equal gives $a_1 = a_0$, $2a_2 = a_1$, $3a_3 = a_2$, $\ldots$, $(n+1)a_{n+1} = a_n$ with $a_0 = y(0) = 1$. Setting $a_0 = a_1 = 1$ gives $a_n = \frac{1}{n!}$ by induction. Thus, we obtain:

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Indeed, this series converges for $-\infty < x < \infty$ (using the Ratio test).

From the differential equation, we can deduce some basic properties, e.g., $e^{a+b} = e^a e^b$.

**Remark 1.1.** *We can also define new functions as solutions of functional equations or difference equations.*

**Example 1.2.** ***(The Gamma Function)***

*The Gamma function* $\Gamma$ *satisfies the functional equation* $\Gamma(x+1) = x\Gamma(x)$*, with* $\Gamma(1) = 1$*. In particular:*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt.$$

## 2 Picard-Lindelöf

Let $U$ be an open subset of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ (i.e., $(x, \vec{y})$) and let $\vec{f} : U \to \mathbb{R}^n$ be a continuous function. Consider the ODE $\vec{y}' = \vec{f}(x, \vec{y})$, with $\vec{y}(x_0) = \vec{y}_0$, where $(x_0, \vec{y}_0) \in U$.

Picard's idea is to transform the differential equation into an integral equation and use the contraction mapping principle:

$$\vec{y}(x) = \vec{y}_0 + \int_{x_0}^{x} \vec{f}(s, \vec{y}(s))\, ds.$$

Thus, we have:

$$\begin{aligned}
\vec{y}_0(x) &= \vec{y}_0, \\
\vec{y}_1(x) &= \vec{y}_0 + \int_{x_0}^{x} \vec{f}(s, \vec{y}_0)\, ds, \\
\vec{y}_2(x) &= \vec{y}_0 + \int_{x_0}^{x} \vec{f}(s, \vec{y}_1(s))\, ds, \\
&\vdots \\
\vec{y}_{n+1}(x) &= \vec{y}_0 + \int_{x_0}^{x} \vec{f}(s, \vec{y}_n(s))\, ds.
\end{aligned}$$

Define an operator $K(\vec{y})(x) := \vec{y}_0 + \int_{x_0}^{x} \vec{f}(s, \vec{y}(s))\, ds$ and introduce a norm in the continuous function space so that $K$ is a contraction with respect to this norm. Then there exists a fixed point $\vec{y}(x) = K(\vec{y})(x)$.

This idea is formalized in the following theorem:

**Theorem 2.1** (Picard-Lindelöf)**.** *If $\vec{f}$ is locally Lipschitz continuous in the second argument (i.e., $\vec{y}$) uniformly in the first argument, then there exists a unique (local) solution $\vec{y}(x)$ to the initial value problem.*

Recall:

$$\sup_{(x,\vec{y}) \neq (x,\vec{y}^*)} \frac{|\vec{f}(x, \vec{y}) - \vec{f}(x, \vec{y}^*)|}{|\vec{y} - \vec{y}^*|} := L$$

where $L$ is finite, and $(x, \vec{y}) \in V$ is compact $\subset U$.

# 3 Ordinary Differential Equations in the Complex Domain

Recall: Suppose we have a function $f : \mathbb{C} \supset U \to \mathbb{C}$ where $U$ is open and $z_0 \in U$. Then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

**Theorem 3.1** (Cauchy-Riemann Equations)**.** *If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, then the first-order partial derivatives of $u$ and $v$ exist at $(x_0, y_0)$ and satisfy:*

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \text{ at } (x_0, y_0).$$

If $U \subset \mathbb{C}$ is open, then $f$ is differentiable in $U$ (i.e., at every $z \in U$) if and only if $u$ and $v$ are differentiable, their partial derivatives are continuous, and the Cauchy-Riemann equations hold.

The Cauchy-Riemann equations imply that $\frac{\partial}{\partial \overline{z}} f = 0$, which means $f(x, y) = f(z, \overline{z}) = f(z)$ is holomorphic. According to Cauchy's theorem, holomorphic functions are analytic, so we can write $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, which converges with $a_n = \frac{f^{(n)}(z_0)}{n!}$ for $z_0 \in U$ open.

**Theorem 3.2** (Existence and Uniqueness of Analytic Solutions)**.** *If $\Omega \subset \mathbb{C} \times \mathbb{C}^n$ is open, $\Lambda \subset \mathbb{C}$ (parameters), and $f : \Omega \times \Lambda \to \mathbb{C}^n$ is analytic in all variables, then for the system*

$$\begin{aligned} \vec{w}' &= f(z, \vec{w}, \lambda), \\ \vec{w}(z_0) &= w_0, \end{aligned}$$

*for $(z_0, w_0, \lambda_0) \in \Omega \times \Lambda$, there exists a unique (local) solution $w(z, w_0; \lambda)$ that is analytic in all variables.*

We can look for solutions in the form of convergent power series using the Method of Frobenius.

# 4 2nd Order Linear ODE in the Complex Domain and the Method of Frobenius

Consider the ODE

$$\begin{aligned} w'' + p(z)w' + q(z)w &= f(z), \\ w &= w(z). \end{aligned}$$

How do we solve this?

1. Solve the homogeneous equation $w'' + p(z)w' + q(z)w = 0$ and find a basis in the vector space of solutions.
2. Solve the non-homogeneous equation (e.g., using the method of variation of parameters).

Assume that $p(z)$ and $q(z)$ are analytic at $z = z_0$ (i.e., in an open set $U$ containing $z_0$):

$$\begin{aligned} p(z) &= \sum_{n=0}^{\infty} p_n (z - z_0)^n, \\ q(z) &= \sum_{n=0}^{\infty} q_n (z - z_0)^n. \end{aligned}$$

Then, look for the analytic solution $w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and find the recurrence relation for the coefficients $a_n$:

$$\begin{aligned} w'' + p(z)w' + q(z)w &= 0, \\ w(z_0) &= w_0, \\ w'(z_0) &= w_1. \end{aligned}$$

Setting $a_0 = w_0$, $a_1 = w_1$, and without loss of generality assuming $z_0 = 0$,

$$\begin{aligned} w(z) &= a_0 + a_1 z + \cdots + a_n z^n + \cdots , \\ w'(z) &= a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1} + (n+1)a_{n+1} z^n + \cdots , \\ w''(z) &= 2a_2 + 2 \cdot 3a_3 z + \cdots + n(n-1)a_n z^{n-2} + n(n+1)a_{n+1} z^{n-1} + (n+1)(n+2)a_{n+2} z^n + \cdots . \end{aligned}$$

By brute force substitution, we obtain:

- At $z^0$: $2a_2 + p_0 \cdot a_1 + q_0 \cdot a_0 = 0$ gives $a_2$.
- At $z^1$: $2 \cdot 3a_3 + 2a_2 p_0 + a_1 p_1 + q_1 a_0 + q_0 a_1 = 0$ gives $a_3$.

and so on.

**Example 4.1** (Airy Equation)**.** *Consider the differential equation* $w'' = zw$*. Setting* $w(z) = \sum_{n=0}^{\infty} a_n z^n$*, at* $z^n$*:*

$$n(n-1)a_n z^{n-2} = a_n z^{n+1} \implies (n+2)(n+1)a_{n+2} = a_n.$$

*This gives the recurrence relation*

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}.$$

*Note:* $a_2 = \frac{a_{-1}}{1\cdot 2} = 0$ *because* $a_{-1} := 0$*.*

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}$$

*which implies* $a_2 = a_5 = a_8 = \cdots = a_{3k+2} = 0$ *for* $k = 0, 1, 2, \ldots$*.*

*If we start with* $a_0$*, we get*

$$a_3 = \frac{a_0}{2\cdot 3} = \frac{a_0}{3!}, \quad a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{3!\cdot 5\cdot 6} = \frac{4a_0}{6!}, \quad a_9 = \frac{4\cdot 7}{9!}a_0, \text{ etc..}$$

*If we start with* $a_1$*,*

$$a_4 = \frac{a_1}{3\cdot 4} = \frac{2}{4!}a_1, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{2\cdot 5}{7!}a_1, \text{ etc.}$$

*Thus, we can write*

$$\begin{aligned} w(z) &= a_0\left(1 + \frac{z^3}{3!} + \frac{4z^6}{6!} + \frac{4\cdot 7\cdot z^9}{9!} + \frac{4\cdot 7\cdot 10\cdot z^{12}}{12!} + \cdots\right) \\ &\quad + a_1\left(z + \frac{2z^4}{4!} + \frac{2\cdot 5\cdot z^7}{7!} + \frac{2\cdot 5\cdot 8\cdot z^{10}}{10!} + \cdots\right). \end{aligned}$$

## 5 The Airy Equation and the Airy Functions

The Airy functions are our first example of special functions determined by an ODE:

$$w''(z) = zw \implies w(z) = c_0 \text{Ai}(z) + c_1 \text{Bi}(z),$$

where

- For $\text{Ai}(z)$, $a_0 = \text{Ai}(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}$, $a_1 = \text{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}$.
- For $\text{Bi}(z)$, $a_0 = \text{Bi}(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}$, $a_1 = \text{Bi}'(0) = -\frac{1}{3^{1/6}\Gamma\left(\frac{1}{3}\right)}$.

The point $x = 0$ is a "turning point". To the left of this, we have oscillations, and to the right, we have exponential growth/decay. In particular, Ai and Bi are the only functions that satisfy these conditions.

Consider the ODE

$$w'' + p(z)w' + q(z)w = 0.$$

We've seen that the solution $w = w(z)$ is analytic whenever $p(z)$ and $q(z)$ are analytic. What about singularities?

Without loss of generality, suppose $z_0 = 0$ is an isolated singular point.

Recall: If $f$ is analytic in $U = \{z \in \mathbb{C} \mid 0 < |z| < r\}$, then, using the Cauchy Integral Theorem, we can write

$$f(z) = \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

as a convergent Laurent series. Thus:

- $f(z)$ is analytic if $c_{-1} = c_{-2} = \cdots = 0$.
- $f(z)$ has a pole of order $k \in \mathbb{Z}^+$ if $c_{-k} \neq 0$ and $c_{-k-1} = c_{-k-2} = \cdots = 0$.
- If infinitely many $c_k \neq 0$ for $k < 0$, then $f(z)$ has an essential singularity.

**Example 5.1** (Euler (or Equidimensional) Equation)**.** *Consider*

$$w'' + \frac{a}{z}w' + \frac{b}{z^2}w = 0.$$

*The coefficients are singular at $z_0 = 0$.*

*Rewrite as $z^2 w'' + azw' + bw = 0$. Define $\partial = \frac{d}{dz}$ and introduce the Euler operator $\delta := z\frac{d}{dz}$. The eigenfunctions are:*

$$\partial w = kw \implies w(z) = e^{kz} = \sum \frac{(kz)^n}{n!}$$

*and*

$$\delta w = kw \implies w(z) = z^k = e^{\log(z)\cdot k}.$$

**Exercise 5.2.** *Show that $[\partial, z] = 1$, $[\partial, \delta] = \partial$, and $z^k \partial^k = \delta(\delta-1)(\delta-2)\cdots(\delta-k+1)$.*

# 6 Methods of Frobenius

We seek solutions to $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ in the form of power series near singular points $z_0 = 0$.

The Euler equation $z^2 w'' + p_0 z w' + q_0 w = 0$ can be rewritten as

$$w'' + \frac{1}{z} p_0 w' + \frac{1}{z^2} q_0 w = 0,$$

which shows the first and second order poles. Using Euler's operator $\delta = z\frac{d}{dz}$, we have:

$$\delta z^k = k z^k, \quad \partial = \frac{d}{dz}.$$

Thus,

$$z^2\partial^2 = z(z\partial)\partial = z(\delta\partial) = z(\delta\partial - \partial) = z\partial\delta - z\partial = \delta^2 - \delta.$$

By induction, $z^k \partial^k = \delta(\delta-1)(\delta-2)\cdots(\delta-k+1)$. Therefore, we can rewrite

$$\delta(\delta-1)w + p_0\delta w + q_0 w = 0 \implies (\delta^2 + (p_0-1)\delta + q_0)w = 0.$$

Comparing $aw'' + bw' + cw = 0$ and $(a\partial^2 + b\partial + c)w = 0$, we see $L = a\partial^2 + b\partial + c = L_1 \circ L_2$. For example, if $a = 1$,

$$(\partial - r_1)(\partial - r_2)w = 0,$$

which can be rewritten as

$$\partial^2 - (r_1 + r_2)\partial + r_1 r_2.$$

This gives the solution

$$w(z) = c_1 e^{r_1 z} + c_2 e^{r_2 z}.$$

**Example 6.1.** *For $w'' + w' - 6w = 0$, we rewrite the LHS as*

$$(\partial^2 + \partial - 6)w = (\partial + 3)(\partial - 2)w,$$

*so*

$$\partial w = 2w \implies w(z) = c_1 e^{2z}$$

*and*

$$\partial w = -3w \implies w(z) = c_2 e^{-3z},$$

*which gives the general solution*

$$w(z) = c_1 e^{2z} + c_2 e^{-3z}.$$

**Example 6.2.** *For $z^2w'' + 2zw' - 6w = 0$, we rewrite the LHS as*

$$(\delta(\delta-1) + 2\delta - 6)w = (\delta^2 + \delta - 6)w = (\delta+3)(\delta-2)w,$$

*so*

$$\delta w = 2w \implies w(z) = c_1 z^2$$

*and*

$$\delta w = -3w \implies w(z) = c_2 z^{-3},$$

*which gives the general solution*

$$w(z) = c_1 z^2 + c_2 z^{-3}.$$

In both examples, there are two distinct real roots of the characteristic equation.

**Example 6.3.**

- *If we take $w'' + w' - 6w = 0$, trying $w(z) = e^{rz} \implies r^2 + r - 6 = 0$ is the characteristic equation.*
- *If we take$w''+4w'+5w = 0$, the characteristic equation is $r^2+4r+5 = (r+2)^2 + 1 = 0$, which gives general solution $w(z) = c_1e^{-2+iz} + c_2e^{(-2-i)z} = e^{-2z}(\tilde{c_1}\cos(z) + \tilde{c_2}\sin(z))$.*
- *If we take $z^2w'' + 5zw' + 5w = 0 \implies (\delta^2 + 4\delta + 5)w = 0 \implies w(z) = c_1z^{-2+i} + c_2z^{-2-i}$.*
- *If we take $w'' + 4w' + 4w = 0$ we can rewrite the LHS as $(\partial^2 + 4\partial + 4)w = (\partial + 2)^2 w(z) = ce^{-2z}$.*

Now try variation of parameters, $w(z) = u(z)e^{-2z}$.

$$(u''e^{-2z} + 2u'(-2e^{-2z}) + u(z)4e^{-2z})) + 4(u'e^{-2z} + u(z)(-2e^{-2z}) + 4u(z)e^{-2z} = 0$$

and then cancelling everything gives $u'' = 0 \implies u(z) = c_1z + c_2$, which gives solution $w(z) = (c_1z + c_2)e^{-2z} = c_1ze^{-2z} + e_2e^{-2z}$. We can think of the second term as the eigenfunction of $\partial$ and the first as the generalized eigenfunction.

Consider the differential operator applied to a function:

$$(\partial + z)(ze^{-2z}) = e^{-2z}.$$

**Regarding** $\delta$:

**Example 6.4.** *Consider the differential equation*

$$z^2w'' + 5zw' + 4w = 0.$$

*The solution is*

$$w(z) = c_1z^{-1}\log(z) + c_2z^{-2}.$$

Reason: An Euler equation transforms into a linear differential equation with constant coefficients under a change of the independent variable

$$\zeta = \log(z),$$

where $\log(z) = \ln|z| + i\arg(z)$ with the principal branch $-\pi < \mathrm{Arg}(z) < \pi$.

By the change of variables $\zeta = \log(z)$, i.e., $z = e^{\zeta}$, we have

$$\frac{d}{d\zeta} = \frac{dz}{d\zeta}\frac{d}{dz} = z\frac{d}{dz}.$$

Let $\mu(\zeta) := w(e^{\zeta})$. Then the equation

$$z^2 w'' + p_0 z w' + q_0 w = 0$$

transforms to

$$u'' + (p_0 - 1)u' + q_0 u = 0.$$

The characteristic equation is

$$r^2 + (p_0 - 1)r + q_0 = 0 \implies r_{1,2} = \frac{1}{2}\left(1 - p_0 \pm \sqrt{(p_0 - 1)^2 - 4q_0}\right).$$

If $r_1 \neq r_2$, then the solution is

$$u(\zeta) = c_1 e^{r_1 \zeta} + c_2 e^{r_2 \zeta},$$

which translates back to

$$w(z) = c_1 z^{r_1} + c_2 z^{r_2}.$$

If $r_1 = r_2$, then the solution is

$$u(\zeta) = (c_1 \zeta + c_2)e^{r\zeta},$$

which translates back to

$$w(z) = c_1 z^r \log(z) + c_2 z^r.$$

We seek solutions near an isolated singular point $z_0 = 0$ of the form

$$w(z) = z^r h(z),$$

where $h$ is holomorphic.

**Lemma 6.5.** *A first-order differential equation $w'(z) + p(z)w(z) = 0$ has a solution $w(z) = z^r h(z)$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $h_0 = 1$ if and only if $p(z)$ has at most a simple pole at $z_0 = 0$, specifically $p(z) = \frac{p_{-1}}{z} + p_0 + \dots$ with $r = -p_1 = -\lim_{z\to 0} zp(z)$.*

*Proof.* Substitute $w(z) = z^r h(z)$ into the differential equation $w' + p(z)w = 0$:

$$p(z) = -\frac{w'}{w} = -\frac{rz^{r-1}h(z) + z^r h'(z)}{z^r h(z)} = -\frac{r}{z} + \frac{h'(z)}{h(z)},$$

where $\frac{h'(z)}{h(z)}$ is holomorphic with $h(0) = h_0 = 1$.

Conversely, if $w' = -p(z)w$, then

$$\begin{aligned} w(z) &= c\exp\left(-\int p(z)\,dz\right) \\ &= w(0)\exp\left(-\int_0^z p(s)\,ds\right) \\ &= -p_{-1}\log z + C + p_0 z + \dots \\ &= w(0)z^{-p_{-1}} \cdot \text{holomorphic function.} \end{aligned}$$

for some constant $C$. ☐

**Definition 6.6.** *A point $z_0$ is called a **regular singular point** of the differential equation $w'' + \tilde{p}(z)w' + \tilde{q}(z)w = 0$ if $\tilde{p}$ and $\tilde{q}$ are meromorphic at $z_0$ and $p(z) = (z - z_0)\tilde{p}(z)$ and $q(z) = (z - z_0)^2\tilde{q}(z)$ are holomorphic at $z_0$. If $p(z)$ and $q(z)$ are holomorphic at $z_0$, we say that $z_0$ is an **ordinary point**.*

Assume $z_0 = 0$. Then, for the equation

$$z^2 w'' + zp(z)w' + q(z)w = 0$$

with $p(z) = p_0 + p_1 z + \dots$ and $q(z) = q_0 + q_1 z + \dots$, we have solutions of the form

$$w(z) = z^r h(z)$$

for a holomorphic function $h$.

The associated Euler equation is

$$z^2 w'' + zp_0 w' + q_0 w = 0,$$

with the characteristic equation

$$r(r-1) + p_0 r + q_0 = 0,$$

which is called the indicial equation because the solutions are exponential at the singularity. The solutions are

$$r_{1,2} = \frac{1}{2}\left(1 - p_0 \pm \sqrt{(p_0 - 1)^2 - 4q_0}\right).$$

There are two cases: $r_1 - r_2 \notin \mathbb{Z}$ and $r_1 - r_2 \in \mathbb{Z}$.

Consider some representative examples.

**Example 6.7.** *For $4zw'' + 2w' + w = 0$, the associated Euler's equation is $4z^2w'' + 2zw' = 0$, and the indicial equation is*

$$4r(r-1) = 2r \implies 4r^2 - 2r = 2r(2r-1) = 0 \implies r_1 = \frac{1}{2}, r_2 = 0.$$

*Thus, $w_1(z) = z^{\frac{1}{2}}h(z)$ and $w_2(z) = h(z)$. Substitute $w(z) = \sum_{n=0}^{\infty} h_n z^{n+r}$:*

$$\sum_{n=0}^{\infty} \left[(n+r)(n+r-1)h_n z^{n+r-1} + (n+r)h_n z^{n+r-1} + h_n z^{n+r}\right].$$

*Shifting the index down by 1 gives the recurrence relation*

$$\begin{aligned}(4(n+r)(n+r-1) + 2(n+r))h_n + h_{n-1} = 0\\ 2(n+r)(2n+2r-1)h_n = -h_{n-1}.\end{aligned}$$

*Letting $F(n+r) = 2(n+r)(2n+2r-1)$, we have at $n = 0$:*

$$F(r) \cdot (h_0 = 1) = -h_{-1} = 0 \text{ and } F(r) = 2r(2r-1) = 0$$

*$n = 0 : F(r) \cdot (h_0 = 1) = -h_{-1} = 0$ and $F(r) = 2r(2r-1) = 0$ is exactly the indicial equation above (the characteristic equation for the approximating Euler's equation). Then for $r_1 = \frac{1}{2} > r_2 = 0$,*

$$h_n = \frac{-1}{(2n+2r)(2n+2r-1)}h_{n-1} = \ldots = \frac{(-1)^n}{(2n+2r)(2n+2r-1)\ldots(2r+2)(2r+1)}$$

**Remark 6.8.** *Products like the one above are sometimes called shifted factorials:*

- *Lower $x_n := x(x-1)(x-2)\ldots(x-n+1) = (x)_n$*
- *Upper $x^n := x(x+1)(x+2)\ldots(x+n-1) = (x)^n$*

**Definition 6.9.** *The Pochhammer symbol $(\alpha)_n := \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$. Note: $(1)_n = n!, (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ since $\Gamma(z+1) = \Gamma(z)$. And $h_n(r) = \frac{(-1)^n}{(2r+1)_{2n}}$.*

*For $r = \frac{1}{2}$:*

$$h(z) = z^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)_{2n}} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\sqrt{z}\right)^{2n+1}}{(2n+1)!} = \sin\left(\sqrt{z}\right).$$

*For $r = 0$:*

$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1)_{2n}} z^n = \cos\left(\sqrt{z}\right).$$

Recap: $4zw'' + 2w' + w = 0$ has the following general solution:

$$w(z) = c_1 \sin\left(\sqrt{z}\right) + c_2 \cos\left(\sqrt{z}\right)$$

Check:

$$\begin{aligned}4z\left(c_1\left(-\frac{1}{4z\sqrt{z}}\cos\left(\sqrt{z}\right) - \frac{1}{4z}\sin\left(\sqrt{z}\right)\right) + c_2\left(\frac{1}{4z\sqrt{z}}\sin\left(\sqrt{z}\right) - \frac{1}{4z}\cos\left(\sqrt{z}\right)\right)\right)\\ +2\left(c_1\left(\frac{1}{2\sqrt{z}}\cos\left(\sqrt{z}\right)\right) + c_2\left(-\frac{1}{2\sqrt{z}}\sin\left(\sqrt{z}\right)\right)\right)\\ + \left(c_1 \sin\left(\sqrt{z}\right) + c_2\cos\left(\sqrt{z}\right)\right)\\ = 0\end{aligned}$$

**Example 6.10.** *$zw'' + w' - w = 0$ has the indicial equation $r(r-1) + r = 0 \implies r^2 = 0$ so it has a repeated root. The power series*

$$\sum_{n=0}^{\infty}\left(\left(n+r\right)\left(n+r-1\right)h_n z^{n+r-1} + \left(n+r\right)h_n z^{n+r-1} - h_n z^{n+r}\right) = 0$$

*Note that $\left(n+r\right)\left(n+r-1\right)h_n z^{n+r-1} + \left(n+r\right)h_n z^{n+r-1} = (n+r)^2 h_n = F(n+r)$ and $F(r) = r^2 = 0$.*

*$n = 0$ gives $r^2 = 0 \implies F(n) = n^2$ and*

$$h_n = \frac{1}{F(n)}h_{n-1} = ... = \frac{1}{n^2(n-1)^2...1^2} = \frac{1}{(n!)^2}.$$

$w_1(z) = z^0 \sum_{n=0}^{\infty} \dfrac{z^n}{(n!)^2} = (1 + z + \dfrac{z^2}{4} + \dfrac{z^3}{36} + ...)$

*To find $w_2(z)$, try $w_2(z) = w_1(z)\log(z) + z^r h(z)$. If $w_2(z) = w_1(z)\log(z) + z^r h(z)$. Let $L[w] = 4zw'' + 2w' + w = 0$ for easier typesetting. Then*

$$L[w_2(z)] = L[w_1(z)\log(z)] + L[z^r h(z)]$$

*Since*

$$\begin{aligned}L[w_1(z)\log(z)] &= z\left(w_1''\log(z) + 2w_1'\cdot\frac{1}{z} - w_1\frac{1}{z^2}\right) + \left(w_1'\log(z) + w_1\cdot\frac{1}{z}\right) - w_1\log(z)\\ &= \log(z)\,L\left[w,(z)\right] + 2w_1' - w_1\cdot\frac{1}{z} + w_1\cdot\frac{1}{z}\\ &= \log(z)\,L\left[w,(z)\right] + 2w_1'\end{aligned}$$

*and*

$$L[z^r h(z)] = -2w_1'$$

*At $r = 0$, this is equal to*

$$z\left(2h_2 + 2\cdot 3zh_3 + 3\cdot 4z^2h_4 + ...\right) + \left(h_1 + 2h_2z + 3h_3z^2 + ...\right) - \left(h_0 + h_1z + h_2z^2 + ...\right) = -2\left(1 + \frac{1}{2}z + \frac{1}{12}z^2 + ...\right)$$

*We have*

$$\begin{aligned} h_1 &= -2 \\ 2\cdot 2h_2 - h_1 &= -1 \\ 3\cdot 3h_3 - h_2 &= -\frac{1}{6} \\ &\vdots \\ n^2h_n - h_{n-1} &= -2\frac{n}{n!n!} = -\frac{2}{(n-1)!n!} \end{aligned}$$

*So*

$$h_n = n\frac{1}{n^2}\left(h_{n-1} - \frac{2}{(n-1)!n!}\right)$$

*can be solved for $h_n$.*

*Furthermore,*

$$w_2(z) = w_1(z)\log(z) + \left(-2z + \frac{3}{4}z^2 - \frac{11}{108}z^3 - ...\right)$$

**Example 6.11.** *Consider $zw'' + w = 0$. Look for $w(z) = z^r h(z)$, where $h(z)$ is analytic (holomorphic) near $z = 0$, ie. $h(z) = \sum_{n=0}^{\infty} h_n z^n$ with $h_0 \neq 1$.*

*Then $w(z) = \sum_{n=0}^{\infty} h_n z^{n+r} = z^r = h_1z^{r+1} + h_2z^{r+2} + ... + h_nz^{r+n}$. Plugging this into the equation gives*

$$\begin{aligned} &z\Big[r(r-1)z^{r-2} + (r+1)rh_1z^{r-1} + (r+2)(r+1)h_2z^r \\ &\quad + \cdots + (r+n)(r+n-1)h_nz^{r+n-2} + \cdots\Big] \\ &\quad + \left[z^r + h_1z^{r+1} + h_2z^{r+2} + \cdots\right] = 0. \end{aligned}$$

- *At $z^{r-1}$, the indicial equation is $r(r-1) = 0$. Alternatively, we can get here through the associated Euler equation: $z^2w'' = 0$, try $w(z) = z^r$ which gives $r(r-1)z^r = 0$. Solving the indicial equation gives $r_1 = 1, r_2 = 0$.*
- *At $z^r$, we have $r(r+1)h_1 + 1 = 0$. In general, we have $w_1(z) = z^{r_1}h(z)$ and $w_2(z) = z^{r_2}h(z)$ but this second doesn't exist because if $r_2 = 0$, then $0 \cdot h_1 + 1 = 0$, which is impossible, so there is no second solution in this form. But the first one always exists.*

- *At $z^{r+n-1}$, we have $(r+n)(r+n-1)h_n + h_{n-1} = 0$. If $r_1 - r_2 = m \in \mathbb{N}_{\geq 0}$, then $F(r_2+m) = F(r_1) = 0$, and we have $0 \cdot h_m + h_{m-1} = 0$ which is impossible unless $h_{m-1}$ happens to be zero. Let's construct the first solution $w_1(z)$. Take the recurrence relation $F(n+r)h_n = -h_{n-1}$. Use $r_1 = 1 : F(n+1) = (n+1)\cdot n$. Then*

$$h_n = \frac{-1}{n(n+1)}h_{n-1} = ... = \frac{(-1)^n}{(n+1)\cdot n \cdot ... \cdot 3 \cdot 2 \cdot 2 \cdot 1}h_0 = \frac{(-1)^n}{(n+1)(n!)^2}.$$

  *This gives*

$$w_1(z) = z^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(n!)^2} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\frac{z^{n+1}}{n+1}.$$

  *How to find $w_2(z)$? Look for the solution in the form $w_2(z) = z^{r_2}h(z) + cw_1(z)\log(z)$. Substitute $L[w] = zw'' + w = 0$, which gives*

$$L[w_2(z)] = L[z^{r_2}h(z)] + cL[w_1(z)\log(z)] = 0$$

  *so $L[z^{r_2}h(z)] = -cL[w_1(z)\log(z)]$, which can be rewritten as*

$$\begin{aligned}\sum_{n=1}^{\infty}(F(n)h_n + h_{n-1})z^{n-1} &= -c\left(z\left(w_1''(z)\log(z) + w_1' \cdot \frac{1}{z} - w_1\frac{1}{z^2}\right)\right) + w_1(z)\log(z)\\ &= -c\left(2w_1' - \frac{1}{z}w_1\right)\\ &= -c\left(-c\left(1 - \frac{3}{z}z + \frac{5}{(2!)^2\cdot 3}z^2 + ...\right)\right)\end{aligned}$$

  *So $c = -1$ and we have $2h_2 + h_1 = -\frac{3}{2}, 6h_3 + h_2 = \frac{5}{12}, ...$*

  *The first will give us a solution of the form $z^0(h_1(z...)) = z(h_1(1+...)) = h_1 \cdot w_1(z)$, ie. is a multiple of $w_1(z)$, nothing new can put $h_1 = 0$. Then $h_2 = -\frac{3}{4}, h_3 = \left(\frac{5}{12} + \frac{3}{4}\right)\frac{1}{6} = \frac{7}{36}$, ... or*

$$h_n = (-1)^n \frac{2n-1}{F(n)n!(n+1)!}.$$

  *This gives our second solution*

$$w_2(z) = \left(1 - \frac{3}{4}z^2 + \frac{7}{36}z^3 - ...\right) - w_1(z)\log(z).$$

**Example 6.12.** *Consider the equation $z^2w'' + zw' + \left(z^2 - \frac{1}{4}\right)w = 0$. The indicial equation is $r(r-1) + r - \frac{1}{4} = 0$, and $F(r) = r^2 - \frac{1}{4} \implies r_1 = \frac{1}{2}, r_2 = -\frac{1}{2}$. The recurrence relation is $F(n+r)h_n + h_{n-1} = 0$.*

*At* $r_1 = \frac{1}{2}$*:* $F\left(n+\frac{1}{2}\right) = n(n+1)$*. Since* $h_n = -\frac{1}{n(n+1)}h_{n-2}$*,* $h_1 = h_3 = ... = h_{2k+1}$*. Furthermore,* $h_{2k} = \frac{(-1)^k}{2k(2k+1)}$ *so we get the solution*

$$w_1(z) = z^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!} = \frac{\sin(z)}{\sqrt{z}}.$$

*What about the second solution* $w_2(z)$*? We have*

$$w(z) = z^{r_2} \sum_{n=0}^{\infty} h_n z^n + c w_1(z) \log(z),$$

*which we can rewrite as*

$$L[w(z)] = L\left[z^{-\frac{1}{2}} \sum_{n=0}^{\infty} h_n z^n\right] + cL[w_1(z)\log(z)] = 0.$$

*Note that*

$$\begin{aligned} L\left[z^{-\frac{1}{2}} \sum_{n=0}^{\infty} h_n z^n\right] &= -cL[w_1(z)\log(z)] \\ &= -c\left(z^2\left(w_1''\log(z) + 2w_1'\frac{1}{z} - w\frac{1}{z^2}\right) + z\left(w_1'\log(z) + w_1\frac{1}{z}\right) + \left(z^2 - \frac{1}{4}\right)w_1\log(z)\right) \\ &= -c \cdot \frac{2w_1'}{z} \end{aligned}$$

*At the same time,*

$$\begin{aligned} L\left[z^{-\frac{1}{2}} \sum_{n=0}^{\infty} h_n z^n\right] &= \sum\left(F\left(n-\frac{1}{2}\right)h_n + h_{n-2}\right)z^{n-\frac{1}{2}} \\ &= (0\cdot h_0 + h_{-2})z^{-\frac{1}{2}} + (0\cdot h_1 + h_{-1})z^{\frac{1}{2}} + (2h_2 + h_0)z^{\frac{3}{2}} + ... \\ &= -c\left(z^{\frac{1}{2}} - \frac{5}{6}z^{\frac{5}{2}} + ...\right) \end{aligned}$$

*so* $c = 0$*. Take* $h_1$ *arbitrary. The recurrence relation gives*

$$h_{2k} = \frac{(-1)^k}{(2k)!}h_0, \qquad h_{2k+1} = \frac{(-1)^k}{(2k+1)!}h_1$$

*so*

$$\begin{aligned} w_2(z) &= z^{-\frac{1}{2}}\left(h_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}z^{2k} + h_1 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}z^{2k+1}\right) \\ &= \frac{1}{\sqrt{z}}\cos(z) + h_1\frac{\sin(z)}{\sqrt{z}} \end{aligned}$$

*Finally, our solutions are* $w_1(z) = \frac{\sin(z)}{\sqrt{z}}, w_2(z) = \frac{\cos(z)}{\sqrt{z}}$.

**Theorem 6.13.** *Consider the equation* $w'' + \tilde{p}(z)w' + \tilde{q}(z)w = 0$ *where* $\tilde{p}(z)$ *has at most a simple pole at* $z_0 = 0$ *and* $\tilde{q}(z)$ *has at most a double pole at* $z_0 = 0$, *ie.* $z^2w'' + wp(z)w' + q(z)w = 0$, *where* $p(z) = z\tilde{p}(z), q(z) = z^2\tilde{q}(z)$, *where* $p(z)$ *and* $q(z)$ *are analytic near* $z_0 = 0$, *ie.* $p(z) = \sum_{n=0}^{\infty} p_n z^n, q(z) = \sum_{n=0}^{\infty} q_n z^n$.

*Then, if* $r_1$ *and* $r_2$ *are the characteristic exponents (ie. roots of the indicial equation* $r(r-1) + rp_0 + q_0 = 0$*), the following two cases can occur:*

- *If* $r_1 - r_2 \notin \mathbb{Z}$, *a fundamental system (ie. a basis) of solutions is given by* $w_j(z) = z^{r_j} h_j(z)$ *with* $h_j(z) = \sum_{k=0}^{\infty} h_{j,k} z^k$ *(convergent near* $z_0 = 0$*) with* $h_{j,0} = 1$.
- *If* $r_1 - r_2 = m \in \mathbb{N}$, *then a fundamental system of solutions is given by* $w_1(z) = z^{r_1} h_1(z), w_2(z) = z^{r_2} h_2(z) + c\log(z) w_1(z)$, *where the constant* $c \in \mathbb{C}$ *might be zero (unless* $r_1 = r_2$, *then* $c \neq 0$*).*

*Proof.* We'll provide a sketch.

Let $z^2w'' + p(z)\cdot zw' + q(z)\cdot w = 0$, with $w(z) = z^r \sum_{n=0}^{\infty} h_n z^n = \sum_{n=0}^{\infty} h_n z^{n+r} = z^r + \sum_{n=1}^{\infty} h_n z^{n+r}$ where $h_0 = 1$, and

$$\begin{aligned} p(z) &= \sum_{n=0}^{\infty} = \left(p_0 + \sum_{n=1}^{\infty} p_n z^n\right) \\ q(z) &= \sum_{n=0}^{\infty} = \left(q_0 + \sum_{n=1}^{\infty} q_n z^n\right). \end{aligned}$$

Then

$$\begin{aligned} q(z)w(z) &= \left(q_0 + \sum_{k=1}^{\infty} q_k z^k\right) z^r \left(1 + \sum_{\ell=1}^{\infty} h_\ell z^\ell\right) \\ &= z^r \left(q_0 + \sum_{n=1}^{\infty} \left(\sum_{k+\ell=n} q_k h_\ell\right)\right) \\ &= z^r \left(q_0 + \sum_{n=1}^{\infty} \left(q_0 h_n + \sum_{k=1}^{n} q_k h_{n-k}\right)\right) \\ &= q_0 z^r + \sum_{n=1}^{\infty} \left(q_0 h_n + \sum_{k=1}^{n} q_k h_{n-k}\right) z^{n+r} \end{aligned}$$

so

$$p(z)zw'(z) = p_0 \cdot rz^r + \sum_{n=1}^{\infty} \left(p_0(n+r)h_n + \sum_{k=1}^{n} p_k(n+r-k)h_{n-k}\right) z^{n+r}.$$

If we substitute

$$(r(r-1)+p_0r+q_0)z^r + \sum_{n=1}^{\infty} \left([(n+r)(n+r-1) + p_0(n+r) + q_0]h_n + \sum_{k=1}^{n} ((p_k(n+r-k) + q_k)h_{n-k}\right) z^{n+r} = ($$

note that $r(r-1)+p_0r+q_0$ is the indicial equation $F(r) = r^2+(p_0-1)r+q_0 = 0$, so the coefficients will have the form

$$F(n+r)h_n + \sum_{k=1}^{n}(p_k(n+r-k)+q_k)h_{n-k} = 0.$$

Let's solve for the exponents $r_1$ and $r_2$ for at the singularity and then solve for $h_n$ in terms of $h_0, ..., h_{n-1}$.

Solve $F(r) = r^2 + (p_0-1)r + q_0 = 0$ to get

$$r_{1,2} = \frac{1}{2}\left(1 - p_0 \pm \sqrt{(p_0-1)^2 - 4q_0}\right).$$

Furthermore, $r_1 + r_2 = 1 - p_0, r_1r_2 = q_0$, so

$$F(n+r)h_n + \sum_{k=1}^{n}\left(p_k(n+r-k)+q_k\right)h_{n-k} = 0.$$

So $F(n+r) = (n+r-r_1)(n+r-r_2)$. Additionally, $F(n+r_1) = n(n+r_1-r_2) \neq 0$ for $n \geq 0$, and $F(n+r_1) = (n+r_2-r_1)n$ can be equal to zero if $r_1 - r_2 = m$ for $m \in \mathbb{N}$.

In the case when $r_1 - r_2 = m \in \mathbb{N}_{\geq 0}$, use variation of parameters technique:

$$w_2(z) = u(z)w_1(z)$$

which can be rewritten as

$$\begin{aligned} L[w_2(z)] &= u(z)L[w_1(z)] + z^2(u''(z) + 2u'(z)w_1'(z)) + p(z)\cdot zu'(z)w_1(z) \\ &= z^2u''(z)z^{r_1}h_1(z) + zu'(z)z^2(r_1z^{r_1-1}h_1(z) + z^{r_1}h_1'(z)) + p(z)\cdot z\cdot u'(z)z^{r_1}h_1(z) \\ &= z^2\cdot z^{r_1}h_1(z)\left(u''(z) + 2u'(z)\cdot\frac{r_1}{z} + 2u'(z)\frac{h_1'(z)}{h(z)} + \frac{p(z)}{z}\cdot u'(z)\right) \end{aligned}$$

which we want to equal zero. So we want

$$u''(z) + \left(\frac{2r_1}{z} + 2\frac{h_1'(z)}{h_1(z)} + \frac{p(z)}{z}\right)u'(z) = 0.$$

and the term in parenthesis is equal to $\frac{2r_1}{z} + \frac{p_0}{z} + \text{holomorphic}$.

Put $v(z) := u'(z)$. Then we get

$$v' + \left(\frac{1+r_1-r_2}{z} + \text{holomorphic}\right)v(z) = 0.$$

and

$$\begin{aligned} v(z) &= \exp\left(-\int\frac{1+r_1-r_2}{z}\,dz + \text{holomorphic}\right) \\ &= z^{r_2-r_1-1}\sum_{n=0}^{\infty}f_nz^n \end{aligned}$$

with $f_0 \neq 0$. Since $v(z) = u'(z)$,

$$u(z) = \int \sum_{n=0}^{\infty} f_n z^{n+r_2-r_1-1}\, dz = \sum_{n=0}^{\infty} f_n \frac{z^{n+r_2-r_1}}{n+r_2-r_1}$$

if $n \neq r_2 - r_1$, and if $r_1 - r_2 = m$ then the last term has $+f_m \log(z)$.

Now, we get our final formula

$$w_2(z) = z^{r_2} \left( \sum_{n=0}^{\infty} w_n z^n \right) + f_m w_1(z) \log(z).$$

Proving the convergence of the resulting series is not difficult - see the textbook.

□

# 7 Bessel Differential Equation and the Bessel Functions

There are several ways in which these arise:

- Vibrations of a circular membrane: separation of variables on the wave equation gives the Bessel equation.

**Definition 7.1.** *Differential equations of the form*

$$z^2w'' + zw' + (z^2 - v^2)w = 0$$

*with* $v \in \mathbb{C}$ *are the* ***Bessel differential equations of over*** $v$*.*

Assume $\mathrm{Re}(v) \geq 0$. Let's solve this using the method of Frobenius: Assume there is a solution of the form

$$w(z;r) = z^r h(z) = z^r \sum_{n=0}^{\infty} h_n z^n = \sum_{n=0}^{\infty} h_n(r) z^{n+r}$$

with $h_0 = 1$. Now we substitute:

$$L[w(z;r)] = \sum_{n=0}^{\infty}[(n+r)(n+r-1)h_n(r)z^{n+r} + (n+r)h_n(r)z^{n+r} - v^2 h_n(r)z^{n+r} + h_n(r)z^{n+r+2}]$$

which we want to equal 0. Let's perform the index shift on the last term to $h_{n-2}(r)z^{n+r}$, with $h_{-2} = h_{-1} = 0$. The reccurence relation is

$$F(n+r)h_n(r) + h_{n-2}(r) = 0.$$

where $F(n+r) = \left[(n+r)^2 - v^2\right]$.

- At $n = 0$, we get the indicial equation is $F(r) \cdot 1 = r^2 - v^2 = 0, r = \pm v, \mathrm{Re}(r_1 = v \geq r_2 = -v)$. If $v = 0$, the indicial equation has a repeated root.
- At $n = 1$, we get $F(1+r) = (1+r-v)(1+r+v)$, so

$$F(1+r) \cdot h_1(r) + h_{-1}(r) = 0$$

which gives at $r_1 = v : F(n+v) = n(n+2v)$ and at $r_2 = -v, F(n-v) = (n-2v)n$. By assumption $\mathrm{Re}(r_1 = v) \geq 0$, so $F(n+v)$ at $r_1$ can never be zero but at $r_2$ is can be zero (when $n = 2v$). So either $F(1+r)$ or $h_1(r) = 0 \implies h_{2k+1}(r) = 0$. At $r_2 = -v : F(1-v) = (1-2v)\cdot$.

There are three special cases:

- $v = 0$: repeated roots
- $v = \frac{1}{2}$: odd terms
- $v = 2m$ for $m \in \mathbb{N}_{\geq 0}$: roots differ by an integer.

The generic case is when $v \neq 2m$ or $\frac{1}{2}$. Here, there are two roots $r_1 = v$ and $r_2 = -v$. Then

$$F(n+v) = n(n+2v) \text{ and } F(n-v) = (n-2v)n$$

so

$$F(n+v)h_n = -h_{n-2}...h_{2k+1} = 0$$

where

$$\begin{aligned}
h_{2k} &= \frac{-1}{2k(2k+2v)} h_{2(k-1)} \\
&= ... \\
&= \frac{(-1)^k}{(2k)(2k-2)...(2)(2k+2v)(2k+2v-2)...(2k+2v)} \\
&= \frac{(-1)^k}{2^k \cdot k! \cdot (v+1)_k}.
\end{aligned}$$

We always have the solution

$$\begin{aligned}
w_1(z) &= z^v \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(v+1)}{2^{2k} k! \Gamma(v+k+1)} z^{2k} \\
&= \Gamma(v+1) \cdot 2^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(v+k+1)} \left(\frac{z}{2}\right)^{2k+v}.
\end{aligned}$$

**Definition 7.2.** *The* ***Bessel function of the 1st kind*** *is defined as*

$$J_v = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(v+k+1)} \left(\frac{z}{2}\right)^{2k+v}.$$

In the generic case, changing $v$ to $-v$ gives

$$\begin{aligned}
w_2(z) &= z^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1-v)}{2^{2k} k! \Gamma(1+k-v)} \\
&= \frac{\Gamma(1-v)}{2^v} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(1+k-v)} \left(\frac{z}{2}\right)^{2k-v}.
\end{aligned}$$

For special cases, at $v = \frac{1}{2}$ we considered earlier:

$$w_1(z) = \frac{\sin(z)}{\sqrt{z}} \text{ and } w_2(z) = \frac{\cos(z)}{\sqrt{z}}.$$

With some additional normalization, we get

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin(z) \text{ and } J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos(z).$$

For repeated roots at $v = 0$, differentiate wrt the parameter $r$. Recall

$$w(z;r) = \sum_{n=0}^{\infty} h_n(r) z^{n+r},$$

so

$$L[w(z;r)] = F(r)z^r + \sum_{n=1}^{\infty} \left( F(n+r)h_n(r) + \sum_{k=1}^{n} (p_k(n+r-k)+q_k)h_{n-k} \right) z^{n+r}.$$

Let's try to solve the recurrence

$$\left( F(n+r)h_n(r) + \sum_{k=1}^{n} (p_k(n+r-k)+q_k)h_{n-k} \right) \qquad (\star)$$

for $h_n(r)$.

If $h_n(r)$ satisfy the recurrence $(\star)$: $L[w(z;r)] = F(r)z^r$ and if $F(r) = (r-r_1)^2$, ie. the repeate root case, then $L[w(z;r)] = (r-r_1)^2 z^4$ which $= 0$ if $r = r_1$ but also:

$$\begin{aligned} \frac{\partial}{\partial r}|_{r=r_1} L[w(z;r)] &= L\left[ \frac{\partial}{\partial r}|_{r=r_1} w(z;r) \right] \\ &= \frac{\partial}{\partial r}|_{r=r_1} (r-r_1)^2 z^r \\ &= 0 \end{aligned}$$

So

$$\begin{aligned} w_2(z) &= \frac{\partial}{\partial r}|_{r=r_1} w_1(z;r) \qquad\qquad = w_1(z;r) \\ &= \frac{\partial}{\partial r}|_{r=r_1} \left( z^r \sum_{n=0}^{\infty} h_n(r) z^n \right) \\ &= \log(z) z^{r_1} \sum_{n=0}^{\infty} h_n(r_1) z^n + z^r \sum_{n=0}^{\infty} h_n'(r_1) z^n. \end{aligned}$$

Applying $F(r) = r^2$ on

$$L[w] = z^2 w'' + z w' + z^2 w = 0$$

gives

$$F(r)z^r + \sum_{n=1}^{\infty} \left( F(r+n)h_n + h_{n-2} \right) z^{r+n} = 0$$

where $h_{2k+1} = 0$. Then

$$h_{2k}(r) = \frac{-1}{(r+2k)^2} h_{2k-2} = ... = \frac{(-1)^{\frac{1}{2}}}{(r+2)^2(r+4)^2...(r+2k)^2}.$$

The repeated root is $r = 0$:

$$h_{2k}(0) = \frac{(-1)^k}{2^2 \cdot 4^2 \cdot ... \cdot (2k)^2} = \frac{(-1)^k}{2^{2k}(k!)^2}$$

and

$$w_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} = J_0(z).$$

Since

$$w_2(z) = J_0(z)\log(z) + \sum_{n=0}^{\infty} h_n'(0) z^n = J_0(z)\log(z) + \sum_{k=1}^{\infty} h_{2k}'(0) z^{2k}$$

and

$$h_{2k}(r) = \frac{(-1)^{\frac{1}{2}}}{(r+2)^2(r+4)^2...(r+2k)^2},$$

we have

$$\frac{h_{2k}'(r)}{h_{2k}(r)} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + ...\frac{1}{r+2k}\right)$$

and at $r = 0$

$$\begin{aligned} h_{2k}' &= -2\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2k}\right) h_{2k}(0) \\ &= -\left(1 + \frac{1}{2} + ... + \frac{1}{k}\right) h_{2k}(0) \\ &= H_k h_{2k}(0) \end{aligned}$$

where $H_k$ is the $k$-th harmonic number. This gives

$$w_2(z) = J_0(z)\log(z) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{2^{2k}(2k)!} z^{2k}.$$

Usually, there is a little bit of change in the second solution

$$\begin{aligned} Y_0 &= \frac{2}{\pi}\left[w_2(z) + (\gamma - \ln(2))J_0(z)\right] \\ &= \frac{2}{\pi}\left[(\gamma + \log\left(\frac{z}{2}\right))J_0(z) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{(2k)!} \left(\frac{z}{2}\right)^{2k}\right] \end{aligned}$$

where $\gamma$ is the Euler constant

$$\gamma = \lim_{n\to\infty}(H_n - \ln(n)) \approx 0.5772$$

For bessel equations of order $m \in \mathbb{Z}_{>0}$, we have

$$z^2 w'' + z w' + (z^2 - m^2) w = 0$$

and

$$F(r) = r^2 - m^2 = (r-m)(r+m)$$

so

$$\sum_{n=0}^{\infty}(F(r+n)h_n + h_{n-2})z^{r+n} = 0$$

When $r+m : F(r_1+n) = F(m+n) = n(n+2m)$ and

$$h_{2k} = \frac{(-1)^k}{(2k)\cdot \ldots \cdot 2 \cdot (2k+2m)\cdot \ldots \cdot (2+2m)} = \frac{(-1)^k}{2^{2k}k!(m+1)_k}$$

which has solution

$$\begin{aligned} w_1(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+1)_k}\left(\frac{z}{2}\right)^{2k} \\ &= \Gamma(m+1)2^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)}\left(\frac{z}{2}\right)^{2k+m} \\ &= \Gamma(m+1)2^m J_m(z). \end{aligned}$$

For the other solution,

$$w_2(z) = z^{-m}h_2(z) + c\log(z)w_1(z)$$

so

$$L\left[z^{-m}h_2(z)\right] + c\left(z^2\left(w_1''\log(z) + 2w_1'\frac{1}{z} - w_1\frac{1}{z^2}\right) + z\left(w_1'\log(z) + w_1\frac{1}{z}\right) + \left(z^2 - m^2\right)\log(z)\,w_1\right) = 0$$

and then spam cancellation gives

$$\sum_{n=0}^{\infty}\left[F(n-m)\,h_n + h_{n-2}\right]z^{n-m} = L\left[z^{-m}h_2(z)\right] = -2c\frac{1}{z}w_1'(z).$$

For simplicity, take $m=1$:

$$w_1(z) = z\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(2)_k}\left(\frac{z}{2}\right)^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{k!\,(k+1)!2^{2k}}$$

Then $F(n-m) = n(n-2m)$ or $F(n-1) = n(n-2)$ and

$$\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{k!\,(k+1)!2^{2k}}$$

so

$$0 \cdot h_0 z^{-1} + (-1) \cdot h_1 + (0 \cdot h_2 + h_0)\, z + (3 \cdot 1 + h_1)\, z^2 + (4 \cdot 2 + h_2)\, z^3 + ...$$
$$= -2c\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k\,(2k+1)\,z^{2k}}{k!\,(k+1)!2^{2k}}\right)$$

To finish our discussion of Bessel functions of order $n \in \mathbb{N}_{>0}$, take $n = 1$:

$$z^2 w'' + zw' + (z^2 - 1)w = 0.$$

The indicial equation is

$$F(r) = r(r-1) + r - 1 = r^2 - 1 = (r-1)(r+1) = 0$$

so $r_1 = 1 > r_2 = -1$. We have

$$w(z) = \sum_{n=0}^{\infty} h_n z^{n+r}$$

so

$$\sum_{n=0}^{\infty} [F(r+n)h_n + h_{n-1}] z^{n+r} = 0.$$

At $n = 0 : F(r)h_0 = 0$, and we get $h_0 = 1, F(r) = 0$.

At $n = 1 : F(r+1)h_1 = 0$ so $F(r+1) \neq 0 \implies h_1 = 0 \implies h_{\text{odd}} = 0$.

At $r_1 = 1, F(n+1) = n(n=2)$ so $F(2k+1)h_{2k} + h_{2(k-1)} = 0$. This tells us

$$h_{2k} = -\frac{1}{(2k)(2k+2)} h_{2(k-1)} = ... = \frac{(-1)^k}{2^{2k}k!(k+1)!} h_0$$

and

$$w_1(z) = z^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}k!(k+1)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2^{2k}k!(k+1)!}$$

What about the second solution? $w_2(z) = z^{-1} \sum_{n=0}^{\infty} h_n z^n + c w_1(z) \log(z)$.

We have

$$L[w_2(z)] = -2c \cdot z w_1'(z)$$

where

$$w_2(z) = \sum_{n=0}^{\infty} [F(n-1)h_n + h_{n-2}] z^{n-1} = -2c \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{2^{2k}k!(k+1)!} z^{2k+1}.$$

But note that

$$\begin{aligned}F(n-1) &= n(n-2) \\ &= (0\cdot h_1)z^{-1} + (-1)h_1 + (0\cdot h_2 + 1)z + (3\cdot 1h_3 + h_1)z^2 + .. \\ &= -2c\left(z - \frac{3}{2^2\cdot 1\cdot 2}z^3 + ...\right)\end{aligned}$$

Now, we can see that $-2c = 1, h_{\text{odd}} = 0, h_2$ is "free", and $w_2(z) = z^{-1}(1 + h_2z^2 + h_4z^4)$ so changing $h_2$ adds a multiple of $w_1(z)$.

Furthermore, we get

$$(2k+2)(2k)h_{2k+2} + h_{2k} = \frac{(-1)^k(2k+1)}{k!(k+1)!2^{2k}}$$

for $k \geq 1$. Let $\tilde{g}_k = \text{RHS}$ and $y_k := h_{2k}$. Then

$$2^2(k)(k+1)y_{k+1} + y_k = \tilde{g}_k$$

with initial condition $y_1 = h_2$. Notice that this is a 1st order linear non-homogeneous difference equation, and can be written:

$$y_{k+1} + \frac{1}{2^2k(k+1)}y_k = g_k = \frac{(-1)^k(2k+1)}{2^{2(k+1)}((k+1)!)^2\cdot k}$$

To solve:

**Step 1**: Solve the homogeneous equation

$$x_{k+1} + \frac{1}{2^2k(k+1)}x_k = 0$$

with $x_1 = x_2$. We get

$$x_{k+1} = \frac{-1}{2^2\cdot k(k+1)}x_k = ... = \frac{(-1)^k}{(2^2)^kk!(k+1)!}h_2.$$

**Step 2**: Solve non-homogeneous recurrence using variation of parameters: set $y_k = u_kx_k$, which gives $y_1 = u_1x_1 = h_2$, ie. $u_1 = 1$. So

$$y_{k+1} + \frac{1}{2^2k(k+1)}y_k = u_{k+1}x_{k+1} - u_kx_{k+1} + \left(u_kx_{k+1} + \frac{1}{2^2k(k+1)}u_kx_k\right) = g_k.$$

The parenthesis term is equal to $u_k(0)$. Now we have $(u_{k+1} - u_k)x_{k+1} = g_k$, which we can solve to get

$$u_{k+1} = u_k + \frac{g_k}{x_{k+1}} = ... = u_1 + \frac{g_1}{x_2} + ... + \frac{g_k}{x_{k+1}}.$$

Is there a closed form? We have

$$\frac{g_k}{x_{k+1}} = \frac{(-1)^k(2k+1)}{2^{2(k+1)}((k+1)!)^2 \cdot k} : \frac{(-1)^k}{(2^2)^k k!(k+1)!} h_2 = \frac{1}{h_2}\left(\frac{1}{2^2} \cdot \frac{2k+1}{k(k+1)}\right).$$

We get

$$\begin{aligned} u_{k+1} &= u_1 + \frac{1}{h_2} \cdot \frac{1}{2^2}\left(\left(\frac{1}{1} + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + ... + \left(\frac{1}{k} + \frac{1}{k+1}\right)\right) \\ &= \frac{1}{2^2 h_2}\left(h_2 \cdot 2^2 + \left(1 + \frac{1}{2} + ... + \frac{1}{k}\right) + \left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k+1}\right)\right) \end{aligned}$$

It's convenient to take $h_2 = \frac{1}{2^2}$, and let $H_k := \left(1 + \frac{1}{2} + ... + \frac{1}{k}\right)$, $H_{k+1} = \left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k+1}\right)$, which allows us to write:

$$\begin{aligned} u_{k+1} &= \frac{1}{2^2 h_2}\left(H_k + H_{k+1}\right) = H_k + H_{k+1} \\ y_k &= u_k x_k = \left(H_{k-1} + H_k\right) \frac{(-1)^{k-1}}{(2^2)^{k-1} k!(k-1)!} h_2 \\ h_{2k} &= y_k \\ w_1(z) &= \sum_{n=0}^{\infty} \frac{(-1)^k}{k!(k+1)!}\left(\frac{z}{2}\right)^{2k+1} \\ w_2(z) &= z^{-1}\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(H_{k-1} + H_k)}{2^{2k}(k-1)!k!} z^{2k}\right) + \left(-\frac{1}{2}\right) w_1(z)\log(z). \end{aligned}$$

**Exercise 7.3.** *Show that for Bessel Equation of order $n \in \mathbb{N} > 0$:*

$$J_n(z) := \frac{1}{2^n}\Gamma(n+1) w_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n}$$

*and*

$$\begin{aligned} Y_n(z) : &= -\frac{2^n(n-1)!}{\pi} w_2(z) + \frac{\gamma - \ln(2)}{2^{n-1}\pi n!} w_1(z) \\ &= \frac{2}{\pi}\left(\gamma + \ln\left(\frac{z}{2}\right)\right) J_n(z) - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(-1)^k(k-1)!}{k!(1-n)_k}\left(\frac{z}{2}\right)^{2k-n} - \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k(H_{k+n} + H_k)}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k-n}. \end{aligned}$$

**Remark 7.4.** *For generic $\nu$, the basis of solutions of Bessel's equation is*

$$J_\nu(z)$$

*and the* ***Hankel functions***

$$Y_\nu(z) = \frac{\cos(\pi\nu) J_\nu(z) - J_{-\nu}(z)}{\sin(\pi\nu)}.$$

## 8 The Legendre Equation

**Definition 8.1.** *The* ***Legendre equation*** *of order $\alpha$ is of the form*

$$(1-z^2)w'' - 2zw' + \alpha(\alpha+1)w = 0.$$

It is clear that $z_0 = 0$ is an ordinary point.

**Exercise 8.2.** *Use the method of power series to find the basis (ie. the fundamental system) of solutions near $z_0 = 0$; find the radius of convergence of the resulting power series. Show that for some $\alpha$ the series becomes polynomial.*

What about singular points? There are three of them. Look at the coefficient at $w''$: $1 - z^2 = 0 \implies z = \pm 1$. We should consider the point at infinity!

At $z_1 = 1$, we can recenter using $t = z-1, z = t+1$, so $\frac{d}{dz} = \frac{d}{dt}$. This gives

$$\begin{aligned} -(1-(t^2+2t+1))w'' - 2(t+1)w' + \alpha(\alpha+1)w &= 0 \\ t(t+2)w'' + 2(t+1)w' + \alpha(\alpha+1)w &= 0 \\ t^2w'' + 2\frac{t+1}{t+2}tw' + \frac{t(\alpha)(\alpha+1)}{t+2}w &= 0 \end{aligned}$$

so $r(r-1) + 2 \cdot \frac{1}{2}r + 0 = 0$, so $r = 0$ with multiplicity 2.

**Exercise 8.3.** *Solve with for $z = -1$.*

At $z = \infty$, change variables to $Z = \frac{1}{z}$. Then we get

$$\begin{aligned} w(z) &= w\left(\frac{1}{z}\right) = \tilde{w}(z) \\ \frac{dw}{dz} &= w'\left(\frac{1}{Z}\right)\frac{d}{dz}Z = \tilde{w}'(Z) \cdot -\frac{1}{Z^2} = \tilde{w}'(Z)(-Z^2) \\ \frac{d}{dz^2}w &= \tilde{w}''(Z)\left(-\frac{1}{Z^2}\right)^2 + \tilde{w}'(Z)\frac{2}{Z^3} = \tilde{w}''(Z)(Z'') + 2\tilde{w}'(Z)Z^3 \end{aligned}$$

$$\begin{aligned} \left(1-\frac{1}{z^2}\right)\left(z''\tilde{w}''(Z) + 2Z^3\tilde{w}'(Z)\right) - 2\frac{1}{Z}(-Z^2)\tilde{w}'(Z) + \alpha(\alpha+1)\tilde{w}(Z) &= 0 \\ (Z^4 - Z^2)\tilde{w}'' + (2Z - 2Z)\tilde{w}' + \alpha(\alpha+1)\tilde{w} &= 0 \\ Z^2\tilde{w}'' + \frac{\alpha(\alpha+1)}{Z^2-1}\tilde{w} &= 0 \end{aligned}$$

where $\dfrac{\alpha(\alpha+1)}{Z^2-1}$ is holomorphic at $Z=0$. So

$$\begin{aligned} r(r-1)-\alpha(\alpha+1)&=0\\ r^2-r-\alpha^2-\alpha&=0\\ (r-\alpha-1)(r+\alpha)&=0 \end{aligned}$$

which implies $r_1=\alpha+1, r_2=-\alpha$.

The Legendre equation has three singular points, each one is regular, and the exponents at singularity are described as

$$\begin{Bmatrix} z_1=1 & z_2=-1 & z_3=\infty \\ 0 & 0 & \alpha+1 \\ 0 & 0 & \alpha \end{Bmatrix}$$

known as the **Riemann scheme**.

**Definition 8.4.** *A linear ODE in the complex domain*

$$w^{(r)}(z)+p_1(z)w^{r-1}(z)+...+p_r(z)w(z)=0$$

*is called* ***Fuchsian*** *if all of its singular points, including the point at infinity, are regular.*

**Definition 8.5.** *A 2nd order Fuchsian ODE is called a Riemann equation if it has exactly three singular points.*

**Example 8.6.** *The Legendre equation is an example of a Riemann equation.*

**Exercise 8.7.** *Is the Bessel equation Fuchsian, ie. is $\infty$ a regular singular point?*

Recall $\partial=\frac{d}{dz}, \delta=z\frac{d}{dz}, [\partial,\delta]=\partial$ and that $z_0=0$ is an regular singular point of

$$L[w]=w^{(r)}(z)+p_1(z)w^{r-1}(z)+...+p_r(z)w(z)=0$$

if $zp_1(z), z^2p_2(z), ..., z^rp_r(z)$ are holomorphic at $z_0=0$. Since

$$L=\partial^r+p_1(z)\partial^{r-1}+...+p_r(z)\cdot 1 \implies z^rL=z^r\partial^r+(zp_1(z))z^{r-1}\partial^{r-1}+...+z^rp_r(z)$$

so $z^k\partial^k$ can be rewritten in terms of $\partial$:

$$\begin{aligned} z\partial&=\partial\\ z^2\partial^2&=z(\delta)\partial=z(\partial\delta-\partial)=z\partial(\delta-1)=\delta(\delta-1)=\delta^2-\delta\\ z^3\partial^3&=...=\delta^3-3\delta^2+2\delta. \end{aligned}$$

We can use this to show

$$\begin{aligned} z^r L &= z^r \sum_{k=0}^{r} p_{r-k}(z) \cdot \partial^k \\ &= \sum_{k=0}^{\infty} p_{r-k}(z) z^{r-k} z^k \partial^k \\ &= \sum_{k=0}^{\infty} (p_{r-k}(z) z^{r-k})(\delta)(\delta-1)...(\delta-k+1) \\ &= \delta^r + q_1(z)\delta^{r-1} + ... + q_r(z) \end{aligned}$$

where $q_j(z)$ are holomorphic at $z_0 = 0$.

**Exercise 8.8.** *Show that*

$$q_1(z) = zp_1(z) - \frac{r(r-1)}{2}.$$

So regular singularity points occur for equations that can be written in terms of $\delta$ with holomorphic coefficients:

$$\begin{aligned} z\partial &= \delta & \delta &= z\delta \\ z^2\partial^2 &= \delta^2 - \delta & \delta^2 &= z^2\partial^2 + z\partial \\ z^3\partial^3 &= \delta^3 - 3\delta^2 + 2\delta & \delta^3 &= z^3\partial^3 + 3(z^2\partial^2 + z\partial) - 2(z\delta) \end{aligned}$$

and so on.

# 9 Riemann Equations

Recall that Fuchsian equations are equations such that all singular points, including the point at $z = \infty$, including the point at $z = \infty$, are regular. Define the differential operator

$$L = \partial^n + a_1(z)\partial^{n-1} + \cdots + a_n(z)$$

where $\partial = \frac{d}{dz}$. We have

$$L[w] = w^{(n)} + a_1(z)w^{(n-1)} + \cdots + a_n(z)w = 0$$

so

$$\begin{aligned} z^n L &= z^n \partial^n + a_1(z) z^n \partial^{n-1} + \cdots + z^n a_n(z) \\ &= \delta(\delta-1)(\delta-2)\dots(\delta-n+1) + a_1(z)\delta(\delta-1)\dots(\delta-n+2) + \cdots + a_n(z) \\ &= \delta^n + b_1(z)\delta^{n-1} + \cdots + b_n(z) \end{aligned}$$

where $\delta = z\frac{d}{dz}$. We know that $L[w] = \sum_{j=0}^n a_j(z)\partial^{n-j}w$ is Fuchsian if and only if $z^n L[w] = \sum_{j=0}^n b_j(z)\delta^{n-j}w$ and all coefficients $b_j(z)$ are holomorphic (analytic).

**Remark 9.1.** *For* $b_1(z) = za_1(z) - \frac{n(n+1)}{2}$,

$$\delta(\delta-1)(\delta-2)\dots(\delta-m+1) = \delta^m + (1-1-2\cdots-(m-1))\delta^{m-1} + \cdots + \frac{m(m-1)}{2}$$

**Lemma 9.2.** *The equation* $L[w] = 0$ *is Fuchsian with regular singular points* $z_1, z_2, \dots, z_m, z_{m+1} = \infty$ *if and only if the coefficients* $a_k(z)$ *have the form*

$$a_k(z) = \frac{p_k(z)}{(z-z_1)^k(z-z_2)^k\dots(z-z_m)^k}$$

*and* $p_k(z)$ *is a polynomial of degree at most* $k(m-1)$.

*Proof.* The Fuchsian condition at $z = z_j$ is that $(z-z_j)^k a_k(z)$ is holomorphic.

At $z = \infty$: put $\zeta = \frac{1}{z}$. Note that $\frac{d}{dz} = \frac{d\zeta}{dz}\frac{d}{d\zeta} = -\frac{1}{z^2}\frac{d}{d\zeta} = -\zeta^2\frac{d}{d\zeta}$, which gives $\delta = z\frac{d}{dz} = -\zeta\frac{d}{d\zeta} = -\theta$ where $\theta = \zeta\frac{d}{d\zeta}$. The equation $z^n L[w] = \sum_{j=0}^n b_j(z)\delta^{n-j}w$ gives $z = \frac{1}{\zeta}$.

We also have $\sum_{j=0}^n b_j\left(\frac{1}{\zeta}\right)(-\theta)^{n-j}w = 0$. This gives

$$b_j(z) = z^j a_j(z) + ...$$

$$
\begin{aligned}
b_j\left(\frac{1}{3}\right) &= \frac{1}{\zeta^j} a_j \left(\frac{1}{\zeta}\right) \\
&= \frac{1}{\zeta^j} a_j \frac{P_j\left(\frac{1}{\zeta}\right)}{(\frac{1}{\zeta}-z_1)^k ... (\frac{1}{\zeta}-z_m)^j} \\
&= \frac{\zeta^{jm-j} p_j\left(\frac{1}{3}\right)}{(1-\zeta z_1)^k ... (1-\zeta z_m)^j}
\end{aligned}
$$

where $p_j\left(\frac{1}{3}\right)$ is a polynomial of degree $d$, $p_j\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^d \tilde{p}_j(\zeta)$, $\zeta^{j(m-1)-d} \geq 0$, and $d = \deg p_j(z) \leq j(m-1)$.

□

**Definition 9.3.** *A Riemann scheme for a Fuchsian equation is the following table*

$$
\begin{pmatrix}
z_1 & z_2 & ... & z_m & z_{m+1=\infty} \\
r_1^1 & r_2^1 & ... & r_m^1 & r_{m+1}^1 \\
\vdots & \vdots & & \vdots & \vdots \\
r_1^n & r_2^n & ... & r_m^n & r_{m+1}^n
\end{pmatrix}
$$

*where the first row are regular singlular points and the first column is the exponents at the singularity (ie. roots of the indicial equation).*

**Lemma 9.4** (Fuchs Relation)**.**

$$
\sum_{i=1}^{m+1} \sum_{j=1}^{n} r_i^j = \frac{(m-1) \cdot n \cdot (n-1)}{2}
$$

*where $m$ is the number of finite singular points and $n$ is the order of the equation.*

Let

$$
L = \sum_{j=0}^{n} a_j(z) \partial^{n-j} = \partial^n + a_1(z)\partial^{n-1} + ...
$$

where $a_1(z) = \frac{\alpha_1}{z-z_1} + ... + \frac{\alpha_n}{z-z_m}$.

At $z = z_i$:

$$
(z-z_i)^n L = (z-z_i)^n \partial^n + ... = \delta_i^n + \left( (z-z_i) \prod_{j=1}^{m} \frac{\alpha_j}{z-z_j} - \frac{n(n-1)}{2} \right) \delta_i^{n-1} + ...
$$

where $\delta_i^n = (z-z_i)^n \partial^n$. The associated Euler equation is

$$
r^n + \left( \alpha_i - \frac{n(n-1)}{2} \right) r^{n+1} + ... = 0
$$

where $r^n = (r-r_i^1)...(r-r_i^n)$ and $r_i^1 + ... + r_i^n = -\alpha_i + \frac{n(n-1}{2}$.

At $z_{m+1} = \infty$:

$$z^n L(z) = \zeta^n L(\zeta) = (-1)^n \left( \theta^n + \left( -\frac{1}{3} a_1 \left( \frac{1}{3} \right) + \frac{n(n-1)}{2} \right) \theta^{n-1} + ... \right) - \left( \sum_{i=1}^{m} \frac{\alpha_i}{1 - \zeta z_i} \right)$$

We have

$$\sum_{j=1}^{n} r_{m+1}^{j} = \sum_{i=1}^{m} \alpha_i - \frac{n(n-1)}{2}$$

and

$$\begin{aligned} \sum_{i=1}^{m+1} \sum_{j=1}^{n} r_i^j &= -\sum_{i=1}^{m} \alpha_i + m\frac{n(n-1)}{2} + \sum_{i=1}^{m} \alpha_i - \frac{n(n-1)}{2} \\ &= \frac{(m-1)n(n-1)}{2} \end{aligned}$$

In particular, if we consider $n = 2$: $\sum \sum r_i^j = (m-1)$ where $m$ is the number of finite regular singular points.

Let's restrict to second order equations. If $w'' + p(z)w' + q(z)w = 0$ has solutions of the form $w(z) = z^r h(r)$ with $h(z)$ analytic at $z-0$, $h(0) = 1$, then the equation has to be Fuchsian at $z = 0$, ie, $z = 0$ is a regular singular point, ie, $p(z)$ has at most first order pole at $z = 0$ and $q(z)$ has at most second order pole at $z = 0$.

Suppose $w_1(z) = z^{r_1} h_1(z)$ and $w_2(z) = z^{r_2} h_2(z)$ are two solutions

$$\begin{cases} w_1'' + p(z)w_1' + q(z)w_1 = 0 \\ w_2'' + p(z)w_2' + q(z)w_2 = 0 \end{cases}$$

where

$$\begin{bmatrix} w_1' & w_1 \\ w_2' w_2 & \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = - \begin{bmatrix} w_1'' \\ w_2'' \end{bmatrix}$$

Then, using Cramer's Rule,

$$p(z) = \frac{\begin{vmatrix} w_1'' & w_1 \\ w_2'' & w_2 \end{vmatrix}}{\begin{vmatrix} w_1' & w_1 \\ w_2' & w_2 \end{vmatrix}} = -\frac{d}{dz} \log \begin{vmatrix} w_1' & w_1 \\ w_2' & w_2 \end{vmatrix}.$$

We can check that

$$\begin{vmatrix} w_1' & w_1 \\ w_2' & w_2 \end{vmatrix} = W(w_1, w_2) \neq 0$$

as

$$\begin{vmatrix} w_1' & w_1 \\ w_2' & w_2 \end{vmatrix} = w_1' w_2 - w_1 w_2' = w_2^2 \left( \frac{w_1}{w_2} \right)'.$$

Let

$$v(z) = \left(\frac{w_1(z)}{w_2(z)}\right)'.$$

Then we have

$$p(z) = -\frac{d}{dz}\log\bigl(w_2^2 v(z)\bigr) = -\frac{v'(z)}{v(z)} - 2\frac{w_2'(z)}{w(z)}.$$

Both $v(z)$ and $w(z)$ are of the form $f(z) = z^r h(z)$ where $h(0) \neq 0$, so

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)} = \frac{r}{z} + \frac{h'(z)}{h(z)}$$

where $\frac{r}{z}$ is at most a first order pole. We can now conclude that $p(z)$ has at most a first order pole at $z = 0$ and $q(z) = -\frac{w_1''}{w_1} - p(z)\frac{w_1'}{w_1}$ has at most a second order pole at $z = 0$.

Let's move on to discuss Riemann equations.

**Definition 9.5.** *A second order Fuchsian ODE on the Riemann sphere is called a **Riemann equation** if it ihas only three regular singular points (including $\infty$).*

The Riemann scheme for singular points $a_1, a_2, a_3$ is

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

The solutions of a Riemann equation (ie, a general solution) are denoted by

$$P\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}; z\right)$$

## 10 Mobius Transformations: The Group of Automorphisms $\mathrm{PSL}_2$ on $\mathbb{P}^1_{\mathbb{C}}$

Define $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C}^2 - \{(0,0)\}/\sim$ where $(x,y) \sim (\lambda x, \lambda y)$ for $\lambda \in \mathbb{C}^\times$. Denote $[(x,y)] = [x : y]$ as the homogeneous coordinates of $\mathbb{P}^1_{\mathbb{C}}$. For $x \neq 0$, we have $[x : y] = \left[1 : \frac{y}{x}\right]$ and similarly for $y \neq 0$ we have $[y : x] = \left[\frac{x}{y} : 1\right]$.

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2 = \mathrm{Aut}(\mathbb{C}^2)$. We have

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

and

$$A \cdot [x : y] = [ax + by : cx + dy].$$

**Definition 10.1.** *For $z = \frac{x}{y}$ with $y \neq 0$,*

$$A \cdot z = \frac{ax+by}{cx+dy} = \frac{az+b}{cz+d}$$

*is the fractional linear transformation or Möbius transformation.*

Additionally, let's define $\mathrm{PGL}_2 = \mathrm{GL}_2/\sim$.

**Proposition 10.2.** *For any three points $a_1, a_2, a_3 \in \mathbb{C}$, there exists a unique Möbius transformation $\varphi$ mapping $\langle a_1, a_2, a_3\rangle \mapsto \langle 0, 1, \infty\rangle$ given by*

$$\varphi(z) = \frac{(a_2 - a_3)}{(a_2 - a_1)} \frac{z - a_1}{z - a_3}.$$

Using a Möbius transformation, we can always change a Riemann equation with regular singular point $a_1, a_2, a_3$ to the one with regular singular point $0, 1, \infty$ and vice versa. Without loss of generality, consider

$$\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0 & \sigma_1 & \sigma_\infty \\ \tau_0 & \tau_1 & \tau_\infty \end{pmatrix}.$$

Then there exists a unique Riemann equation with this Riemann scheme (satisfying the Fuchs relation $(\sigma_0 + \tau_0) + (\sigma_1 + \tau_1) + (\sigma_\infty + \tau_\infty) = 1$).

For $w'' + a_1(z)w' + a_2(z) = 0$, we have

$$a_1(z) = \frac{\text{linear}}{z(z-1)} = \frac{A_0}{z} + \frac{A_1}{z-1}$$

$$a_2(z) = \frac{\text{quadratic}}{z^2(z-1)^2} = \frac{B_0}{z^2} + \frac{B_1}{(z-1)^2} + \frac{B_2}{z(z-1)}.$$

Near $z=0$: we have $A_1 = 1-\sigma_1-\tau_1, B_2 = \sigma_1\tau_1$ and

$$w'' + \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right) w' + \left(\frac{B_0}{z^2} + \frac{B_1}{(z-1)^2} + \frac{B_2}{z(z-1)}\right) w = 0.$$

The indicial equation is

$$r(r-1) + A_0 r + B_0 = (r-\sigma_0)(r-\tau_0) = 0$$

so $A_0 = 1-\sigma_0-\tau_0$ and $B_0 = \sigma_0\tau_0$ with $A_0 = 1-\sigma_0-\tau_0, B_0 = \sigma_0\tau_0$

Near $z=1$:

$$z^2\left(w'' + \left(\frac{A_0}{z} + \frac{A}{z-1}\right) w' + \left(\frac{B_0}{z^2} + \frac{B_1}{(z-1)^2} + \frac{B_2}{z(z-1)}\right) w = 0\right)$$

$$z^2\partial^2 + \left(A_0 + A_1\frac{z}{z-1}\right) z\partial + \left(B_0 + B_1\left(\frac{z}{z-1}\right)^2 + B_2\left(\frac{z}{z-1}\right)\right) w = 0$$

where $z^2\partial^2 = \delta(\delta-1) = -\theta(-\theta-1)$ so

$$\theta^2 + \theta - \left(A_0 + A_1\frac{1}{1-\zeta}\right)\theta + \left(B_0 + B_1\left(\frac{1}{1-\zeta}\right)^2 + B_2\frac{1}{1-\zeta}\right) w = 0.$$

The indicial equation is

$$r^2 + (1-A_0-A_1)r + (B_0+B_1+B_2) = (r-\sigma_\infty)(r-\tau_\infty)$$

Since $1-A_0-A_1 = -\sigma_\infty-\tau_\infty$, $A_0 = 1-\sigma_0-\tau_0, A_1 = 1-\sigma_1-\tau_1$, which gives the Fuchs relation

$$1 = 1-\sigma_0-\tau_0+1-\sigma_1-\tau_1-\sigma_\infty-\tau_\infty.$$

In addition, $B_1 = \sigma_1\tau_1, B_0 = \sigma_0\tau_0$ so

$$B_0+B_1+B_2 = \sigma_\infty\tau_\infty$$

and

$$B_2 = \sigma_\infty\tau_\infty - \sigma_0\tau_0 - \sigma_1\tau_1.$$

Finally,

$$w''+\left(\frac{1-\sigma_0-\tau_0}{z} + \frac{1-\sigma_1-\tau_1}{z-1}\right) w'+\left(\frac{\sigma_0\tau_0}{z^2} + \frac{\sigma_1\tau_1}{(z-1)^2} + \frac{\sigma_\infty\tau_\infty-\sigma_0\tau_0-\sigma_1\tau_1}{z(z-1)}\right) w = 0$$

**Exercise 10.3.**

$$z^\nu(1-z^\mu)P\left(\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0 & \sigma_1 & \sigma_\infty \\ \tau_0 & \tau_1 & \tau_\infty \end{pmatrix}; z\right) = P\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0+\nu & \sigma_1+\mu & \sigma_\infty-\nu-\mu \\ \tau_0+\nu & \tau_1+\mu & \tau_\infty-\nu-\mu \end{pmatrix}$$

We can focus on

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix}$$

where $\alpha = \sigma_0 + \sigma_1 + \sigma_\infty$, $\beta = \sigma_0 + \sigma_1 + \tau_\infty$.

**Definition 10.4.** *The* ***Gauss hypergeometric function*** *is*

$$E(\alpha, \beta, \gamma) : z(1-z)\frac{d^2\omega}{dz^2} + (\gamma - (\alpha+\beta+1)z)\,\frac{dw}{dz} - \alpha\beta w = 0.$$

# 11 Gauss Hypergeometric Equation

Let's discuss Gauge transformations. We are interested in functions $w = (wz)$, where $z \in \mathbb{C} \subseteq \mathbb{P}^1_{\mathbb{C}}$, and $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}) = \mathrm{PGL}_2 =$ Möbius transformations act on $w$ by $w \to (z-a)^k w$. The Riemann equations have 3 removable singular points:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & \infty \\ \alpha_1' & \alpha_2' & \alpha_3' \end{pmatrix}$$

where $g \in \mathrm{PGL}_2$.

Assume that the removable singular points are $z_0 = 0, z_1 = 1$, and $z_\infty = \infty$. We still have the "residual" $\mathrm{PGL}_2$ action permuting these points.

**Exercise 11.1.** *Describe this group $\simeq S_3$ as linear fractional transformations.*

We can write

$$w(z) = z^a (z-1)^b u(z)$$

where $w(z)$ is a solution of the form

$$\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0 & \sigma_1 & \sigma_\infty \\ \tau_0 & \tau_1 & \tau_\infty \end{pmatrix}$$

Earlier, we explicitly wrote down the differential equation that $w$ satisfies:

$$\frac{d^2w}{dz^2} + \left(A_0 + \frac{A_1}{z}\right)\frac{dw}{dz} + \left(\frac{B_0}{z(z-1)} + \frac{B_1}{(z-1)} + \frac{B_2}{z}\right) w = 0.$$

with coefficients

$$A_0 = 1 - \sigma_0 - \tau_0$$
$$A_1 = 1 - \sigma_1 - \tau_1$$
$$B_0 = \sigma_0\tau_0$$
$$B_1 = \sigma_1\tau_1$$
$$B_2 = \sigma_\infty\tau_\infty - \sigma_0\tau_0 - \sigma_1\tau_1.$$

Additionally, we have

$$\frac{dw}{dz} = w(z)\left(\frac{a}{z} + \frac{b}{z-1} + \frac{u'(z)}{u(z)}\right) = z^a(z-1)^b\left(\left(\frac{a}{z} + \frac{b}{z-1}\right)u(z) + u'(z)\right)$$

and

$$\begin{aligned}\frac{d^2w}{dz^2} &= z^a(z-1)^b\left(\left(\frac{a}{z}+\frac{b}{z-1}\right)\left(\left(\frac{a}{z}+\frac{b}{z-1}\right)u(z)+u'(z)\right)\right.\\&\left.+\left(-\frac{a}{z^2}-\frac{b}{(z-1)^2}\right)u(z)+\left(\frac{a}{z}+\frac{b}{z-1}\right)u'(z)+u''(z)\right)\\&= z^a(z-1)^b\left(u''(z)+\left(\frac{2a}{z}+\frac{2b}{z-1}\right)u'(z)\right.\\&\left.+\left(\left(\frac{a}{z}+\frac{b}{z-1}\right)^2-\frac{a}{z^2}-\frac{b}{(z-1)^2}\right)u(z)\right)\end{aligned}$$

so

$$\begin{aligned}&\frac{d^2u}{dz^2}+\left(\frac{A_0+2a}{z}+\frac{A_1+2b}{z-1}\right)\frac{du}{dz}\\&+\left(\frac{B_0}{z^2}+\frac{B_1}{(z-1)^2}+\frac{B_2}{z(z-1)}+\frac{a^2-a}{z^2}+\frac{b^2-b}{(z-1)^2}+\frac{2ab}{z(z-1)}+\left(\frac{A_0}{z}+\frac{A_1}{z-1}\right)\left(\frac{a}{z}+\frac{b}{z-1}\right)\right)u\\&=0\end{aligned}$$

which can be simplified into

$$\begin{aligned}&\frac{d^2u}{dz^2}+\left(\frac{A_0+2a}{z}+\frac{A_1+2b}{z-1}\right)\frac{du}{dz}\\&+\left(\frac{B_0+A_0a+a^2-a}{z^2}+\frac{B_1+A_1b+b^2-b}{(z-1)^2}+\frac{B_2+A_0b+A_1a+2ab}{z(z-1)}\right)u\\&=0\end{aligned}$$

This gives the relations

$$1-\alpha_0-\beta_0=1-\sigma_0-\tau_0+2a$$

$$1-\alpha_1-\beta_1=1-\sigma_1-\tau_1+2b$$

$$\begin{aligned}\alpha_0\beta_0 &= \sigma_0\tau_0+(1-\sigma_0-\tau_0)a+a^2-a\\&=\sigma_0\tau_0-(\sigma_0+\tau_0)a+a^2\\&=(\sigma-a)(\tau_0-a)\end{aligned}$$

where $\alpha_0=\sigma_0-a$ or $\sigma_0=\alpha_0+a, \tau_0=\beta_0+a, \sigma_1=\alpha_1+b, \tau_1=\beta_1+b$. We can now write

$$\alpha_\infty\beta_\infty-\alpha_0\beta_0-\alpha_1\beta_1=\sigma_\infty\tau_\infty-\sigma_0\tau_0-\sigma_1\tau_1+(1-\sigma_0-\tau_0)b+(1-\sigma_1-\tau_1)a+2ab$$

so

$$\begin{aligned}
\alpha_\infty\beta_\infty &= \sigma_\infty\tau_\infty + (\sigma_0 - a)(\tau_0 - a) + (\sigma_1 - b)(\tau_1 - b) - \sigma_0\tau_0 - \sigma_1\tau_1 + (1 - \sigma_0 - \tau_0)b + (1 - \sigma_1 - \tau_1)a + 2ab \\
&= \sigma_\infty\tau_\infty - a(\sigma_0 + \tau_0) + a^2 - b(\sigma_1 + \tau_1) + b^2 + (1 - \sigma_0 - \tau_0)b + (1 - \sigma_1 - \tau_1)a + 2ab \\
&= \sigma_\infty\tau_\infty + (1 - \sigma_0 - \tau_0 - \sigma_1 - \tau_1)a + (\sigma_\infty + \tau_\infty)b + (a + b)^2 \\
&= \sigma_\infty\tau_\infty + (\sigma_\infty + \tau_\infty)a + (\sigma_\infty + \tau_\infty)b + (a + b)^2 \\
&= (\sigma_\infty + a + b)(\tau_\infty + a + b
\end{aligned}$$

with $\sigma_\infty = \alpha_\infty - a - b$ and $\tau_\infty = \beta_\infty - a - b$.

We can now write

$$P\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0 = \alpha_0 + a & \sigma_1 = \alpha_1 + b & \sigma_\infty = \alpha_\infty - a - b; z \\ \tau_0 = \beta_0 + a & \tau_1 = \beta_1 + b & \tau_\infty = \beta_\infty - a - b \end{pmatrix} = z^a(z-1)^b P\begin{pmatrix} 0 & 1 & \infty \\ \alpha_0 & \alpha_1 & \alpha_\infty; z \\ \beta_0 & \beta_1 & \beta_\infty \end{pmatrix}$$

so

$$\begin{aligned}
P\begin{pmatrix} 0 & 1 & \infty \\ \sigma_0 & \sigma_1 & \sigma_\infty; z \\ \tau_0 & \tau_1 & \tau_\infty \end{pmatrix} &= z^{\sigma_0}(z-1)^{\sigma_1} P\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \sigma_\infty + \sigma_0 + \sigma_1; z \\ \tau_0 - \sigma_0 & \tau_1\sigma_1 & \tau_\infty + \tau_0 + \tau_1 \end{pmatrix} \\
&= z^{\sigma_0}(z-1)^{\sigma_1} P\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha\,; z \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{pmatrix}.
\end{aligned}$$

The Gauss hypergeometric equation $E(\alpha, \beta, \gamma)$ is the Riemann equation with the Riemann scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1 - \gamma & \gamma - \alpha - \beta & \beta \end{pmatrix}$$

We have

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\alpha + \beta + 1 - \gamma}{z - 1}\right)\frac{dw}{dz} + \frac{\alpha\beta}{z(z-1)}w = 0.$$

For $E(\alpha, \beta, \gamma)$, we have

$$z(1 - z)w''(z) + (\gamma - (\alpha + \beta + 1)z)w' - \alpha\beta w = 0.$$

Now, we discuss the regular singular points at $z = 0, z = 1$, and $z = \infty$.

At $z = 0$,

$$w(z) = z^r \sum_{h=0}^{\infty} h_n z^n = \sum_{n=0}^{\infty} h_n z^{n+r}$$

where $h_0 = 1$. Using the index shift $n \rightsquigarrow n + 1$, we have

$$\sum_{n=0}^{\infty} \begin{bmatrix} (n+r)(n+r-1)h_n z^{n+r-1} - (n+r)(n+r-1)h_n z^{n+r} \\ \gamma(n+r)h_n z^{n+r-1} - (\alpha + \beta + 1)(n+r)h_n z^{n+r} - \alpha\beta h_n z^{n+r} \end{bmatrix} = 0$$

is equivalent to showing that

$$[((n+r)(n+r+1)+\sigma(n+r+1))\,h_{n+1}-((n+r)(n+r+\alpha+\beta)+\alpha\beta)\,h_n]\,z^{n+1}=0$$

or

$$(n+r+1)(n+r+\gamma)h_{n+1}-(n+r+\alpha)(n+r+\gamma)h_n=0.$$

The indicial equation at $n=-1$ is $r(r-1+\gamma)h_0=0$ with $r_1=0, r_2=1-\gamma$. We assume the non-resonant case, i.e. $\gamma\notin\mathbb{Z}$. Then

$$\begin{aligned} h_{n+1}(r) &= \frac{(n+r+\alpha)(n+r+\beta)}{(n+r+1)(n+r+\gamma)}h_n \\ &= \\ &\vdots \\ &= \frac{(n+r+\alpha)(n+r+\beta)...(r-\alpha)(r+\beta)}{(n+r+1)(n+r+\gamma)...(r+1)(r+\gamma)}h_0 \\ &= \frac{(r+\alpha)(r+\alpha+1)...(r+\alpha+r)(r+\beta)n+1}{(r+1)_{n+1}(r+\gamma)_{n+1}} \end{aligned}$$

which gives

$$h_n(r)=\frac{(r+\alpha)_n(r+\beta)_n}{(r+1)_n(r+\gamma)_n}.$$

At $r=0$:

$$h_0=\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n}$$

where $(1)_n=n!$ and

$$w_1(z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n}z^n$$

which is analytic if $|z|<1$ and analytic when $|z|\geq 1$ if $\mathrm{Re}(\gamma-\alpha-\beta)>0$.

**Definition 11.2.** *The Gauss Hypergeometric function is*

$${}_2F_1(\alpha,\beta;\gamma;z):=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n}z^n$$

We have $r_2=1-\gamma$,

$$h_n(1-\gamma)=\frac{(1-\gamma+\alpha)_n(1-\gamma+\beta)_n}{(2-\gamma)_n(1)_n}$$

so

$$w_2(z)=z^{1-\gamma}\sum_{n=0}^{\infty}\frac{(1-\gamma+\alpha)_n(1-\gamma+\beta)_n}{(2-\gamma)_n(1)_n}z^n=z_2^{1-\gamma}F_1(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z.$$

Our expression for $P$ is

$$P\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix} = c_1\,{}_2F_1(\alpha,\beta;\gamma;z)+c_z\,z^{1-\gamma}\,{}_2F_1(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z)$$

Near $z=1$:

$$z(1-z)w'' + (\gamma-(\alpha+\beta+1)z)w' - \alpha\beta w = 0.$$

Put