BIMSA: Affine Lie Algebras and Affine Quantum Groups

Lectures by Bart Vlaar, Notes by Gary Hu

In Spring 2024, Bart Vlaar taught Affine Lie Algebras and Affine Quantum Groups at BIMSA.

This an unofficial set of notes scribed by Gary Hu, who is responsible for all mistakes. If you do find any errors, please report them to: gh7@williams.edu

Contents

1 Introduction

A rough plan: the course is split into two halves

1. Certain infinite-dimensional Lie algebras (affine Lie algebras): basic definitions and structure theory, representation theory, applications to specific functions (Macdonald identities).
2. The q-deformations, certain associative algebras called affine quantum groups introduced since 1980s: basic definitions and strucutre theory, representation theory, and applications to quantum integrable systems (construction of solutions to parameter dependent braid relation, aka Yang-Baxter equation).

We will focus on $\mathfrak{sl}_2$ case. In general, these "affine" algebraic structures are related to special functions, canonical bases/crystal bases, cluster algebras, vertex operator algebras, string theory, and integrable systems.

The reference material is as follows: [@kac] for part 1 and [@chari] for part 2.

Prerequisites for the course: basic notions of algebra and representation theory; basic facts about representation theory of simple finite dimensional Lie algebras ($\mathfrak{sl}_n, \mathfrak{so}_n$) over $\mathbb{C}$.

Professor Vlaar hopes that students will gain the following from attending these lectures:

- Basic working knowledge on these algebraic structures
- Platform for you own research/deeper study

Students should ask questions and do homework if they want to get the most out of this course.

2 Lie Algebra Basics

Let k be a field.

Definition 2.1. *A k-**algebra** is a k-linear space A with bilinear product: $A \times A \to A$*

Definition 2.2. *A **Lie algebra** (over k) is a k-algebra with a Lie bracket $[\cdot, \cdot]$ such that*

1. *Alternating property:* $[x, x] = 0$
2. *Jacobi identity* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \forall x, y, z \in \mathfrak{g}$

For $x \in \mathfrak{g}$, define $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$, the left adjoint map. The Jacobi identity $\iff \forall x \in \mathfrak{g}$, $\mathrm{ad}(x)$ is a **derivation** on $\mathfrak{g}$ if

$$\mathrm{ad}(x)([y, z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)].$$

Remark 2.3. *A k-algebra $\mathfrak{g}, [\cdot,\cdot]$ such that $\forall x \in \mathfrak{g}$ $ad(x)$ is a derivation is called a* ***Leibniz algebra.***

Definition 2.4. *A* ***Lie algebra homomorphism*** *if a k-linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ ($\mathfrak{g}, \mathfrak{h}$ are Lie algebras over k) such that $\varphi([x, y]) = [\varphi(x), \varphi(y)] \forall x, y \in \mathfrak{g}$. If φ is invertible, called* ***isomorphism****. If $\mathfrak{h} = \mathfrak{g}, \varphi$ is called *endomorphism of $\mathfrak{g}$. If $\mathfrak{h} = \mathfrak{g}$ and φ invertible, φ is called* ***automorphism*** *of $\mathfrak{g}$*

If $\mathfrak{h}, \mathfrak{h}'$ are two subsets of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{h}'] = \mathrm{span}_k\{[x, y] | x \in \mathfrak{h}, y \in \mathfrak{h}'\} \subseteq \mathfrak{g}$.

Definition 2.5. *A* ***Lie subalgebra*** *$\mathfrak{h}$ is a k-linear subspace of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.*

Definition 2.6. *An* ***ideal*** *i is a k-linear subspace of $\mathfrak{g}$ such that $[g, i] \subseteq i$ (equivalently $[i, \mathfrak{g}] \subseteq i$)*

Exercise 2.7. *Prove the following are ideals:*

1. *$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ derived subalgebra.*
2. *$Z(\mathfrak{g}) := \{z \in g | [x, y] = 0 \forall y \in \mathfrak{g}\}$ center of $\mathfrak{g}$.*
3. *hom$\varphi : \mathfrak{g} \to \mathfrak{h}, ker(\varphi) := \{x \in \mathfrak{g} | \varphi(x) = 0\} \subseteq \mathfrak{g}$.*

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra over k. $\mathfrak{g}$ is called **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$ (maximal centre and maximal derived subalgebra). $\mathfrak{g}$ is called **simple** if only ideals are 0 and $\mathfrak{g}$, and $\mathfrak{g}$ is not abelian (minimal center and maximal derived subalgebra).

Simple Lie algebras are not assumed to be finite dimensional.

Definition 2.8. *A* ***spanning set*** *of $\mathfrak{g}$ is the same as a spanning set of underlying vector spaces (linear independence spanning set is called a* ***basis****). A* ***generating set*** *of $\mathfrak{g}$ is a subset $S \subseteq \mathfrak{g}$ such that the smallest subalgebra of $\mathfrak{g}$ contains S is $\mathfrak{g}$ itself.*

Example 2.9. $\mathfrak{sl}_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{C}\right\}$ *have spanning set (also a basis)*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and a generating set $\{e, f\}$ (since $h = [e, f]$)

Definition 2.10. *The* ***dimension*** *of $\mathfrak{g}$ is simply the dimension of the underlying vector space.*

Remark 2.11. *If $\dim \mathfrak{g} = \infty$, arbitrary elements of $\mathfrak{g}$ are finite k-linear combinations of any given basis.*

3 Familiar Examples of Lie Algebras

Example 3.1. *Let $(A, \cdot)$ be any associative k-algebra*

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \forall x, y, z \in A$$

The ***commutator*** $[x, y] := xy - yx$ *defines a Lie bracket on* A, $(A, [\cdot, \cdot])$ *is a Lie algebra. (Associativity* $\implies$ *Jacobi identity)*

Example 3.2. *Let* V *be any* k*-linear space. Then* $End_k(V) = \{k-\text{linear maps}: V \to V\}$*. The Lie algebra is often written* $gl(V)$ *or* $gl(V, k)$*.*

Example 3.3. *Let* (A_j) *be any* k*-algebra.*

$$der(A) = \{\varphi \in gl(A) | \varphi(x \cdot y) = \varphi(x) \cdot y + x \cdot \varphi(y)\}$$

Suppose $\varphi, \psi \in der(A)$*. For* $x, y \in A$*,*

$$\begin{aligned}(\varphi \circ \psi)(xy) &= \varphi(\psi(x) \cdot y + x \cdot \psi(y)) \\ &= \varphi(\psi(x)) \cdot y + \psi(x) \cdot \varphi(y) + \varphi(x) \cdot \psi(y) + x \cdot \varphi(\psi(y))\end{aligned}$$

Hence

$$(\psi \circ \varphi)(xy) = \psi(\varphi(x)) \cdot y + \psi(x) \cdot \varphi(y) + \varphi(x) \cdot \psi(y) + x \cdot \psi(\varphi(y))$$

Note that

$$\begin{aligned}[\varphi, \psi](xy) &= \varphi(\psi(x)) \cdot y + x \cdot \varphi(\psi(y)) - \psi(\varphi(x)) \cdot y - x \cdot psi(\varphi(y)) \\ &= [\varphi, \psi](x) \cdot y + x \cdot [\varphi, \psi](y)\end{aligned}$$

which implies that $der(A)$ *is a Lie algebra.*

Example 3.4. *Let* $U \subset \mathbb{R}^n$ *be open.* $C^\infty(U) := \{\text{smooth functions } U \to \mathbb{R}\}$*. A* ***(smooth) vector field*** *on* U *is* $C^\infty(U)$ *of the form*

$$X = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

with $a_i \in C^\infty(U)$*.*

Remark 3.5. *View* $\frac{\partial}{\partial x_i}$ *as a "direction vector" in tangent space at* $x = (x_1, x_2, ..., x_n)$*.*

Exercise 3.6. *Let* $X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}$ *with* $a_i, b_i \in C^\infty(U)$*. Then show* $[X, Y] = \sum_{i=1}^n c_i(x) \frac{\partial}{\partial x_i}$ *with* $c_i(x) = \sum_{j=1}^n (a_j(x) \frac{\partial}{\partial x_j} b_i(x) - b_j(x) \frac{\partial}{\partial x_i} a_i(x))$*. Conclude that* $VF(U) = \{\text{vector fields on } U\}$ *is a Lie algebra over* $\mathbb{R}$*.*

Remark 3.7. *May replace* U *with any small* n*-dimensional manifold* M*. Use "local" coordinates* $x_1, ..., x_n$ *on each*

Example 3.8. *In particular, let* G *be a Lie group (group and* n*-dimensional real manifold such that product and inverse are smooth maps).* G *acts on* $C^\infty(G)$ *by "lect translation"* $\forall \varphi \in C^\infty(G), \forall g \in G$*, define* $\varphi_g \in C^\infty(G)$ *by* $\varphi_g(h) := \varphi(gh) \forall h \in G$*. Call* $X \in VF(G)$ ***left-invariant*** *if* $\forall \varphi \in C^\infty(G)$*,* $X_\varphi \in C^\infty(G)$ *satisfies*

$$(X_\varphi)_g = X(\varphi_g) \forall g \in G$$

The left invariant vector fields on G *form a Lie algebra over* $\mathbb{R}$ $Lie(G)$ *"the Lie algebra of* G*".*

Exercise 3.9. *Use the notation of the previous exercise to show* $Lie(G)$ *is really a Lie algebra.*

Remark 3.10. *Can view vector fields* $X \in VF(G)$ *as a smoothly varying family of tangent vectors* $(X_g)_{g \in G}$ *for* $g \in G$*. Then the map* $f : Lie(G) \to T_e(G), X \mapsto X_e$ *is a* $\mathbb{R}$*-linear isomorphism, with* $e = id$ *of* G*.*

Example 3.11. $gl_n(k) = gl_n(k^n) = \{n \times n \text{ matrices over } k\}$ *is a Lie algebra. If* $k = \mathbb{R}, gl_n(k) = Lie(GL_n(k))$ *where*

$$GL_n(k) = \{n \times n \text{ invertible matrices over } k\}$$

This is not simple: $Id_{n\times n} \in Z(gl_n(k))$*. (if* $n > 1$*, not abelian). One special case of this is* $sl_n(k) = \{X \in gl_n(k) | Tr(X) = 0\}$*. When* $k = \mathbb{R}$*, this is* $= Lie(SL_n(k))$*, where* $SL_n(k) = \{n \times n \text{ matrices over } k, \det 1\}$*.*

Exercise 3.12. *Prove* $sl_2(k)$ *is simple if* $\operatorname{char} k \neq 2$*.*

4 Some Representation Theory Basics

Definition 4.1. *A* ***representation*** *of a Lie algebra* $\mathfrak{g}$ *(over* k*) is a Lie algebra homomorphism* $\pi : \mathfrak{g} \to gl(V)$*,* V *some* k*-linear space. Also call* (π, V) *a* ***representation*** *of* $\mathfrak{g}$ *and call* V *a* $\mathfrak{g}$***-module****. The* ***dimension*** *of* $(\pi, V) = \dim(V)$*.*

Example 4.2.

- $\pi : \mathfrak{g} \to gl(V)$*, with* $V = \{0\}$*, the* ***zero representation****.*
- $\pi : \mathfrak{g} \to gl(V)$ *is called* ***trivial*** *if* $\pi(x) = 0 \in gl(V) \forall x \in \mathfrak{g}$*.*
- $adL\mathfrak{g} \to gl(\mathfrak{g}), x \mapsto ad(x) = [x, \cdot]$ *is called the* ***adjoint representation****.*
- $\mathfrak{g} = \mathfrak{sl}_2(k) = k\langle e, f, h | [h, e] = 2e, [h, f] = -2f, [e, f] = -h\rangle$*. Take* $n \in \mathbb{Z}_{\geq 0}$*. Choose basis* $v_0, v_1, ..., v_n$ *of* k^{n+1}*.* $\pi_n : \mathfrak{g} \to gl\left(k^{n+1}\right)$ *defined by* $e \cdot v_{r+1} = (r+1)(n-r)v_r, f \cdot v_r = v_{r+1}, e \cdot v_0 = f \cdot v_n = 0, v \cdot v_r = (n-2r)v_r$ *for* $r \in \{0, ..., n-1\}$*.*

Exercise 4.3. *Verify that the fourth bullet point is indeed a representation of* $\mathfrak{sl}_2(k)$*.*

Definition 4.4. *(ρ, W) is a **subrepresentation** of a given $\mathfrak{g}$-rep (π, V) if $W \subseteq V$ is a subspace and*

$$\rho(x) = \pi(x)|_W \forall x \in \mathfrak{g}, \text{ preserves } W.$$

Remark 4.5. *We also call W a $\mathfrak{g}$-submodule of V.*

Example 4.6.
- *The zero representation is a subrepresentation of any representation.*
- *The map (π, V) itself is a subrepresentation of (π, V).*

Definition 4.7. *If (π, V) is not the zero representation and it has no subrepresentations except the zero representation and itself, call (π, V) **irreducible** and call V **simple**.*

Example 4.8.
- *If V is a $\mathfrak{g}$-module, $\dim(V) = 1$, then V is simple.*
- *$(ad, \mathfrak{g})$ is irreducible if and only if $\mathfrak{g}$ is simple or $\dim(\mathfrak{g}) = 1$.*

Definition 4.9. *Let V be a $\mathfrak{g}$-module. Call $v \in V$ **cyclic** if every $\mathfrak{g}$-submodule containing $v \in V$ equals V. Call $v \in V$ **cocyclic** if v is contained in every nonzero submodule of V.*

Lemma 4.10. *Let V be a $\mathfrak{g}$-module. If $\exists v \in V$ cyclic and cocylic, then V is simple.*

Remark 4.11. *If V is simple, all nonzero elements are cyclic and cocyclic.*

Proof. Let $v \in V$ be cyclic and cocylic. Let W be any nonzero submodule $W \subset V$. Then $v \in W$ by cocyclicity. By cyclicity, $W = V$. □

Exercise 4.12. *Verify π_n is irreducible representation of $\mathfrak{sl}_2(k)$ if $char(k) = 0$.*

Lemma 4.13. *If $\dim(\mathfrak{g}) = \infty$ and $\mathfrak{g}$ is simple, and (π, V) is a $\mathfrak{g}$-representation with $\dim(V) < \infty$ then (π, V) is trivial.*

Proof. Use Rank-Nullity:

$$\dim(\text{Ker}(\pi)) + \dim(\text{im}(\pi)) = \dim(\mathfrak{g})$$

Since $\dim(\text{im}(\pi)) < \infty$ and $\dim(\mathfrak{g}) = \infty$, then $\ker(\pi)$ is ∞-dimensional ideal of $\mathfrak{g}$. Hence $\ker(\pi) = \mathfrak{g}$. □

Lemma 4.14. *If $\mathfrak{g}$ is any Lie algebra, and V is a nonzero $\mathfrak{g}$-module with $\dim(V) < \infty$, then V contains an irreducible submodule.*

Proof. By induction on $\dim(V)$.If $\dim(V) = 1$, then V is simple. Let $\dim(V) = n \in \mathbb{Z}_{>1}$, and assume all $\mathfrak{g}$-modules V' with $\dim(V')$ has simple submodules. If V is simple, we are done. Otherwise, choose any submodule V' with $0 < \dim(V') < n$. V' has a simple submodule U. Clearly, U is a submodule of V. □

Exercise 4.15. *Find a $\mathfrak{g}$-module V without any simple submodules.*

Hint: $\dim(\mathfrak{g}) = 1$. *Canonical solution:* $V = k[x]$, *action of some nonzero element of* $\mathfrak{g}$ *is multiplication by* x.

5 Infinite Dimensional Lie Algebras

We'll look at a simple example and some general/historical comments.

Now, let's discuss the Witt algebra

- An important example in its own right
- Not an example of an affine Lie algebra but plays an important role in representation theory
- Develop some intuition for ∞-dimensional Lie algebras

This is a real/complex Lie algebra that arises naturally in three ways:

1. Derivations of algebra $k[x, x^{-1}]$
2. Vector fields on circle
3. "Tangent space of identity" of a group of diffeomorphisms of a circle.

5.1 Derivations of Algebra

Describe $\mathrm{der}(k[z, z^{-1}])$ if $\mathrm{char}(k) = 0$. Let $D \in \mathrm{der}(k[z, z^{-1}])$. We have

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = D(1) + D(1) \implies D(1) = 0$$

and

$$D(z^n) = D(z \cdot z^{n-1}) = D(z)z^{n-1} + zD(z^{n-1})$$

for $n \in \mathbb{Z}_{>0}$. Repeatedly add to get $D(z^n) = nz^{n-1}D(z)$. Then,

$$0 = D(1) = D(z \cdot z^{-1}) = D(z)z^{-1} + zD(z^{-1}) \implies D(z^{-1}) = -z^{-2}D(z)$$

which gives for $n \in \mathbb{Z}_{>0}$,

$$D\left(z^{-n}\right) = ... = -z^{-n-1}D(z) + z^{-1}D\left(z^{1-n}\right)$$

so

$$D\left(z^n\right) = nz^{n-1}f(z) \qquad (\star)$$

for all $n \in \mathbb{Z}$, where $f(z) = D(z)$

Exercise 5.1. *Conversely, if D is linear and satisfies $(\star)$ for some $f(z) \in k[z, z^{-1}]$, show it's a derivation.*

$\{z^n\}_{n\in\mathbb{Z}}$ is a basis for $k[z,z^{-1}] \implies$

$$\mathrm{der}(k[z,z^{-1}]) = \{f(z)\frac{d}{dz}|f \in k[z,z^{-1}]\} = k[z,z^{-1}]\frac{d}{dz}$$

which gives $\dim(\mathrm{der}(k[z,z^{-1}])) = \infty$.

Convenienet basis of $k[z,z^{-1}]$: $\{-z^{n+1}\}_{n\in\mathbb{Z}}$. Set $Ln := -z^{n+1}\frac{d}{dz}$ for $n \in \mathbb{Z}$. Then this is a basis for $\mathrm{der}(k[z,z^{-1}])$!

What is the Lie algebra structure?

For $m,n \in \mathbb{Z}, p \in k[z,z^{-1}]$, then

$$\begin{aligned}(Lm \circ Ln)(p) &= z^{m+1}\frac{d}{dz}\left(z^{n+1}\frac{d}{dz}p(z)\right)\\ &= z^{m+1}\left((n+1)z^n p' + z^{n+1}p''\right)\\ &= (n+1)z^{m+n+1}p' + z^{m+n+2}p''\end{aligned}$$

which gives

$$\begin{aligned}[Lm,Ln](p) &= (n=1)z^{m+n+1}p' - (m+1)z^{m+n+1}p'\\ &= (n-m)z^{m+n+1}\frac{d}{dz}p\\ &= (m-n)Lm+n\end{aligned}$$

$\forall m,n \in X$.

5.2 Vector Fields On A Circle

Take $S^1 = \{e^{i\theta}|\theta \in \mathbb{R}/2\pi\mathbb{Z}\} \subseteq \mathbb{C}$ the 1-dimensional real-manifold (smooth). Then

$$\mathrm{VF}(S^1) = \{f\frac{d}{d\theta}|f \in C^\infty(S^1)\}$$

is the finite $\mathbb{R}$-linear (Fourier) combination of $\cos(n\theta), \sin(n\theta)$.

Complexification. Let V be any $\mathbb{R}$-linear space.

$$V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C} \cong V \oplus iV$$

where $i^2 = -1$. Then

$$\mathrm{VF}(S^1)_\mathbb{C} = \{f\frac{d}{dt}|t \text{ is a finite } \mathbb{C}-\text{linaer combination of } e^{in\theta}\}$$

for $n \in \mathbb{Z}$.

The convenient basis over $\mathbb{C}$ is $ie^{in}\frac{d}{d\theta} = -z^{n+1}\frac{d}{dz} = Ln$. Same relations for Ln ($n \in \mathbb{Z}$).

5.3 "Tangent Space At Identity"

Let $G = \text{Diff}_+(S^1) = \{\text{orientation-preserving diffeomorphisms of } S^1\}$. $\gamma \in G$ acts on $f \in C^\infty(S^1, \mathbb{C})$ by

$$(\gamma \cdot f)(z) = f\left(\gamma^{-1} z\right)$$

for $z \in S^1$. Take γ "close to identity":

$$\gamma(z) = z(1 + \epsilon(z))$$

where $\epsilon(z) \in \mathbb{C}^\infty(S^1, \mathbb{C})$ is small. Then

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n z^n$$

where $\epsilon_n = \epsilon\lambda_n, \epsilon$ small, and only finitely many λ_n nonzero. This allows us to write

$$\gamma^{-1}(z) = z - \epsilon \sum_n \lambda_n z^{n+1} + \mathcal{O}(\epsilon^2).$$

Exercise 5.2. *By using Fourier decomposition of f to show $(\gamma \cdot f)(z) = \left(1 + \epsilon \sum_{n \in \mathbb{Z}} \lambda_n Ln\right) f(z) + \mathcal{O}(\epsilon^2)$ where $Ln = -z^{n+1} \frac{d}{dz}$.*

This implies

$$\lim_{\epsilon \to 0} \frac{(\gamma\epsilon f)(z) - f(z)}{\epsilon} = \left(\sum_{n \in \mathbb{Z}} \lambda_n Ln \right) f(z)$$

"element in tangent space of id" = "general element of same Lie algebra".

6 Structure Theory

Define Witt $= \mathbb{C}\langle \{Lm\}_{m \in \mathbb{Z}} | [Lm, Ln] = (m - n)Lm + n \rangle$ for $m, n \in \mathbb{Z}$. The relations are quadratic to generators, so $\{Lm\}_{m \in \mathbb{Z}}$ is a spanning set. By studying $\pi : \text{Witt} \to \text{der}(k[z, z^{-1}])$ observe this is a basis.

Lemma 6.1. *Let $i \in$ Witt be an ideal, nonzero. Then $\exists n \in \mathbb{Z} : Ln \in i$.*

Proof. Let $0 \neq x \in \sum_{m \in M} c_m Lm, M \in \mathbb{Z}$ finite nonempty, $c_m \neq 0 \forall m \in M$.

Note: $[Lm, Lm']$ is nonzero if $m \neq n'$.

If $|M| = 1$, we are done. Otherwise, pick $m_1 \in M$. Then $[x_1, Lm_1] = \sum_{m \in M \setminus \{m_1\}} c_m (m - m_1) Lm + m_1$ where $c_m(m - m_1)$ is nonzero. Repeatedly do this will get a set of the element. □

Theorem 6.2. *Witt is simple.*

Remark 6.3. *No nontrivial finite dimensional representations, but $\exists$ interesting ∞-dimensional representations.*

Proof. It suffices to show $Lm \in i \forall m \in \mathbb{Z}$. By previous lemma, $\exists n \in \mathbb{Z}, Ln \in i$. First show $L0 \in i$. If $n = 0$, done. Otherwise $L0 = \frac{1}{2n}[Ln, L-n] \in i$. Now $Lm \in i$ (any m). If $m = 0$, done. $Lm = \frac{1}{m}[Lm, L0] \in i$ for $m \neq 0$. $\square$

Two fun facts about Witt.

Exercise 6.4. *Show Witt is finitely generated ($\exists$ set $S \subset$ Witt such that $\langle S \rangle =$ Witt, $|S| < \infty$). Find S with $|S|$ minimal.*

Exercise 6.5. *Show Witt is isomorphic to a proper Lie subalgebra.*

Remark 6.6. *If $k = \mathbb{F}_p$ with p prime, Witt (1930s) studied der$(k[z]/(z^p)) = \bigoplus_{m=-1}^{p-2} kLm$ where $Lm = -z^{m+1}\frac{d}{dz}$.*

The Witt algebra appears in the rep theory of loop algebras, especially in central extensions (Virasoro, affine Lie algebras, etc.). We'll give an overview of important families of ∞-dimensional Lie algebras.

1. ∞-dimensional Lie algebras of vector fields on finite dimensional spaces (including Witt).

This was originally introduced by Cartan in 1907, but mostly forgotten for a couple decades. Then in 1960, mathematicians such as Guillemin, Quillen, Singer, Sternberg, Weisfeiler revived this area of math through algebraic approaches, with the key concept being filtered/graded Lie algebras.

1. Lie algebras of smooth maps form a manifold to a finite dimensional Lie algbebra (loop algebras).

2. Lie algebra of operators on Banach spaces (motivated by quantum field theory)

3. Kac-Moody Algebras, such as (extended) affine Lie algebras.

This was introduced by Kac and Moody independently in '68 - we'll follow Kac's approach which was done to generalize 1960s work on (1) towards arbitrary $\mathbb{Z}$-graded Lie algebras $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ with $\dim(\mathfrak{g}_j) < \infty$ and $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$. Kac realized it would be really nice if $\dim(\mathfrak{g}_j) <$ (polynomial in j).

The topic for the first half of the course are these affine Lie algebras $\hat{g}$. We'll discuss the relationship between $L\mathfrak{g} \twoheadleftarrow \hat{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}}$, the loop algebra, affine Lie algebra, and the Kac-Moody algebra, where the first arrow is a central extension and the second is a extension by derivation.

7 Loop Algebras

Let $\mathfrak{g}$ be a Lie algebra over k. Let R be a commutative, associative k-algebra. Then $\mathfrak{g} \otimes_k R$ is naturally a Lie algebra with

$$[x \otimes r, y \otimes s]_{\mathfrak{g} \otimes R} := [x, y]_{\mathfrak{g}} \otimes rs$$

where $x, y \in \mathfrak{g}, r, s \in R$.

Exercise 7.1. *Show indeed this defines a Lie algebra structure on* $\mathfrak{g} \otimes R$.

Remark 7.2.

1. *If* V *is a* $\mathfrak{g}$*-module, then* $V \otimes_k R$ *is a module over* $\mathfrak{g} \otimes_k R$.
2. *If* R *is unital (commutative ring containing* k*), then we have the Lie algebra embedding* $\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes R, x \mapsto x \otimes 1$.

Definition 7.3. *The* ***loop algebra*** *of* $\mathfrak{g}$ *is the Lie algebra*

$$L\mathfrak{g} := \mathfrak{g} \otimes_k k\left[z, z^{-1}\right]$$

Remark 7.4. *If* $k = \mathbb{C}$*, we can view elements of* $L\mathfrak{g}$ *as Lie-algebra valued small functions on* S^1*. In fact,*

$$\sum_{r \in \mathbb{Z}} x_r \otimes z^r \Longleftrightarrow map\left(\theta \to \sum_{r \in \mathbb{Z}} e^{2\pi i r \theta} x_r\right)$$

with $x_r \otimes z^r \in L\mathfrak{g}$ *and* $\theta \in S^1$.

Notation: $x^{(m)} := x \otimes z^m$ for $x \in \mathfrak{g}, m \in \mathbb{Z}$. Then the defining the relations for $L\mathfrak{g}$: $\left[x^{(m)}, y^{(n)}\right] = [x, y]^{(m+n)}$ where $x, y \in \mathfrak{g}, m, n \in \mathbb{Z}$.

Example 7.5. *For* $\mathfrak{sl}_2(k)$,

$$\mathfrak{g} = k\langle e, f, h | [h, e] = 2e, [h, f] = -2f, [e, f] = h\rangle$$

so the relations are

$$\left[h^{(m)}, e^{(n)}\right] = 2e^{m+n}, \left[h^{(m)}, f^{(n)}\right] = -2f^{m+n}, \left[e^{(m)}, f^{(m)}\right] = h^{m+n}.$$

and

$$\left[e^{(m)}, e^{(n)}\right] = \left[f^{(m)}, f^{(n)}\right] = \left[h^{(m)}, h^{(n)}\right] = 0$$

for $m, n \in \mathbb{Z}$.

Exercise 7.6. *If char* $\neq 2$*, show* $L\mathfrak{g}$ *(*$\mathfrak{g} = \mathfrak{sl}_2$*) is generated by* $\{e^{(0)}, f^{(0)}, e(-1), f(1)\}$.

Proposition 7.7. $\dim(L\mathfrak{g}) = \infty$.

Proposition 7.8. $L\mathfrak{g}$ *is not simple.*

Proof. Let $p(z) \in k[z]$ nonzero monic polynomial with $z \nmid p(z)$ and consider the ideal $(p(z)) \subset k[z, z^{-1}]$. Then $\mathfrak{g} \otimes_k (p(z))$ is an ideal of $L\mathfrak{g}$ of codimension $(\deg(p)) \times (\dim(\mathfrak{g}))$. □

Remark 7.9. *If* $\mathfrak{g}$ *is simple,* $\dim(\mathfrak{g}) < \infty$*, all nonzero ideals of* $L\mathfrak{g}$ *are of this form. See [Kac, Lemma 8.6]* $\implies$ *all proper quotients of* $L\mathfrak{g}$ *are finite-dimensional. The existence of these ideals in* $L\mathfrak{g}$ *of finite codimension allows for existence of nontrivial finite dimensional reps.*

Definition 7.10. *Fix $a \in k^\times$. Then the **evaluation map** is defined as*

$$\begin{aligned} ev_a : L\mathfrak{g} &\to \mathfrak{g} \\ x \otimes p(z) &\mapsto p(a)x \end{aligned}$$

for $x \in \mathfrak{g}, p(z) \in k\left[z, z^{-1}\right]$.

Note: $\ker(\mathrm{ev}_a) = g \otimes (z - a)$. Let $\pi : \mathfrak{g} \to \mathrm{gl}(V)$ by a $\mathfrak{g}$-rep.

Definition 7.11. *The **evaluation rep** is*

$$\pi_a = \pi \circ ev_a : L\mathfrak{g} \to gl(V)$$

More generally, for $(a_1, ..., a_\ell) \in (k^\times)^\ell$, define

$$\begin{aligned} ev_{(a_1,...,a_\ell)} : L\mathfrak{g} &\to g^{\oplus l} (= \mathfrak{g} \oplus \mathfrak{g} ... \oplus \mathfrak{g}) \\ x \otimes p(z) &\mapsto (p(a_1)x, p(a_2)x, ..., p(a_\ell)x) \end{aligned}$$

Remark 7.12. *This is surjective if and only if $\dim(\mathfrak{g}) = 1$ or $\ell = 1$.*

Proposition 7.13. *If $(\pi_1, V_1), ..., (\pi_ell, V_\ell)$ are $\mathfrak{g}$-reps, we can construct a new rep $(\pi_1 \otimes ... \otimes \pi_\ell, V_1 \otimes ... \otimes V_\ell)$, where*

$$\begin{aligned} \pi_1 \otimes ... \otimes \pi_ell : \mathfrak{g}^{\oplus \ell} &\to gl(V_1 \otimes ... \otimes V_\ell) \\ (x_1, ..., x_\ell) &\mapsto \sum_{i=1}^{\ell} Id_{V_1} \otimes ... \otimes Id_{V_i - 1} \otimes \pi_i(x_i) \otimes Id_{V_i+1} \otimes ... \otimes Id_{V_\ell} \end{aligned}$$

Now, compose with $\mathrm{ev}_{(a_1,...,a_\ell)}$, we get the evaluation rep $L\mathfrak{g} \to \mathrm{gl}(V_1 \otimes ... \otimes V_\ell)$

$$(\pi_{1,a_1} \otimes ... \otimes \pi_{\ell,a_\ell}, V_1 \otimes ... \otimes V_\ell) := (\pi_1 \otimes ... \otimes \pi_\ell) \circ \mathrm{ev}_{a_1,...,a_\ell}.$$

Exercise 7.14. *Prove this is a rep of $L\mathfrak{g}$.*

Example 7.15. *Take $\mathfrak{g} = \mathfrak{sl}_2(k)$, $\pi_i = \pi^{(1)} : \mathfrak{g} \to \mathfrak{gl}(k^2)$ and*

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $\ell = 2$,

$$\pi^{(1)} \otimes \pi^{(1)} : (x_1, x_2) \mapsto \pi^{(1)}(x_1) \otimes Id_V + Id_V \otimes \pi^{(2)}(x_2)$$

for $a_1, a_2 \in k^\times$, eg.

$$\left(\pi^{(1)}_{a_1} \otimes \pi^{(2)}_{a_2}\right) = a_1^m \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes Id_V + a_2^m Id_V \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for $m \in \mathbb{Z}$, and similarly for f and h.

Exercise 7.16. *Take char*$(k) = 0, \mathfrak{g} = \mathfrak{sl}_2(k)$*, fix* $a_1, a_2 \in k^\times$*. Show that* $\pi_{a_1}^{(1)} \otimes \pi_{a_2}^{(1)}$ *os an irreducible* $L\mathfrak{g}$*-rep if and only if* $a_1 \neq a_2$*. Hint: consider the vector* $\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$*. If* $a_1 = a_2$*, show this is a 3-dimensional submodule and 1-dimensional submodule.*

Now, let's discuss the connection with Witt as a derivation of $L\mathfrak{g}$. We have

$$Lm\left(x^{(r)}\right) = -x^{m+r} \qquad (\star)$$

for $x \in \mathfrak{g}, r \in \mathbb{Z}$. This implies the "semidirect sum" of Lie algebras: Witt $\ltimes L\mathfrak{g}$.

8 Variations of Loop Algebras

Now, let's discuss variations of loop algebras. See [@senesi] for more.

Take $k = \mathbb{C}, \mathfrak{g}$ finite dimensional and simple. Choose outer automorphism σ of $\mathfrak{g}$ of order r. Lift σ to a map on $L\mathfrak{g}$:

$$\overline{\sigma}(x \otimes z^m) = Z^{-m}\sigma(x) \otimes z^m$$

where $x \otimes z^m \in L\mathfrak{g}$ and Z is a dual primitive r-th root of unity. $(L\mathfrak{g})^{\overline{\sigma}}$ is a realization of a twisted loop algebra. See [@senesi], [@kac, chapter 8], or [@carter, chapter 18] for more information.

Here are some variations

- Multi-loop algebra: $\mathfrak{g} \otimes k[z_1^{\pm 1}, z_2^{\pm 2}, ..., z_n^{\pm 1}]$
- Current Lie algebra $g \otimes k[z]$, see [Makedonskyi's BIMSA Spring 2023 course](https://bimsa.net/activity/repliealg/)
- Loop group. Roughly, $L\mathfrak{g} = \text{Lie}(LG)$ for some ∞-dimensional. We can think of this as smooth maps $S^1 \to G$, with $\mathfrak{g} = \text{Lie}(G)$ and G a simple Lie group.

9 More General Representation Theory

Let $(\pi_1, V_1), (\pi_2, V_2)$ be $\mathfrak{g}$-reps, $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ the direct sum of k-linear spaces. Then $(\pi_1 \oplus \pi_2)(x) = (\pi_1(x), 0) + (0, \pi_2(x)) \forall x \in \mathfrak{g}$ and $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$.

Definition 9.1. *A* $\mathfrak{g}$*-rep* (π, U) *is* ***indecomposable*** *if it is not a direct sum of nonzero submodules* $(\dim(V) > 0)$*.*

Irreducible (no nonzero submodules) $\implies$ indecomposable.

Definition 9.2. *If* $\mathfrak{g}$ *is a Lie algebra such that all indecomposable finite dimensional* $\mathfrak{g}$*-reps are irreducible, then all its finite dimensional reps are* ***completely***

reducible *(or* ***semisimple****), ie. can be written as a direct sum of irreducible* $\mathfrak{g}$*-reps*

Exercise 9.3. *Prove it (hint: induction)*

Example 9.4.
- *Abelian Lie algebra. Irreducibles are 1-dim, all finite-dimensional reps are direct sum of irreducibles.*
- *Finite dimensional semisimple Lie algebras (review for* $\mathfrak{sl}_2$ *for later)*

Counterexample: the subalgebra $f = \langle h, e\rangle \subset \mathfrak{sl}_2(k)$ has 2-dim $\mathfrak{sl}_2$-rep that restricts to a finite f-rep which is reducible bt indecomposable:

$$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{e} \begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{e} \begin{pmatrix}0\\0\end{pmatrix}.$$

$h \circlearrowleft \begin{pmatrix}1\\0\end{pmatrix}$ is a submodule but $h \circlearrowleft \begin{pmatrix}0\\1\end{pmatrix}$ is not.

Lemma 9.5. *Let* $\varphi : \mathfrak{h} \to \mathfrak{g}$ *be a surjective Lie algebra homomorphism. Let* $\pi : \mathfrak{g} \to gl(V)$ *be a rep.*

1. $\pi \circ \varphi$ *is irreducible if and only if* π *is irreducible.*
2. $\pi \circ \varphi$ *is indecomposable if and only if* π *is indecomposable.*

Proof. Suppose $\exists y \in \mathfrak{h}$: $\pi(\varphi(y))(W) \subsetneq W$. Let $W \subset V$ be a nonzero $\mathfrak{h}$-submodule. $\forall y \in \mathfrak{h}, \pi(\varphi(y))(W) \subseteq W$. Suppose $\exists x \in \mathfrak{g}$ with $\pi(x)(W) \subseteq W$ for some $y \in \mathfrak{g}$. This is a contradiction, and we are done. □

Example 9.6. *If* (π, U) *is an irreducible* $\mathfrak{g}$*-module and* $a \in k^\times, (\pi_a, U)$ *is an irreducible* $L\mathfrak{g}$ *is an irreducible* $L\mathfrak{g}$*-module.*

Not all finite-dimensional $L\mathfrak{g}$-reps are evaluation reps. "Canonical counterexample" is reducible and indecomposable: let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$, $L\mathfrak{g} = g \otimes \mathbb{Z}[z, z^{-1}]$. Let $R = \mathbb{C}[\epsilon]/(\epsilon^2) = \mathbb{C}\overline{1} \oplus \mathbb{C}\overline{\epsilon}$. Then $- : \mathbb{C}[\epsilon] \twoheadrightarrow R$ is a surjective map. Note $\overline{1} + \overline{\epsilon}$ is invertible: $(\overline{1} + \overline{\epsilon}) + (\overline{1} - \overline{\epsilon}) = \overline{1}$. Surjective unital alg-homomorphism $\mathbb{C}[z, z^{-1}] \to R, z \mapsto \overline{1} + \overline{\epsilon}$ implies surjective Lie algebra homomorphism $\varphi : L\mathfrak{g} \twoheadrightarrow \mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R, x \otimes z^m \mapsto x \otimes (\overline{1} + \overline{\epsilon})^m$ for $x \in \mathfrak{g}, m \in \mathbb{Z}$. Let V be any irreducible $\mathfrak{g}$-module ($1 \leq \dim(V) < \infty$) (ie. adjoint rep). Then $\mathfrak{g}_R$ action on $V_R = V \otimes_{\mathbb{C}} R$ is given by: for $x, y \in \mathfrak{g}, v_1, v_2 \in V$,

$$(x + y\overline{\epsilon}) \cdot (v_1 + v_2\overline{\epsilon}) = (x \cdot v_1) + (x \cdot v_2 + y \cdot v_1)\overline{\epsilon}.$$

Clearly, $V\overline{\epsilon}$ is a nonzero proper submodule of V_R. We will show that V_R is indecomposable.

Proposition 9.7. *Every nonzero submodule* $W \subseteq V_R$ *contains a nonzero element of* $V\overline{\epsilon}$*.*

Proof. Let $v_1+v_2\overline{\epsilon} \in W$ be arbitrary. WLOG $v_1 \neq 0$. Suppose $\forall y \in \mathfrak{g}$, $y \cdot v_1 = 0$. Then $\mathbb{C}v_1$ is a 1-dim $\mathfrak{g}$-submodule of V (V irreducible, $\dim(V) > 1$). Hence $\exists y \in \mathfrak{g}$ and $y \cdot v_1 \neq 0$. Then $y\overline{\epsilon} \in \mathfrak{g}_R$ sends $v_1 + v_2\overline{\epsilon}$ to $(y \cdot v_1)\overline{\epsilon} \in V\overline{\epsilon}\backslash\{0\}$. □

Proposition 9.8. *Every nonzero* $\mathfrak{g}_R$ *submodule* $W \subset V_R$ *contains* $V\overline{\epsilon}$*.*

Proof. Let $v\overline{\epsilon} \in W$ for $v \neq 0$. Consider $\mathfrak{g}$-submodule of V generated by v. It is equal to V. Inclusion of Lie algebra $\mathfrak{g} \hookrightarrow \mathfrak{g}_R, x \mapsto x + 0\overline{\epsilon}$. Hence $V\overline{\epsilon} =$ ($\mathfrak{g}$-submodule of V generated by v)$\overline{\epsilon} \subseteq$ ($\mathfrak{g}_R$-submodule of V generated by v)$\overline{\epsilon} \subseteq V_R$. Suppose $V_R = U_1 \oplus U_2$, with U_1, U_2 nonzero $\mathfrak{g}_R$-submodules. Both contain $V\overline{\epsilon}$, so $U_1 \cap U_2 \neq \{0\}$. □

Lemma 9.9. *$L\mathfrak{g}$ has finite dimensional reducible indecomposable modules.*

Definition 9.10. *A* $\mathfrak{g}$***-intertwiner*** *(or* $\mathfrak{g}$*-module homomorphism, or* $\mathfrak{g}$*-equivariant map) between two* $\mathfrak{g}$*-reps* $(\pi, V), (\rho, W)$ *is a* k*-linear map* $f : V \to W$ *such that* $\forall x \in \mathfrak{g}, \forall v \in V$,

$$f(\pi(x)v) = \rho(x)(f(v)).$$

We will write $f : (\pi, V) \to (\rho, W)$.

Exercise 9.11. *Prove that the* k*-linear combinations of intertwiners* $(\pi, V) \to (\rho, W)$ *are again intertwiners. If* $(\pi, V) = (\rho, W)$*, ten show that the composition of intertwiners is an intertwiner.*

Definition 9.12. $(\pi, V), (\rho, W)$ *are* ***isomorphic*** *if* $\exists$ *invertible intertwiner* $f : (\pi, V) \to (\rho, W)$.

Remark 9.13. *It's easy to see that the inverse is also an intertwiner.*

Lemma 9.14 (Schur's Lemma)**.** *Let* $f : (\pi, V) \to (\rho, W)$ *be an intertwiner between irreducible* $\mathfrak{g}$*-representations. Then* $f = 0$ *or* f *is an isomorphism (in particular* $\dim(V), \dim(W) \geq 1$*).*

Proof. The main claim is that $\ker f \subseteq V$ and $\mathrm{im}(f \subseteq W)$ are $\mathfrak{g}$-submodules.

By irreducibility and rank-nullity, either:

- $\ker f = \{0\}$ and $\mathrm{im} f = W \implies f$ is an isomorphism
- $\mathrm{Ker} f = V$ and $\mathrm{im} f = \{0\} \implies f = 0$.

□

Now, we'll add the condition that $\overline{k} = k$ is algebraically closed and $\dim V, \dim W < \infty$.

Remark 9.15. *Dixmier's lemma can weaken this to* $\dim V, \dim W < |k|$.

Proposition 9.16. *Suppose that in addition* $(\pi, V) = (\pi, W)$*. Then* $f = \lambda id_V$ *for* $\lambda \in k$*.*

Proof. Because $f : V \to V$ is k-linear, $\exists$ eigenvalue $\lambda \in k$, eigenvector $v \in V\backslash\{0\}$. Consider $f' = f - \lambda \mathrm{id}_V$. The intertwiner satisfies $f'(v) = 0$. $\ker f' \subset V$ submodule is not zero, so $f' = 0$. □

Proposition 9.17. *Instead, suppose have two intertwiners $f, g : (\pi, V) \to (\rho, W)$. Then $f = \lambda g$ for some $\lambda \in k^\times$ or one of f, g is 0.*

Proof. Assume both f, g are nonzero. By main statement, f, g are isomorphisms, so $g^{-1} \circ f$ is a $\mathfrak{g}$-intertwiner from (π, V) to itself. Use the previous proposition to deduce $g^{-1} \circ f = \lambda \mathrm{id}_V$. $\lambda = 0$ is impossible. □

Corollary 9.18. *Assume $\overline{k} = k, \dim V < \infty$. Let $c \in Z(\mathfrak{g})$. Let (π, V) be an irreducible $\mathfrak{g}$-rep. Because $[c, y] = 0 \forall y \in \mathfrak{g}, \pi(c)$ is an intertwiner: $(\pi, V) \to (\pi, V)$, Hence $\pi(c) = \lambda Id_V \forall \lambda \in k$. In particular, if $\mathfrak{g}$ is abelian, it follows that $\dim V = 1$.*

10 Universal Enveloping Algebras

See [@carter chapter 9] for proofs.

Fix a field k (typically $k = \mathbb{C}$). Given: Lie algebra $\mathfrak{g}$ over k. We can constuct ∞-dimensional unital associative algebra $\mathcal{U}(\mathfrak{g})$ with the same representation theory of $\mathfrak{g}$.

Define the k-linear space

$$\mathfrak{g}^{\otimes \ell} = \mathfrak{g} \otimes ... \otimes \mathfrak{g}$$

with ℓ tensors for $\ell \in \mathbb{Z}^{\geq 0}$.

Definition 10.1. *The **direct sum** is*

$$T(\mathfrak{g}) = \bigoplus_{\ell \geq 0} \mathfrak{g}^{\otimes \ell}$$

where elements are finite linear combinations of tensor products of elements of $\mathfrak{g}$ of finite length.

Define the product $T(\mathfrak{g}) \times T(\mathfrak{g}) \to T(\mathfrak{g})$ by $\otimes$:

$$(x_1 \otimes ... \otimes x_m) \cdot (y_1 \otimes ... \otimes y_n) = x_1 \otimes ... \otimes x_m \otimes y_1 \otimes ... \otimes y_n$$

and extend bilinearly to get an associative unital algebra with unit $1 \in g^{\otimes 0} = k$. To get an associative algebra with the correct multiplication (compatible with $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$), let $J(y)$ be the 2-sided ideal of $T(\mathfrak{g})$ generated by elements of the form

$$x \otimes y - y \otimes x - [x, y] \forall x, y \in \mathfrak{g}.$$

Definition 10.2. *The* ***universal enveloping algebra***

$$\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/J(\mathfrak{g})$$

is a unital associative algebra (note $1 \in \mathcal{U}(\mathfrak{g})$*).*

Example 10.3. $\mathfrak{g}$ *is abelian,* $\dim \mathfrak{g} = n$*, and*

$$\mathfrak{g} = k\langle \{x_1, ..., x_n\} | [X_i, X_j] = 0 \rangle.$$

$J(\mathfrak{g})$ *is generated by all elements of the form* $x_i \otimes x_j - x_j \otimes x_i$ *for* $1 \leq i < j \leq n$*.* $\mathcal{U}(\mathfrak{g})$ *is commutative and generated (as an unital algebra) by images of* $x_1, ..., x_n$*. This implies*

$$\mathcal{U}(\mathfrak{g}) \cong k[X_1, ..., X_n].$$

We have linear maps $\mathfrak{g} \to T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$. Composition is denoted ι. $\mathcal{U}(\mathfrak{g})$ is generated by $\iota(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$. For better notation, the elements of $\mathfrak{g}$ are $x, y, ...$ and the elements of $\mathcal{U}(\mathfrak{g})$ are $X, Y, ...$ with $X = \iota(x), Y = \iota(y)$.

Theorem 10.4 (Universal Property of UEAs)**.** *Let A be any unital associative k-algebra (Lie algebra with $[x, y] = xy - yx$). Let $\varphi : \mathfrak{g} \to A$ be a Lie algebra homomorphism. Then $\exists!\ \Phi : \mathcal{U}(\mathfrak{g}) \to A$ (hom of unital algebras) such that $\Phi \circ \iota = \varphi$.*

Proof. Here's a sketch.

1. Extend $\varphi : \mathfrak{g} \to A$ to a homomorphism of unital algebras $\varphi : T(\mathfrak{g}) \to A, x_1 \otimes ... \otimes x_\ell \mapsto \varphi(x_1)\varphi(x_2)...\varphi(x_\ell)$ for $x_1, ..., x_\ell \in \mathfrak{g}$.
2. Check $\varphi(x \otimes y - y \otimes x - [x, y]) = 0 \forall x, y \in \mathfrak{g}$ so $J(\mathfrak{g}) \subseteq \ker\varphi$. Get unital algebra homomorphism $\Phi : T(\mathfrak{g})/J(\mathfrak{g}) \to A$ such that

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Check, by restricting φ to $\mathfrak{g}$ that indeed $\Phi \circ \iota = \varphi$.

1. To show uniqueness, take second homomorphism $\Phi' : \mathcal{U}(\mathfrak{g}) \to A$ with same properties. CAn show it coincides with Φ on $\iota(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$.

□

Exercise 10.5. *Show that $\mathcal{U}(\mathfrak{g})$ is the unique (up to isomorphism) algebra with this universal property for $\mathfrak{g}$.*

Apply universal property to $A = \mathrm{End}(V)$ where V is any linear space. This gives a Lie algebra rep $\pi : \mathfrak{g} \to \mathrm{gl}(V)$ that extends uniquely to a rep of associative algebra $\Pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathfrak{g})$ and $\pi = \Pi \circ \iota$. This shows

$$\{\text{reps of } \mathfrak{g}\} \overset{1:1}{\Longleftrightarrow} \{\text{reps of } \mathcal{U}(\mathfrak{g})\}$$
$$(\pi, V) \mapsto (\Pi, V)$$

The basis of $\mathfrak{g}$ gives a natural basis of $\mathcal{U}(\mathfrak{g})$.

Theorem 10.6 (Poincaré-Birkhoff-Witt)**.** *Let $\{x_i | i \in I\}$ be a basis for $\mathfrak{g}$, and choose a total order $<$ on I. Let $X_i := \iota(x_i) \in \mathcal{U}(\mathfrak{g})$. The elements $X_{i_1}, X_{i_2}, ..., X_{i_\ell}$ where $\ell \in \mathbb{Z}_{\geq 0}$ with $i_1 \leq i_2 \leq ... \leq i_\ell$ form a basis for $\mathcal{U}(\mathfrak{g})$.*

Example 10.7. *For $\mathfrak{sl}_2(\mathbb{C})$,*

$$\mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}\langle E, F, H | HE - EH = 2E, HF - FH = -2F, EF - FE = H\rangle.$$

Choose basis (f, h, e) of $\mathfrak{sl}_2(\mathbb{C})$, ie. $f < h < e$ gives

$$\mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) = \bigoplus_{r,s,t \geq 0} \mathbb{C}F^r H^s E^t.$$

Proof. Let's start by proving that these monimials span $\mathcal{U}(\mathfrak{g})$.

1. Set of products $x_{j_1} \otimes ... \otimes x_{j_\ell}$ for $\ell \in \mathbb{Z}_{\geq 0}, j_1, ..., j_\ell \in I$ span $T(\mathfrak{g})$. Quotient out $J(\mathfrak{g})$, the images $X_{j_1} \cdot X_{j_2} \cdot ... \cdot X_{j_\ell}$ span $\mathcal{U}(\mathfrak{g})$.
2. Show every $X_{j_1} \cdot ... \cdot X_{j_\ell}$ is a linear combination of $X_{i_1} \cdot ... \cdot X_{i_n}$ with $i_1 \leq ... \leq i_n \in I$. By induction on ℓ_1, use $X_i X_j = X_j X_i + \iota([x_i, x_j])$.

Now let's show linear independence. The idea is to construct a linear map $\theta : T(\mathfrak{g}) \to k[\{z_i\}_{i \in I}]$ with

$$\theta(x_{i_1} \otimes ... \otimes x_{i_n}) = z_{i_1} ... z_{i_n}$$

if $i_1 \leq ... \leq i_n$ and θ is such that $J(\mathfrak{g}) \subset \ker\theta$. See [@carter lemma 9.5]. Use linear independence of order monomials $k[\{z_i\}_{i \in I}]$ to deduce linear independence of $\{X_{i_1} \cdot ... \cdot X_{i_n} | i_1 \leq ... \leq i_n\} \subset \mathcal{U}(\mathfrak{g})$. □

Corollary 10.8. *$\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective (because $\{X_i\}_{i \in I} \subset \mathcal{U}(\mathfrak{g})$ is linearly independent). Hence $\iota(\mathfrak{g}) \subseteq \mathcal{U}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$. We will view $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$.*

Exercise 10.9. *If v is a cyclic vector of a $\mathfrak{g}$-module V, then $V = \mathcal{U}(\mathfrak{g}) \cdot v = \{X \cdot v | X \in \mathcal{U}(\mathfrak{g})\}$.*

Exercise 10.10. *Take $\mathfrak{g}_1, \mathfrak{g}_2$ Lie algebras over k. $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie algebra with $[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2])$. Show $\mathcal{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2)$.*

11 $\mathfrak{sl}_2$ Rep Theory Over $\mathbb{C}$

To describe the finite dimensional irreducible $\mathfrak{sl}_2$-reps, we will use the approach of Verma modules.

11.1 Step 1: Highest Weight Modules

Claim: Let (π, V) be a finite dimensional $\mathfrak{sl}_2$-rep. Then $\exists v \in V \backslash \{0\}, \exists \lambda$ such that $E \cdot v = 0, H \cdot v = \lambda v$. This is the **highest weight vector**, denoted v_λ with **weight** λ. Such a module V is called a **highest weight module**.

Theorem 11.1 (Primary Decomposition Theorem)**.** *If $\varphi : V \to V$ is a linear map with $\dim V < \infty$ and distinct eigenvalues of φ are $\mu_1, ..., \mu_r$ for $1 \leq r < \infty$, then*

$$V = \bigoplus_{i=1}^{r} V_{\mu_i}^{gen}$$

where $V_{\mu_i}^{gen} = \{v \in V | \exists m \in \mathbb{Z}_{>0} (\varphi - \mu_i Id)^m(v) = 0\}$.

Exercise 11.2. *Use $HE - EH = 2E$ to show that $E \cdot V_{\mu}^{gen} \subseteq V_{\mu+2}^{gen}$.*

Since $\{\mu + 2i\}_{i \in \mathbb{Z}_{\geq 0}}$ are distinct elements (char 0) and $\dim V < \infty$, $\exists \lambda \in \mathbb{C}$ such that $E \cdot V_{\lambda}^{\text{gen}} = 0$ and $V_{\lambda}^{\text{gen}} = \{0\}$, so $\exists v \in V_{\lambda}^{\text{gen}}$ such that $H \cdot v = \lambda v$.

Remark 11.3. *If V is spanned by weight vectors, it is a weight module (assuming axiom of choice).*

Example 11.4. *The reps $\pi_n : \mathfrak{sl}_2 \to \mathfrak{gl}(k^{n+1})$ because $h \cdot v_r = (n - 2r)v_r$ and $k^{n+1} = \bigoplus_{r=0}^{n} kv_r$.*

Exercise 11.5. *Show that the map ρ defined by*

$$e \mapsto x\frac{\partial}{\partial y}, \quad f \mapsto y\frac{\partial}{\partial x}, \quad h \mapsto x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

gives a representation of $\mathfrak{sl}_2$ on $V = \{$holomorphic functions $\mathbb{C} \times \mathbb{C} \to \mathbb{C}\}$. Show that the submodule

$$V_n = span_{\mathbb{C}}\{(x, y) \mapsto x^k y^l \mid k + l = n, \ k, l \in \mathbb{Z}_{\geq 0}\}$$

is isomorphic to (π_n, k^{n+1}).

Example 11.6. *Let W be ∞-dimensional submodule of V of polynomial functions. Spanned by $x^l y^m$, which are weight vectors of weight $l - m$.*

Example 11.7. *Here's a non-example. Consider the submodule*

$$We^{x+y} = \{(x, y) \mapsto p(x, y)e^{x+y} | p(x, y) \text{ polynomial}\}.$$

h acts as $h \cdot (p(x, y)e^{x+y}) = ((h + x - y) \cdot yp(x, y))e^{x+y}$. h preserves total degree of monomial, so the eigenfunction of h-action $\iff$ $(y - x)p(x, y) = h \cdot p(x, y) + \mu p(x, y)$ for $\mu \in \mathbb{C}$. This is not a weight module.

Exercise 11.8. *Let V be a weight module. Show that any submodule and any quotient module of V is a weight module.*

*Hint: Useful to define the **support** of v as follows: let $v = \sum_{\mu \in \mathbb{C}} v_\mu$ with $v_\mu \in V_|mu$. Then $Supp(v) := \{\mu \in \mathbb{C} | v_\mu \neq 0\}$, which is a finite set. Do induction with respect to $|Supp(v)|$ for $v \in W, W \subseteq V$ submodule.*

Definition 11.9. *Let V be an $\mathfrak{sl}_2$-module. Suppose $\exists v \in V \backslash \{0\}$ such that $E \cdot v = 0$ and $H \cdot v = \lambda v$ for some $\lambda \in \mathbb{C}$. Then v is called the **highest weight vector** with **highest weight** λ. And if V is generated by a highest weight vector v (so $V = \mathcal{U}(\mathfrak{sl}_2) \cdot v$), then V is called a **highest weight module**.*

Example 11.10. (π_n, k^{n+1}) *are highest weight modules, generated by a highest weight vector* v_0 *with weight* n *because* $e \cdot v_0 = 0, h \cdot v_0 = nv_0$ *for* $n \in \mathbb{Z}_{\geq 0}$. *All weights of this module has* $\{n, n-2, ..., 2-n, -n\}$.

Lemma 11.11. *Let* V *be a highest weight module. Then* V *is a weight module.*

Proof. Let v_λ be a highest weight vector. Then

$$\begin{aligned} V &= \mathcal{U}(\mathfrak{sl}_2) \cdot v_\lambda \\ &= \sum_{r,s,t \geq 0} c_{r,s,t} F^r H^s E^t \cdot v_\lambda \\ &= \sum_{r \geq 0} \tilde{c_r} F^r \cdot v_\lambda \end{aligned}$$

for $c_{r,s,t}, \tilde{c_r} \in \mathbb{C}$. $F^r \cdot v_\lambda$ is a weight vector of weight $\lambda - 2r$, a consequence of $[H, F] = -2F$. □

Earlier, we used the generalized eigencomposition of H-action on V ($\dim(V) < \infty$) to see that V contained a highest weight vector v_λ of weight λ. If V is irreducible, then $V = \mathcal{U}(\mathfrak{sl}_2) \cdot v_\lambda = \mathcal{U}(\langle f \rangle) \cdot v_\lambda$.

11.2 Step 2: Verma Modules

Let

$$U = \mathcal{U}(\mathfrak{sl}_2) = \bigoplus_{r,s,t \in \mathbb{Z}_{\geq 0}} \mathbb{C} F^r H^s E^t = \bigoplus_{r,s,t \in \mathbb{Z}_{\geq 0}} \mathbb{C} F^r (H - \lambda)^s E^t.$$

Fix $\lambda \in \mathbb{C}$. Let $I(\lambda) \subset U$ be the left ideal generated by $H - \lambda$ and E.

$$I(\lambda) = U \cdot (H - \lambda) + U \cdot E = \bigoplus_{r,s,t \in \mathbb{Z}_{\geq 0}, s > 0 \text{ or } t > 0} \mathbb{C} F^r (H - \lambda)^s E^t.$$

By definition it is a left U-module.

Definition 11.12. *The* ***Verma module*** *is*

$$M(\lambda) := U / I(\lambda).$$

Notice that this is another left U-module with

$$M(\lambda) = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} \mathbb{C} F^r = \mathbb{C}[F]$$

as vector spaces.

For the module structure, let $- : U \to M(\lambda)$ be the quotient map. Consider $\overline{1} \in M(\lambda)$ where $\overline{1} = 1 + I(\lambda)$. Then we have $E \cdot \overline{1} = \overline{0}, H \cdot \overline{1} = \lambda \overline{1}, E \cdot 1 \equiv 0 \pmod{I(\lambda)}, (H - \lambda) \cdot 1 \equiv 0 \pmod{I(\lambda)}$.

Proposition 11.13. *This is a highest weight module*

THe spanning elements of $M(\lambda)$ are $\overline{F^r} = F^r \cdot T$. We can rewrite as follows: let $U^{\geq 0} = \mathcal{U}(\langle e, h\rangle)$ where $\langle e, h\rangle$ is the standard Borel subalgebra of $\mathfrak{sl}_2$. The 1-dim reps $\mathbb{C}(\lambda) : E \cdot 1 = 0, H \cdot 1 = \lambda$.

Then $\mathrm{Ind}_{U^{\geq 0}}^{U}\mathbb{C}(\lambda) := U \otimes_{U^{\geq 0}} \mathbb{C}(\lambda)$ left-module. As $\mathbb{C}$-linear space,

$$U \otimes_{U^{\geq 0}} \mathbb{C}(\lambda) \cong \mathbb{C}[F] \otimes \mathbb{C} \cong M(\lambda).$$

The U-module structure coincides: both are highest weight moduels and the highest weight vectors are identified as follows:

$$1_U \otimes 1_{\mathbb{C}} \mapsto \overline{1} = 1_U + I(\lambda)$$

The highest weight vector in $M(\lambda)$ is denoted 1_λ.

11.3 Step 3: Universal Property of $M(\lambda)$

Let $M(\lambda)$ be the Verma module. Take any highest weight module V with highest weight vector v_λ. Then $\exists!$ $\mathfrak{sl}_2$-intertwiner $\sigma : M(\lambda) \to V$ such that $\sigma(1_\lambda) = v_\lambda$. Namely:

$$\begin{aligned}\sigma\left(\sum_{r\geq 0} c_r F^r \cdot 1_\lambda\right) &= \sum_{r\geq 0} c_r \sigma(F^r \cdot 1_\lambda) \\ &= \sum_{r\geq 0} c_r F^r \sigma(1_\lambda)\end{aligned}$$

an arbitrary element of V.

Suppose V is irreducible. Since $\mathrm{Im}\sigma \subseteq V$ submodule and contains $v_\lambda \neq 0$, $\mathrm{Im}\sigma = V \implies \sigma$ surjective. Every irreducible finite dimensional $\mathfrak{sl}_2$-module is a quotient of $M(\lambda)$ by a ∞-dim maximal proper submodule.

Let's quickly discuss induced modules: for an algebra A, a subalgebra B, and a left B-module M, then $A \otimes_B M$ is a left A module (acts by left multiplication of first factor).

11.4 Step 4: Description of Maximal Proper Submodule of $M(\lambda)$

Proposition 11.14.
1. *If $\lambda \neq \mathbb{Z}_{\geq 0}$, $M(\lambda)$ is irreducible.*
2. *If $\lambda = n \in \mathbb{Z}_{\geq 0}$, then $M(\lambda)$ has a unique proper nonzero submodule, isomorphic to $M(-n-2)$ and of codimension $n+1$.*

Proof. TO prove, note $M(\lambda) = \bigoplus_{r\geq 0} \mathbb{C}m(r)$ where $m(r) = F^r \cdot 1_\lambda$. For $\lambda \in \mathbb{C}$,

$$\begin{aligned} F \cdot m(r) &= m(r+1) \\ H \cdot m(r) &= (\lambda - 2r)m(r) \\ E \cdot m(r) &= r(\lambda - r + 1)m(r-1) \end{aligned}$$

with $m(-1) := 0$. Let $S \subset M(\lambda)$ be a proper submodule. By the previous exercise, S is a weight module. $S = \text{span}_{\mathbb{C}}\{m(n+1), m(n+2), ...\}$ for some $n \in \mathbb{Z}_{\geq 0}$. Then $E \cdot m(n+1) = 0 \iff \lambda = n$.

To summarize: $M(\lambda)$ irreducible $\iff \lambda \notin \mathbb{Z}_{\geq 0}$.

We have

$$H \cdot m(n = 1) = (n - 2(n+1))m(n+1) = (-n-2)m(n+1).$$

The universal property of Verma modules $\implies$ $\exists \sigma : M(-n-2) \to S \subseteq M(n), 1_{-n-2} \mapsto m(n+1)$. Note that $-n-2 \in \mathbb{Z}_{\geq 0}$, so $M(-n-2)$ is irreducible. Hence σ is injective, so $M(-n-2) \cong S$ and $\dim_{\mathbb{C}} (M(n)/M(-n-2)) = n + 1$. ☐

Exercise 11.15. *Prove the relations in the proof for $\mathcal{U}(\mathfrak{sl}_2)$.*

11.5 Step 5: Description of Quotient

Let $n \in \mathbb{Z}_{\geq 0}$ and $V(n) := M(n)/M(-n-2)$. Let $v(r) = m(r) + M(-n-2) \in V(n)$ for $0 \leq r \leq n$. Since $V(n) = \bigoplus_{r=0}^{n} \mathbb{C}v(r)$, we have

$$\begin{aligned} F \cdot v(r) &= v(r+1) \\ H \cdot v(r) &= (n - 2r)v(r) \\ E \cdot v(r) &= r(n - r + 1)v(r-1) \end{aligned}$$

with $v(n+1) := 0$, $v(0) := 0$. Clearly $V(n) \ncong V(n')$ if $n \neq n'$ by comparing dimensions.

Remark 11.16.

$$V(n) \cong U/(H - n, E, F^{n+1})$$

12 Arbitrary Finite Dimensional Semisimple Lie Algebras over $\mathbb{C}$

See [@carter, chapter 10]. Recall all finite-dimensional simple Lie algebras have a **Chevalley presentation**

$$\mathfrak{g} = \mathbb{C}\langle \{e_i, f_i, h_i\}_{i=1}^{r} \rangle$$

with relations described by Cartan matrix $A = (a_{ij})_{1\leq i,j\leq r}$:

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$
- $a_{ij} = 0 \implies a_{ji} = 0$.

The $\mathfrak{sl}_2$ relations: each $\{e_i, f_i, h_i\}$ satisfy $\mathfrak{sl}_2$ relations. The cross relations are in terms of a_{ij} for $i \neq j$.

Example 12.1. *For $\mathfrak{sl}_3$ with Cartan matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, we have*

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ f_1 &= e_1^T \\ h_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} e_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ f_2 &= e_2^T \\ h_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

with the Serre relation $ad(e_1)^2(e_2) = 0$.

The decomposition $\mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ is generalized to $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$ where $\mathfrak{h}$ is abelian, the **triangular decomposition**. We have $n^- = \langle f_i \rangle, \mathfrak{h} = \langle h_i \rangle, n^+ = \langle e_i \rangle$.

Definition 12.2. *A **weight** is an element of $\mathfrak{h}* = \{f : \mathfrak{h} \to \mathbb{C} | linear\}$. A **weight module** is any module M where*

$$M = \bigoplus_{\lambda \in h^*} M_\lambda$$

where $M_\lambda = \{m \in M | H_i \cdot m = \lambda(h_i)m\}$ and H_i is the image of h_i in $\mathcal{U}(\mathfrak{g})$.

Generally, let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra. We can describe the finite dimensional irreducible reps using the same approach as earlier.

Let $\mathfrak{g} = \mathbb{C}\langle\{e_i, f_i, h_i\}_{i=1}^r\rangle$ with relations described by Cartan matrix $A = (a_{ij})_{1\leq i,j\leq r}$, where r is the rank of $\mathfrak{g}$. We have

$$[h_i, h_j] = 0 \qquad [h_i, e_j] = a_{ij}e_j \qquad [h_i, f_j] = -a_{ij}f_j.$$

Exercise 12.3. *For $\mathfrak{g} = \mathfrak{sl}_3$, recover the presentation (derive these relations) using explicit matrices for generators and find the triangular decomposition.*

Let V be any irreducible finite dimensional rep.

12.1 Step 1: Highest Weight Module

By considering generalized eigenspaces of joint action of $h_1, ..., h_r$, we deduce $\exists v_\lambda \in V\backslash\{0\}$ with $e_i \cdot v_\lambda = 0, h_i \cdot v_\lambda = \lambda(h_i)v_x, \mathcal{U}(n^-) \cdot v_\lambda = V$ for $1 \leq i \leq r, \lambda \in \mathfrak{h}^*$. So V is a highest weight module (for $\mathfrak{g}$) and λ is called the highest weight, v_λ called the highest weight vector.

V is a weight module: $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$ where $V_\mu = \{v \in V | H_i \cdot v = \mu(h_i)v, 1 \leq i \leq r\}$. Defined Supp$V := \{\mu \in \mathfrak{h}^* | V_\mu \neq \{0\}\}$. Using $[h_i, f_j] = -a_{ij}f_j$, we get $F_j \cdot V_\mu \subseteq V_{\mu-\alpha_j}$ where $\alpha_j \in \mathfrak{h}^*$ defined by $\alpha_j(h_i) = a_{ij}$. We can show that Supp$V \subseteq \lambda - \text{span}_{\mathbb{Z}_{\geq 0}}\{\alpha_1, \alpha_2, ..., \alpha_r\}$.

Exercise 12.4. *Show that submodules and quotients of any weight module are also weight modules.*

Hint: Let $W \subseteq V$ be a submodule. Take $v \in W, v = \sum_{i=1}^n v_{\mu_i}$ where $v_{\mu_i} \in V_{\mu_i}$. We want to show each $v_{\mu_i} \in W$. Consider $\prod_{i=1,j\neq i}^n (H - \mu_j(h))$ where $1 \leq i \leq r, h \in \mathfrak{h}$.

12.2 Step 2: Verma Modules

We have

$$M(\lambda) := \mathcal{U}(\mathfrak{g})/I(\lambda) \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}(\lambda)$$

where $I(\lambda)$ is the left ideal generated by $\{H_i - \lambda(h_i), E_i\}_{1\leq i\leq r}$ and $\mathfrak{b}^+ = \langle n^+, \mathfrak{h}\rangle \subseteq \mathfrak{g}$ is the standard upper Borel supalgebra. $\mathbb{C}(\lambda)$ is the 1-dim $\mathfrak{b}^+$-module defined by $H_i \cdot 1 = \lambda(h_i), E_i \cdot 1 = 0$ for $1 \leq i \leq r$. Choose a highest weight vector $1_\lambda \in M(\lambda)$.

12.3 Step 3: Universal Property of $M(\lambda)$

For any highest weight module V (of $\mathfrak{g}$) with weight $\lambda \in \mathfrak{h}^*$, highest weight vector v_λ, $\exists!\mathfrak{g}$-intertwiner $\sigma : M(\lambda) \to V$ such that $\sigma(1_\lambda) = v_\lambda$. So V irreducible $\implies \sigma$ surjective.

12.4 Step 4: Submodules of $M(\lambda)$

$\forall\lambda \in \mathfrak{h}^*$, $M(\lambda)$ has unique maximal proper submodule $J(\lambda)$ [@carter chapter 10]. If $M(\lambda)$ is irreducible, $J(\lambda) = 0$.

Exercise 12.5. *Show this if and only if $M(\lambda)$ is reducible. Hint: First show any submodule $V \subsetneq M(\lambda)$ satisfies $\lambda \notin$ Supp(V). Now define $J(\lambda)$ = sum of all proper submodules of $M(\lambda)$.*

12.5 Step 5: Finite Dimensional Quotients of $M(\lambda)$

By step 1 and 3, any irreducible finite dimensional $\mathfrak{g}$-module V must be a quotient of $M(\lambda)$ for some $\lambda \in \mathfrak{h}^*$ by a nonzero proper maximal submodule. Step 4 says that if it exists, it must be $J(\lambda)$. We also need $J(\lambda)$ to be of finite codimension.

Call $\lambda \in \mathfrak{h}^*$ **dominant** if $\lambda(h_i) \in \mathbb{R}_{\geq 0}$ and **integral** if $\lambda(h_i) \in \mathbb{Z} \forall 1 \leq i \leq r$. Let $P_+ = \{\lambda \in \mathfrak{h}^* | \text{dominant and integral}\}$. Set $V(\lambda) = M(\lambda)/J(\lambda)$ and $v_\lambda := 1_\lambda + J(\lambda) \in V(\lambda)$.

Theorem 12.6 ([@carter). *, Theorem 10.20 and Proposition 10.15]* $\forall \lambda \in \mathfrak{h}^*$ *we have*

$$\dim V(\lambda) < \infty \Longleftrightarrow \lambda \in P_+.$$

Furthermore, if $\dim V(\lambda) < \infty$ *then* $F_i^{\lambda(h_i)+1} \cdot v_\lambda = 0$.

Proof. Sketch of proof: To show $\implies$, for each $1 \leq i \leq r$ can show span$\{F_i^m \cdot v_\lambda\}_{m \in \mathbb{Z}_{\geq 0}}$ is a module for subalgebra $\langle e_i, f_i, h_i \rangle$ and use the $\mathfrak{sl}_2$-result.

To show $\Longleftarrow$, note $\dim V(\lambda)_\mu \leq \dim M(\lambda)_\mu < \infty$. The second inequality is because root spaces of $\mathfrak{g}$ (weight spaces for adjoint action) as finite-dimensional.

It removes to show that $\text{Supp}(V(\lambda)) = \{\mu \in \mathbb{C}^\times | V(\lambda)_\mu \neq 0\}$ is a finite set.

The standard approach:

1. Show $\text{Supp}(V(\lambda)) \subset \mathfrak{h}^*$ is preserved by action of Weyl group $W < \text{GL}(\mathfrak{h}^*)$ generated by simple reflections s_i $(1 \leq i \leq r)$, where $s_i(\mu) := \mu - \mu(h_i)\alpha_i$. To show $\text{Supp}(V(\lambda))$ is preserved, requires some study of $V(\lambda)$ as a module of $\mathbb{C}\langle e_i, f_i, \mathfrak{h}\rangle$ and using that support of a finite dimension $\mathfrak{sl}_2$-module is symmetric around 0.

2. Each weight in $\text{Supp}(V(\lambda))$ is in W-orbit of a dominant integral weight (not necessarily λ).

3. The set $(\lambda - \text{span}_{\mathbb{Z}_{\geq 0}}\{\alpha_1, ..., \alpha_r\}) \cap P_+$ is finite.

Exercise 12.7. *Show this.*

1. W is finite. It is because W permutes roots (weights of adjoint rep) and simple roots $\alpha+i$ span $\mathfrak{h}^*$. Hence $|W| \leq$ number of permutations of roots $< \infty$.

□

Remark 12.8. *1. If $\lambda \neq \lambda' \in P^+$, then $V(\lambda) \ncong V(\lambda')$ (because an invertible intertwiner $\varphi : V(\lambda) \to V(\lambda')$ must send highest weight vector to highest weight vector and hence the highest weights must be the same). This implies bijection $P^+ \overset{1:1}{\to} \{$finite dimensional irreducible$\mathfrak{g}$-modules$\}/$isomorphisms.*

2. *If $\lambda \in P^+$,*

$$V(\lambda) \cong \mathcal{U}(\mathfrak{g})/\{H_i - \lambda(h_i), E_i, F_i^{\lambda(h_i)+1}\}_{1 \leq i \leq r}.$$

3. *W is a Coxeter group, ie. $W = \langle s_1, ..., s_r | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$ if $i < j$ for $m_{ij} \in \mathbb{Z}_{\geq 2}$. More precisely, $m_{ij} = \pi / \cos^{-1}\left(\frac{1}{2}\sqrt{a_{ij}a_{ji}}\right)$, which equals $2, 3, 4, 5$ if $a_{ij}a_{ji} = 0, 1, 2, 3$. See [@carter chapter 5] for more on Coxeter groups.*

4. *If $\mathfrak{g}$ is simple and finite dimensional, $ad : \mathfrak{g} \to \mathfrak{g}$ is irreducible. It is isomorphic to $V(\theta)$, where θ is the highest root of $\mathfrak{g}$. Here, $\theta = \sum_{i=1}^r m_i \alpha_i, m_i \in \mathbb{Z}_{\geq 0}$, and $\sum_{i=1}^r m_i$ is maximal.*

   For example, for $\mathfrak{sl}_3$, we have roots $\alpha_1, \alpha_2, \alpha_{1+2} = \theta, -\alpha_1, -\alpha_2, -\theta$ with weights $e_1, e_2, [e_1, e_2], f_1, f_2, [f_1, f_2]$.

5. *Define $\omega_i \in P^+$ by $\omega_i(h_j) = \delta_{ij}$. The fundamental weights $\mathfrak{h} = \bigoplus_{i=1}^r h_j \implies \{w_i\}_{i=1}^r$ are the dual basis of $\mathfrak{h}^*$. $V(\omega_i)$ is called a fundamental rep. The standard N-dim rep of $\mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N$ is $V(\omega_1)$ (for a suitable choice of $r \in \{1, ..., r\}$.)*

13 Complete Reducibility Of Finite Dimensional $\mathfrak{g}$-reps

General result can be found in [@carter chapter 12]. The key ingredient is the Casimir element in $Z(\mathcal{U}(\mathfrak{g})) \subset \mathcal{U}(\mathfrak{g})$. Later on, we will consider the generalized Casimir element (for a large class of Lie algebras). Note: tensor products of irreducible finite dimensional modules

$$V(\lambda) \otimes V(\lambda') = \bigoplus_{\nu \in P^+} c_{\lambda,\lambda'}^{\vee} V(\nu)$$

where $c_{\lambda,\lambda'}^{\vee} \in \mathbb{Z}_{\geq 0}$ is computable using Steinberg's formula.

For $\mathfrak{sl}_2$:

$$V(n) \otimes V(n') \cong V(n + n') \oplus V(n + n' - 2) \oplus ... \oplus V(|n - n'|)$$

for $n, n' \in \mathbb{Z}_{\geq 0}$, so $V(1) \otimes V(1) \cong V(2) \oplus V(0)$.

Exercise 13.1. *For $\mathfrak{sl}_2$, check $C = EF + FE + \frac{1}{2}H^2$ lies in $Z(\mathcal{U}(\mathfrak{sl}_2))$. Use Schur's lemma to deduce it acts on $V(n) \cong (\pi_n, \mathbb{C}^{n+1})$ by multiplication by $\frac{n(n+2)}{2}$.*

14 Irreducible Finite Dimensional Reps of $L\mathfrak{g}$

Consider $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}\left[z, z^{-1}\right]$. For any finite dimensional $L\mathfrak{g}$-module M, we can form a composition series $M = M_0 \supset M_1 \supset ... \supset M_{n-1} \supset M_n = \{0\}$, where M_i/M_{i+1} is irreducible.

We will argue that finite dimensional evaluation modules are "generically" irreducible. First, some general facts about tensor product modules: if $\mathfrak{h}, \mathfrak{k}$ are any Lie algebras over $\mathbb{C}$, V irreducible $\mathfrak{h}$-module, W irreducible $\mathfrak{k}$-module, $1 \leq \dim V, \dim W < \infty$, then we will show $V \otimes W$ is irreducible as a $(\mathfrak{h} \oplus \mathfrak{k})$-module.

Recall: $(x, y) \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (y \cdot w)$ where $x \in \mathfrak{h}, y \in \mathfrak{k}, v \in V, w \in W$.

Remark 14.1. *if $\mathfrak{k} = \mathfrak{h}$, then $V \otimes W$ is also a $\mathfrak{h}$-module via the diagonal embedding $\mathfrak{h} \hookrightarrow \mathfrak{h} \oplus \mathfrak{h} \to gl(V \otimes W), x \mapsto (x, x)$. We don't discuss it here.*

Exercise 14.2. *Show that the embedding $\mathfrak{h} \hookrightarrow \mathfrak{h} \oplus \mathfrak{k}, x \mapsto (x, 0)$ defines an $\mathfrak{h}$-module structure in $V \otimes W$.*

1. *Describe it and show it is reducible.*
2. *Show, if $V \otimes W$ is irreducible as a $(\mathfrak{h} \oplus \mathfrak{k})$-module, then V is irreducible as a $\mathfrak{h}$-module.*

Recall: $\mathcal{U}(\mathfrak{h} \oplus \mathfrak{k}) \sim \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{k})$. Let A be any unital associative $\mathbb{C}$-algebra.

Exercise 14.3. *If S is an irreducible submodule of $\bigoplus_{i=1}^{n} V_i$, where V_i is an irreducible A-module with $\dim V_i < \infty$, show that $S \cong V_i$ $(1 \leq i \leq n)$.*

Hint: use embeddings, projections, and Schur's lemma.

A refinement of this exercise in the case $V_1 \cong ... \cong V_n$.

Lemma 14.4. *Let V be an irreducible finite dimensional A-module. Let $n \in \mathbb{Z}_{\geq 1}$ and $W \subseteq V^{\oplus n}$ be a nonzero A-submodule. Then $W \cong V^{\oplus r}$ where $1 \leq r \leq n$ and the inclusion map $\varphi : W \to V^{\oplus n}$ is of the form*

$$\phi(v_1, ..., v_r) = (v_1, ..., v_r) \cdot X$$

where X is a full-rank $r \times n$ matrix.

Proof. By induction with respect to $n \in \mathbb{Z}_{\geq 1}$. If $n = 1, r = 1, W = V, \phi(v) = xv = v \cdot X$ for $x \in C^{\times}, X = (x)$ which gives a 1×1-matrix.

For the induction step, choose an irreducible submodule $S \subseteq W$. By the previous exercise, $S \cong V, \phi|_S$ must satisfy $\phi(v) = (x_1 v, ..., x_n v)$ for $v \in V \cong S, (x_1, ..., x_n) \in \mathbb{C}^n$ not all zero, and $x_n v = v \cdot X$, where X is a $1 \times n$ matrix. Note that $G = \mathrm{GL}_n(\mathbb{C})$ acts on $V^{\oplus n}$ as

$$(v_1, ..., v_n) \mapsto (v_1, ..., v_n) \cdot g$$

where $g \in G$. Note that left A-action commutes with this right G-action.This implies G acts on a set of A-submodules of $V^{\oplus n}$, and the G-action preserves the desired property. Now choose $g \in G$ so that $(x_1, ..., x_n) \cdot g = (1, 0, ..., 0)$ where $(x_1, ..., x_n) \in \mathbb{C}^n \backslash \{(0, ..., 0)\}$. Now $W \cdot g = V \oplus W'$, where $W' \subseteq V^{\oplus(n-1)}$. So the induction hypothesis implies the desired property. □

Recall that if V is an irreducible A-module, $\forall v \in V \backslash \{0\}, \exists a \in A$ such that $a \cdot v = w$.

Theorem 14.5 (The Density Theorem)**.** *Let V be an irreducible A-module with $\dim V < \infty$. If $\rho : A \to End_{\mathbb{C}}(V)$ is the representation map, then ρ is surjective.*

Proof. Let $\varphi \in \mathrm{End}_{\mathbb{C}}(V)$ be arbitrary. Choose a basis $(v_1, v_2, ..., v_{\dim V})$ of V. It suffices to show $\exists a \in A$ such that $\rho(a)(v_i) = \varphi(v_i)$ for $1 \leq i \leq \dim V$. Suppose $a \in A$ does not exist. THen the image of the map $A \to V^{\oplus n}, a \mapsto (\rho(a)(v_1), ..., \rho(a)(v_n))$ is a proper nonzero submodule W. Hence $\exists r \in \{1, 2, ..., n-1\}$ such that inclusion $W \hookrightarrow V^{\oplus n}$ is given by the full rank $r \times n$ matrix X. $\exists u_1, ..., u_r \in V$ such that

$$(v_1, ..., v_n) = (u_1, ..., u_r) \cdot X \qquad (\star)$$

$(a = 1)$. Because $r < n$, we can choose $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathrm{Ker}\left(X^T\right) \backslash \{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \}$ so $X \cdot (y_1, ..., y_n) = (0, ..., 0)$.

So plugging in gives

$$\sum_{i=1}^{n} y_i v_i = (u_1, ..., u_r) \cdot X \cdot (y_1, ..., y_r) = 0$$

with $(v_1, ..., v_n)$ basis and we are done. □

Let A, B be associative unital algebras.

Theorem 14.6. *Let V be an irreducible A-module, $\dim V < \infty$. Let W be an irreducible B-module, $\dim W < \infty$. Then $V \otimes W$ is an irreducible $(A \times B)$-module.*

Proof. By density theorem, the algebra homomorphisms $A \to \mathrm{End}_{\mathbb{C}} V, B \to \mathrm{End}_{\mathbb{C}} W$ are surjective, so the algebra homomorphism $A \otimes B \to \mathrm{End}_{\mathbb{C}} V \times \mathrm{End}_{\mathbb{C}} W \cong \mathrm{End}_{\mathbb{C}}(V \otimes W)$ is also surjective. By characterization of irreducibliity in terms of the cyclic and cocyclic vectors, $V \otimes W$ is an irreducible $(A \otimes B)$-module. □

Remark 14.7. *The convese also holds, although we will not need it.*

Fix $\ell \in \mathbb{Z}_{\geq 1}$. As a consequence, if $V_1, ..., V_\ell$ are irreducible finite dimension $\mathcal{U}(\mathfrak{g})$-modules, then $V_1 \otimes ... \otimes V_\ell$ is an irreducible $\mathcal{U}(\mathfrak{g})^{\otimes \ell}$-module, so an irreducible $g^{\oplus \ell}$-module.

15 Loop Algebras, Part 2

Recall $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}\left[z, z^{-1}\right]$. Let $a_1, ..., a_\ell \in \mathbb{C}^\times$, $\mathrm{ev}_{a_1,...,a_\ell} : L\mathfrak{g} \to \mathfrak{g}^{\oplus \ell}, x \otimes p(z) \mapsto (p(a_1)x, ..., p(a_\ell)x)$.

Lemma 15.1. *Fix $a_1, ..., a_\ell \in \mathbb{C}^\times$. The evaluation map is surjective if and only if all a_i are distinct.*

Proof. $\implies$: exercise.

$\Longleftarrow$: suppose $\{x_i\}_{i=1}^{\dim \mathfrak{g}}$ is a basis for $\mathfrak{g}$. Then $\{(0, ..., 0, x_i, 0, ..., 0) | 1 \leq i \leq \dim \mathfrak{g}, 1 \leq r \leq \ell\}$ where x_i is in the rth spot is a basis for $g^{\oplus \ell}$.

Claim: such a basis element lies in $\mathrm{ev}_{a_1,...,a_\ell}(L\mathfrak{g})$ with distinct a_i. Set $p(z) = \prod_{s=1, s\neq r}^{\ell} \dfrac{a_s - z}{a_s - a_r}$. Then $p(a_s) = 0, p(a_r) = 1$ if $s \neq r$. So

$$\mathrm{ev}_{a_1,...,a_\ell}(x_i \otimes p(z)) = (0, ..., 0, x_i, 0, ..., 0)$$

as desired. ☐

Recall, for $a_1, ..., a_\ell \in \mathbb{C}^\times, (\pi_1, V_1), ..., (\pi_\ell, V_\ell)$ irreducible finite dimensional $\mathfrak{g}$-reps. The evaluation rep is defined via:

$$\pi_{1,a_1} \otimes ... \otimes \pi_{\ell,a_\ell} = (\pi_1 \otimes ... \otimes \pi_e ll) \circ \mathrm{ev}_{a_1,...,a_\ell} : L\mathfrak{g} \to \mathrm{gl}(V_1 \otimes ... \otimes V_\ell).$$

Let $a_1, ..., a_\ell \in \mathbb{C}^\times$ be distinct. Since $\mathrm{ev}_{a_1,...,a_\ell}$ is surjective and $\pi_1 \otimes ... \otimes \pi_\ell$ is an irreducible $\mathfrak{g}^{\oplus \ell}$-rep, we ge the following result:

Corollary 15.2. *$\pi_{1,a_1} \otimes ... \otimes \pi_{\ell,a_\ell}$ is an irreducible $L\mathfrak{g}$-rep if a_i are distinct nonzero complex numbers.*

16 Classification of Irreducibles

Recall the triangular deompositions of $\mathfrak{g}$ and $L\mathfrak{g}$. These two imply the triangular decomposition of the universal enveloping algebra of the loop algebra:

$$\mathcal{U}(L\mathfrak{g}) \cong \mathcal{U}(Ln^-) \otimes \mathcal{U}(L\mathfrak{h}) \otimes \mathcal{U}(Ln^+).$$

Definition 16.1. *Let V be a $L\mathfrak{g}$-module, $\bigwedge \in L(\mathfrak{h})^*$. Call V a **loop highest weight module** with highest weight $\bigwedge$ if:*

1. *$\exists v \in V$ such that $L(n^+) \cdot v = 0$*
2. *$\forall h \in L(\mathfrak{h}), h \cdot v = \bigwedge(h)v$.*

3. $V = \mathcal{U}(L(\mathfrak{g})) \cdot v$ $(= \mathcal{U}(L(n^-)) \cdot v)$*.*

Remark 16.2. *One can define a "loop Verma module" of $L\mathfrak{g}$ by means of 1-dim module of $L\mathfrak{l}^+$ where $\mathfrak{l}^+ = \langle n^+, \mathfrak{h} \rangle$.*

We will consider a finite dimensional quotient called the Weyl module; they play the role of the universal finite dimensional loop highest weight modules. We first describe the set that parametrizes the Weyl module for $L\mathfrak{sl}_2$.

Consider $p(u) \in \mathbb{C}[u]$ such that $p(0) = 1$, the Drinfeld polynomial, which gives a monoid P under multiplication. For $a \in \mathbb{C}^\times$ define $p_a(u) : 1 - au \in P$, which $\implies \forall p \in P \exists!(a_1, ..., a_\ell) \in (\mathbb{C}^\times)^\ell$ (up to reordering) and $(s_1, ..., s_\ell) \in \mathbb{Z}^\ell_{>0}$ such that $a_i \neq a_j$ for $i \neq j$ and $p = \prod_{j=1}^{\ell} p_{a_j}^{s_j}$. Note that a_i occurs if and only if $p\left(a_i^{-1}\right) = 0$.

Consider polynomials in P whose roots lie in a finite set and order this set as $\{a_1, ..., a_\ell\} \subseteq \mathbb{C}^\times$ where a_i are distinct. To $p \in P$ associate a tuple of dominant integral $\mathfrak{sl}_2$ weights

$$\prod_{j=1}^{\ell} \left(p_{a_j}\right)^{s_j} \mapsto (s, \omega, ..., s_\ell \omega)$$

where $\omega \in \mathfrak{h}^*$ satisfies $\omega(h) = 1$.

Example 16.3.

- $(1 - a_1 u) \mapsto (1)$
- $(1 - a_1 u)^2 \mapsto (2)$
- $(1 - a_1 u)(1 - a_2 u)^2 \mapsto (1, 2)$ *is* $a_1 \neq a_2$.

Additionally, let $\mathrm{ev}_{a_1,...,a_\ell} : L\mathfrak{g} \to \mathfrak{g}^{\oplus \ell}, x^{(m)} \mapsto (a_1^m x, ..., a_\ell^m x)$ be the evaluation homomorphism, where $a_1, ..., a_\ell \in \mathbb{C}^\times$. This is surjective if and only if a_j are all distinct. For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible finite dimensional $\mathfrak{g}$-module with λ as highest weight. The representation π_λ gives an evaluation module $V(\lambda)_a$ where $a \in \mathbb{C}^\times$, with representation $\pi_{\lambda_a} = \pi_\lambda \circ \mathrm{ev}_a$.

For the tensor product $V(\lambda_1)_{a_1} \otimes ... \otimes V(\lambda_\ell)_{a_\ell}$, $x^{(m)} \in L\mathfrak{g}$ acts as

$$\sum_{j=1}^{\ell} \mathrm{id} \otimes ... \otimes \mathrm{id} \otimes a_j^m \pi_{\lambda_j}(x) \otimes \mathrm{id} \otimes ... \otimes \mathrm{id}.$$

This is irreducible if and only if all a_j are distinct. The proof of this relies on the density theorem for arbitrary Lie algebras $k/\mathbb{C}$.

Proposition 16.4. *The unital algebra homomorphism $\mathcal{U}(k) \xrightarrow{\rho} End(U)$ is surjective if (ρ, V) is a finite dimensional irreducible k-reps.*

Exercise 16.5. *Show this is false if $\mathcal{U}(k)$ is replaced by k.*

Now, we will study the converse of the statement: "This is irreducible if and only if all a_j are distinct."

Definition 16.6. *A* ***loop highest weight module*** *of $L\mathfrak{g}$ is a module V such that there exists a $\bigwedge \in (L\mathfrak{h})^*$ (called the* ***loop highest weight****) and there exists a non-zero vector $v \in V$ (called the* ***loop highest weight vector****) such that:*

- $(Ln^+) \cdot v = 0$,
- $h \cdot v = \bigwedge(h)v$ *for all* $h \in L\mathfrak{h}$,
- $V = \mathcal{U}(L\mathfrak{g}) \cdot v = \mathcal{U}(Ln^-) \cdot v$.

Lemma 16.7. *Let V be a finite dimensional irreducible $L\mathfrak{g}$-module. Then V is a loop highest weight module.*

Proof. Let $V^0 = \{v \in V | Ln^+ \cdot v\}$. To see $V^0 \neq \{0\}$, consider the generalized joint eigenspaces of the $\mathfrak{h}$-action (where $\mathfrak{h} \in L\mathfrak{h}$) V_μ^{gen} (where $\mu \in \mathfrak{h}^+$). As usual, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu^{\text{gen}}$. Note $e_i^{(m)}$ (where $1 \leq i \leq r, m \in \mathbb{Z}$) sends V_μ^{gen} to $V_{\mu+\alpha_i}^{\text{gen}}$ (where α_i is a simple root). Because $\dim V < \infty$, we cannot have an infinite chain

$$V_\mu^{\text{gen}} \to V_{\mu+\alpha_i}^{\text{gen}} \to V_{\mu+\alpha_i+\alpha_j}^{\text{gen}} \to \dots$$

This gives a nonzero element of V^0 annihilated by Ln^+ (last nonzero space in this chain).

Exercise 16.8. *Show that $L\mathfrak{h}$-action preserves V^0.*

Hint: $[h_i^{(m)}, e_j^{(n)}] = a_{ij} e_j^{m+n}$.

Deduce $\exists$ common eigenvector v of the $h_i^{(\mu)}$-action $1 \leq i \leq r, m \in \mathbb{Z}$. In particular, $v \neq 0$. The irreducibility of V forces the submodule $\mathcal{U}(L\mathfrak{g}) \cdot v$ to be equal to V.

□

Goal: describe the universal finite dimensional loop highest weight modules, known as the Weyl modules.

Consider $L\mathfrak{sl}_2$ first. Recall $V_{a_1} \otimes V_{a_2}$ for $a_1, a_2 \in \mathbb{C}^\times$, where $V = V(\omega)$ for $\omega \in \mathfrak{h}^*, \omega(h) = 1, \mathfrak{h} = \mathbb{C}h$ is a 2-dim irreducible $\mathfrak{sl}_2$-module equal to $\mathbb{C}^2$.

We have

$$\begin{aligned}
\begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\end{pmatrix} &= a_1^m + a_2^m \\
\begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\end{pmatrix} &= a_1^m - a_2^m \\
\begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\end{pmatrix} &= -a_1^m + a_2^m \\
\begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\end{pmatrix} &= -a_1^m - a_2^m
\end{aligned}$$

Note that v is a loop highest weight vector, with loop highest weight $= a_1^m + a_2^m$.

Exercise 16.9. *Show that these 4 vectors are joint eigenvectors of $h^{(m)}$ where $m \in \mathbb{Z}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that these eigenvalues are indeed equal in the centered equations above. If $a_1 \neq a_2$, show that the vectors $f^{(n)} \cdot v$ where $n \in \mathbb{Z}$ are not eigenvectors of any $h^{(m)}$.*

For $L\mathfrak{sl}_2$, we considered $\mathcal{P} = \{p \in \mathbb{C}[u] | p(0) = 1\}$ monoid under $p_a = 1 - au$ for $a \in \mathbb{C}^\times$. $\forall p \in \mathcal{P} \exists! \{a_1, ..., a_\ell\} \subseteq \mathbb{C}^\times$ without repeats and $\exists! \{s_1, ..., s_\ell\} \subseteq \mathbb{Z}_{>0}$ with $p = \prod_{j=1}^{\ell} (p_{a_k})^{s_j}$.

Definition 16.10. *Let $a_1, ..., a_\ell \in \mathbb{C}^\times$ be distinct and $s_1, ..., s_\ell \in \mathbb{Z}_{>0}$. Set $p = \prod_{j=1}^{\ell} (p_{a_j})^{s_j}$. Let $J(p)$ be the left ideal of $U(L\mathfrak{sl}_2)$ generated by:*

- e
- $h^{(m)} - \bigwedge_p \left(h^{(m)}\right) \forall m \in \mathbb{Z}$, *where* $\bigwedge_p \left(h^{(m)}\right) = \prod_{j=1}^{\ell} a_j^m s_j$.
- f^{degp+1} *where* $degp = \sum_{j=1}^{\ell} s_j$.

*The **Weyl module** is $W(p) = \mathcal{U}(L\mathfrak{sl}_2)/J(p)$.*

Remark 16.11. *Because $e, h^{(m)} - \bigwedge_p \left(h^{(m)}\right) \forall m \in \mathbb{Z}$ are in $J(p)$, $W(p)$ is a loop highest weight module with loop highest weight $\bigwedge_p \in (L\mathfrak{h})^*$ and loop highest weight vector is $1_p = 1_{\mathcal{U}(L\mathfrak{g})} + J(p)$. This relies on checking the following exercise:*

Exercise 16.12. *Show that, if $e = e^{(0)}$ annihilates a joint eigenvector v of all $h^{(m)}$, then all $e^{(n)}$ annihilate v.*

Remark 16.13. *Because $\mathcal{U}(L\mathfrak{g}) \cong \mathcal{U}(Ln^-) \otimes U(L\mathfrak{h}) \otimes \mathcal{U}(Ln^+)$ and $\mathcal{U}(Ln^-) \cong \mathbb{C}\left[\{f^{(m)}\}_{m \in \mathbb{Z}}\right]$, each element of $W(p)$ lies in the span of $f^{(m_1)} ... f^{(m_s)} \cdot 1_p$ for $m_1, ..., m_s \in \mathbb{Z}, m_1 \leq m_2 \leq ... \leq m_s$ for some $s \in \mathbb{Z}_{\geq 0}$.*

Exercise 16.14. *Show that, for fixed $s \in \mathbb{Z}_{\geq 0}$, the subspace*

$$W(p)_s = span_{\mathbb{C}} \{f^{(m_1)} ... f^{(m_s)} \cdot 1 | m_1, ..., m_s \in \mathbb{Z}, m_1 \leq m_2 \leq ... \leq m_s\}$$

is an $L\mathfrak{h}$-module of $W(p)$.

Remark 16.15. *Note that $V(\omega)_{a_1} \otimes V(\omega)_{a_2}$ has loop highest weight $(a_1^m + a_2^m)$ where $m \in \mathbb{Z}$. If $p = (1 - a_1 u)(1 - a_2 u), W(p)$ has loop highest weight $\bigwedge_p \left(h^{(m)}\right)$.*

For general finite dimensional simple $\mathfrak{g}$ over $\mathbb{C}$ or rank r, we need r-tuples of polynomials.

$$\mathcal{P} = \{p = (p_1, ..., p_r) | p_1(0) = ... = p_r(0) = 1\}$$

is a monoid under entrywise multiplication, and the neutral element is $(1, 1, ..., 1)$ of length r.

Furthermore, we have $p_{i_1}^{(u)} := (1, ..., 1, 1 - au, 1, ..., 1)$ for $a \in \mathbb{C}^\times, 1 \leq i \leq r$, where the first string is $i - 1$ long and the second string is $r - i$ long. We have $p = \prod_{j=1}^{\ell} \prod_{i=1}^{r} (p_{i,a_j})^{s_{ij}}$ with distinct $a_j \in \mathbb{C}^\times$ (inverses of roots of $p_1 ... p_r$)

where $s_{ij} \in \mathbb{Z}$ (multiplicity of a_j^{-1} in p_i) and ℓ is the number of distinct roots in $p_1 ... p_r$.

Definition 16.16. *$J(p)$ is the left ideal of $\mathcal{U}(L\mathfrak{g})$ generated by $L(n^+), h_i^{(m)} - \bigwedge_p \left(h_i^{(m)}\right)$ where $\bigwedge_p \left(h_i^{(m)}\right) := \sum_{j=1}^{\ell} a_j^m s_{ij}$, and $f_i^{degp_i+1}$. It has $1_p := 1_{\mathcal{U}(\mathfrak{g})} + J(p)$. The **Weyl module** is $W(p) = U(L\mathfrak{g})/J(p)$.*

Theorem 16.17 ([@chari)**.** *Theorem 1, Proposition 2.1]*

1. *$\dim W(p) < \infty \forall p \in P$.*
2. *Universal property: let V be any loop highest weight module $\dim V < \infty, \exists ! p \in P$ such that V is a quotient of $W(p)$.*
3. *$W(p)$ has unique irreducible quotient $V(p)$.*

Remark 16.18. *For (1), we need to find an efficient spanning set of $W(p)$, which relies on $f_i^{degp_i+1} 1_p = 0$.*

For (3), we roughly need $W(p)$ to have unique maximal proper submodule: sum of all proper submodules. We can do this via weight decomposition and 1-dimensionality of highest weight subspace.

The key idea of the proof is to embed $L\mathfrak{g}$ in a larger Lie algebra $L\mathfrak{g}^{ext}$ by adjoining element a such that $ad(d) = z\frac{d}{dz}, \left[d, x^{(m)}\right] = mx^{(m)}$. Then study integral modules of $L\mathfrak{g}^{ext}$ and relate them to Weyl modules.

We have the following bijections:

$$\begin{aligned} &\mathcal{P} \to \{\text{universal finite dimensional loop highest weight modules of } L\mathfrak{g}\} \\ &\underline{p} \mapsto W(p) \end{aligned}$$

Taking the unique irreducible quotients gives

$V(\underline{p}) \in \{$irreducible finite-dimensional loop highest weight modules of $\mathfrak{g}\}/\text{iso} \Longleftrightarrow \{$evaluation modules defined

Compare with evolution modules. For $p \in \mathcal{P}$, define

$$\underline{p}_{\lambda,a} = ((1-au)^{\lambda(h_1)}, ..., (1-au)^{\lambda(h_r)}) \in \mathcal{P}.$$

Any $\underline{p} \in \mathcal{P}$ factorizes uniquely as

$$\underline{p} = \prod_{j=1}^{\ell} \underline{p}_{\lambda_j, a_j}$$

where $a_j \in \mathbb{C}^\times$ are distinct. Compare to loop highest weights to obtain

$$V(p) \cong V(\lambda_1)_{a_1} \otimes ... \otimes V(\lambda_\ell)_{a_\ell} \forall p \in \mathcal{P}.$$

Theorem 16.19 ([@chari)**.** *Theorem 2 and 3]*

1. *Let* $\underline{p} = (p_1, ..., p_r) \in \mathcal{P}, \underline{p}' \in (p'_1, p'_2, ..., p'_r) \in \mathcal{P}$*. If* p_i *and* p'_j *are coprime for all* $1 \leq i, j \leq r$*, then*

$$W(\underline{p} \cdot \underline{p}') \cong W(\underline{p}) \otimes W(\underline{p}').$$

2. *For simplicity, consider* $\mathfrak{g} = \mathfrak{sl}_N, N = r+1$*. Let* $\underline{p} = (p_1, ..., p_r) \in \mathcal{P}$*. Then* $W(\underline{p}) = V(\underline{p}) \iff p_1(u)...p_r(u)$ *has distinct roots. To generalize, define* $p_\theta(u) = p_1(u)^{m_1}...p_r(u)^{m_r}$ *where* $\theta = \sum_{i=1}^r m_i \alpha_i$ *is the highest root and* $m_i \in \mathbb{Z}_{>0}$*.*

17 Affine Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. We have discussed $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$. We will construct $\hat{\mathfrak{g}}$, a 1-dim central extension of $L\mathfrak{g}$.

17.1 Central Extensions of Groups

Definition 17.1. *A* ***projective representation*** *of a group* G *on a* k*-linear space* V *is a group homomorphism* $\rho : G \to PGL(V) := GL(v) \backslash k^\times Id_V$*.*

Remark 17.2. *These are important in quantum mechanics,* V *is viewed as a space of states of a quantum mechanical system, up to scalar multiples.*

Take the collection of linear maps $\{\rho(g)\}_{g \in G}$ such that

$$\rho(g) \circ \rho(h) = c_{g,h} \rho(gh)$$

for $c_{gh} \in k^\times$ $(\forall gh \in G)$.

Definition 17.3. *If* ρ *is projective, then there is a* ***central extension*** *of* G*, a short exact sequence of groups*

$$1 \to K \xrightarrow{\iota} H \xrightarrow{\tau} G \to 1$$

where ι *injective,* $im(\iota) = ker(\tau)$*,* τ *surjective.*

These conditions imply $G \cong H/\iota(K)$, and $\iota(k) \subseteq Z(G) \implies K$ abelian. $\exists \hat{p} : H \to \mathrm{GL}(V)$ (a genuine rep of H) so that we have a commuting diagram:

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Canonically: $H = \{(g, T) \in G \times \mathrm{GL}(V) | \mathrm{proj}(T) = \rho(g)\}$. $\tau : H \twoheadrightarrow G$ is given by $\tau(g, T) = g$. ι inclusion, $K = \ker\tau = \{e_G\} \times k^\times \mathrm{Id}_V$. $\hat{\rho} : H \to \mathrm{GL}(V)$ is given by $\hat{\rho}(g, T) = T$.

Example 17.4.

$$1 \to \{\pm I\} \hookrightarrow SU(2) \xrightarrow{\tau} SO(3) \to 1.$$

Odd-dim irreducible $SU(2)$ reps descend to $SO(3)$-reos. Even-dim irreducible irreducible $SU(2)$ reps π induces a projective $SO(3)$ rep $\pi^\vee$. We have

$$\pi^\vee(g) \circ \pi^\vee(h) = \pm\pi^\vee(gh)$$

for $g, h \in SO(3)$.

17.2 Central Extensions of Lie Algebras

Definition 17.5. *An **extension** of Lie algebras is a short exact sequence*

$$0 \to \mathfrak{c} \xrightarrow{\iota} \mathfrak{b} \xrightarrow{\tau} \mathfrak{a} \to 0$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are Lie algebras over k, with $\mathfrak{a} = \mathfrak{b}/2(\mathfrak{c})$.

In our case $k = \mathbb{C}, \mathfrak{a} = L\mathfrak{g}$. Given $\mathfrak{a}$ it is interesting to classify extensions. We say "$\mathfrak{b}$ is extension of $\mathfrak{a}$ by $\mathfrak{c}$".

Note, as k-linear spaces $\mathfrak{b} \cong \mathfrak{a} \oplus \mathfrak{c}$ (if $\mathfrak{b}$ has a basis).

Exercise 17.6.
1. *Show that, as a k-linear map, $\tau : \mathfrak{b} \to \mathfrak{a}$ has a "section" $\sigma : \mathfrak{a} \to \mathfrak{b}$ (k-linear) $\tau \circ \sigma = id_{\mathfrak{a}}$.*
2. *Deduce $\mathfrak{b} = \sigma(\mathfrak{a}) \oplus \iota(\mathfrak{c})$ (as k-linear space)/*

Definition 17.7. *An extension is called **central** if $\iota(\mathfrak{c}) \subseteq Z(\mathfrak{b})$.*

In particular, $\mathfrak{c}$ is abelian. Common choice: $\dim(\mathfrak{c}) = 1$.

Given $\mathfrak{a}, \mathfrak{c}$, how do we construct all central extensions? The idea is to measure the failure of σ to be a Lie algbera homomorphism

$$\beta(x, y) := [\sigma(x), \sigma(y)]_{\mathfrak{b}} - \sigma\left([x, y]_{\mathfrak{a}}\right) \in \mathfrak{b}$$

for $x, y \in \mathfrak{a}$. Clearly, β is k-linear, alternating: $\beta(x, x) = 0 \implies \beta(x, 0) = \beta(0, x) = 0, \beta(y, z) = -\beta(x, y)$.

Exercise 17.8.
1. *Show that Jacobi identity: $\beta([x, y], z) + \beta([y, z], z) + \beta([z, x], y) = 0$.*
2. *Show $im(\beta) \subseteq 2(\mathfrak{c})$.*

Consider the space

$$C^2(\mathfrak{a}, \mathfrak{c}) = \{\beta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{c} | \text{ bilinear, alternating Jacobi identity}\}.$$

Given $\beta \in C^2(\mathfrak{a}, \mathfrak{c})$, define a Lie bracket on $\mathfrak{a} \oplus \mathfrak{c}$, and call the resulting Lie algebra $\mathfrak{b}_\beta$. We have

$$[(x, y), (x', y')]_\beta = ([x, x']_{\mathfrak{a}}, \beta(x, x'))$$

where $x, x' \in \mathfrak{a}, y, y' \in \mathfrak{c}$.

One checks this is a well-defined Lie bracket on $\mathfrak{b}$. Also, $\mathfrak{c}$ embeds into $\mathfrak{b}_\beta$ as $\{0\} \oplus \mathfrak{c} \subseteq Z(\mathfrak{b}_\beta)$. Note, $\beta, \beta' \in C^2(\mathfrak{a}, \mathfrak{c})$ may produce $\mathfrak{b}_\beta \cong \mathfrak{b}_{\beta'}$.

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As diagrams of linear maps, $\theta(0, y) = (0, y), y \in \mathfrak{c}$ and $\tau(\theta(x, y)) = x \forall x \in \mathfrak{a}, y \in \mathfrak{c} \implies \theta(x, c) = (x, c + \varphi(x))$ for some $\varphi : \mathfrak{a} \to \mathfrak{c}$ linear.

Exercise 17.9. *Use the fact that θ is s Lie algebra homomorphism to deduce $\beta'(x, y) = \beta(x, y) + \varphi([x, y]) \forall x, y \in \mathfrak{a}$.*

Let $B'(\mathfrak{a}, \mathfrak{c}) = \{\varphi : \mathfrak{a} \to \mathfrak{c}\ k - \text{linear}\}$. Define $(d\varphi)(x, y) = \varphi([x, y])$ for $x, y \in \mathfrak{a}, \varphi \in B'(\mathfrak{a}, \mathfrak{c})$. This implies there is a linear map $a : B'(\mathfrak{a}, \mathfrak{c}) \to C^2(\mathfrak{a}, \mathfrak{c})$ with $\beta' = \beta + d\varphi$.

Proposition 17.10. *$H^2(\mathfrak{a}, \mathfrak{c}) := C^2(\mathfrak{a}, \mathfrak{c})/\alpha B'(\mathfrak{a}, \mathfrak{c}) \stackrel{1:1}{\to} \{$central extensions of $\mathfrak{a}$ over $\mathfrak{c}\}$. with $[\beta] = \beta + \alpha\beta'(\mathfrak{a}, \mathfrak{c}) \mapsto \mathfrak{b}_\beta$.*

Remark 17.11. *Part of Lie algebra cohomology of $\mathfrak{a}$ valued in $\mathfrak{c}$. $C^2(\mathfrak{a}, \mathfrak{c}) =$"2-cocycles", $B'(\mathfrak{a}, \mathfrak{c}) =$"1-cochains", $dB'(\mathfrak{a}, \mathfrak{c}) =$"2-coboundaries".*

Also, if $[\beta] = 0 \in H^2(\mathfrak{a}, \mathfrak{c})$, then $\mathfrak{b}_\beta$ is the "trivial central extension" and $\mathfrak{b}_\beta \cong \mathfrak{a}, \mathfrak{c}$ as Lie algebras.

From now on, $k = \mathbb{C}, \dim(\mathfrak{c}) = 1$. Let $\mathfrak{g} =$ finite dimensional simple Lie algebras over $\mathbb{C}$. We will study $H^2(\mathfrak{c}, \mathbb{C})$ and $H^2(L\mathfrak{g}, \mathbb{C})$.

Definition 17.12. *The **Killing form** of $\mathfrak{g}$ is the bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \kappa(x, y) = Tr(ad(x) \circ ad(y))$.*

The key properties for the Killing form are:

- Nondegeneracy: $\forall x \in \mathfrak{g}\backslash\{0\}, \exists y \in \mathfrak{g}\backslash\{0\}, \kappa(x, y) \neq 0$.
- Invariance: $\forall x, y, z \in \mathfrak{g}, \kappa([x, y], z) = \kappa(x, [y, z])$.
- Symmetry: $\forall x, y \in \mathfrak{g}, \kappa(x, y) = \kappa(y, x)$.

The space of such forms is 1-dim.

Example 17.13. *$\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}\ell \oplus \mathbb{C}f \oplus \mathbb{C}h$, then $ad(e)(e) = 0, ad(e)(f) = h, ad(e)(h) = -2e$. This implies $ad(e) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.*

Similarly, $ad(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$ and $ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We have $\kappa(e, f) = 4 = \kappa(f, e), k(h, h) = 8$, and $\kappa(x, y) = 0$ for all other $(x, y), x, y \in \{e, f, h\}$.

The existence of Killing form $\implies$ all derivations of $\mathfrak{g}$ are inner.

Proposition 17.14. $H^2(\mathfrak{g}, \mathbb{C}) = 0 \implies$ *all 1-dim central extensions of* $\mathfrak{g}$ *are trivial.*

Proof. Let $\beta \in C^2(\mathfrak{g}, \mathbb{C})$. We need to show $\exists \varphi \in B'(\mathfrak{g}, \mathbb{C})$ such that $\beta = d\varphi$. For $x \in \mathfrak{g}$ define $\rho : \mathfrak{g} \to \mathfrak{g}^*$ by $\rho(x)(y) = \beta(x, y)$. Define $\nu : \mathfrak{g} \to \mathfrak{g}^*$ by $\nu(x)(y) = \kappa(x, y)$. Because of non-degeneracy, ν is invertible. So $f := \nu^{-1} \circ \rho : \mathfrak{g} \to \mathfrak{g}$ is $\mathbb{C}$-linear.

Exercise 17.15. 1. *Show* $\forall x, y \in \mathfrak{g}, \kappa(f(x), y) = \beta(x, y)$.

2. *Show* f *is a derivation on* $\mathfrak{g}, f([x, x']) = [f(x), x'] + [x, f(x)] \forall x, x' \in \mathfrak{g}$

Hint: Show, using (1), that $\kappa(f([x, x']), y) = \kappa([f(x), x'] + [x, f(x)]) \forall x, x', y \in \mathfrak{g}$ *and use non-degeneracy of* κ.

Hence $f = \mathrm{ad}(y)$ for some $y \in \mathfrak{g}$.

Compute

$$\begin{aligned} \beta(x, x') &= \rho(x)(x') \\ &= \kappa(\nu(\rho(x)), x') \\ &= \kappa(f(x), x') \\ &= \kappa([y, x], x') \\ &= \kappa(y, [x, x']) \end{aligned}$$

Define $\varphi \in B'(\mathfrak{g}, \mathbb{C})$ by $\varphi(x) = \kappa(y, x)$.

$$\begin{aligned} (d\varphi)(x, x') &= \varphi([x, x']) \\ &= \kappa(y, [x, x']) \\ &= \beta(x, x') \end{aligned}$$

which implies $\beta = d\varphi$. □

17.3 From Loop Algebras to Affine Lie Algebras

Let $L\mathfrak{g} = g \otimes \mathbb{C}[z, z^{-1}], x^{(m)} = x \otimes z^m \in L\mathfrak{g}, x \in \mathfrak{g}, m \in \mathbb{Z}$. To find: a 2-cocycle $\beta : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}$. Use bracket of central extension:

$$[x^{(m)} + \lambda c, y^{(n)} + \mu c] = [x, y]^{(m+n)} + \beta(x, y)c$$

for $m, n \in \mathbb{Z}, x, y \in \mathfrak{g}, \lambda, \mu \in \mathbb{C}$.

Instead of $(x^{(m)}, \lambda)$ I will write $x^{(m)} + \lambda c$. (Identified $L\mathfrak{g} \oplus 0$ with $L\mathfrak{g}$, and $c = (0, 1)$). First note: $\mathrm{Res}\left(\sum_{m \in X} a_m z^m\right) := a_{-1}$ so $\mathrm{Res} : \mathbb{C}[z, z^{-1}] \to \mathbb{C}$.

Exercise 17.16. *Check that the bilinear map* $\beta_2 : \mathbb{C}[z,z^{-1}] \times \mathbb{C}[z,z^{-1}] \to \mathbb{C}$ *defined by* $\beta_2(f_1,f_2) = Res(f_1'(z)f_2(z)), f_1,f_2 \in \mathbb{C}[z,z^{-1}]$*, satisfies* $\beta_2(f,f) = 0, \beta_2(f_1f_2,f_3) + \beta_2(f_2f_3,f_1) + \beta_2(f_3f_1,f_2) = 0$.

Hint: $Res(f') = 0 \forall f \in \mathbb{C}[z,z^{-1}]$.

Note $\beta_2(z^m,z^n) = \text{Res}(mz^{m+n-1}) = \begin{cases} m \text{ if } m+n=0 \\ 0 \text{ otherwise} \end{cases}$. Fix any nondegenerate invariant symmetric bilinear form $(\cdot|\cdot) : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ (scalar multiple of Killing form). Extend β_2 to a bilinear map: $L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}$:

$$\beta(x_1 \otimes f_1, x_2 \otimes f_2) = (x_1|x_2)\beta_2(f_1f_2)$$

with $x_1,x_2 \in \mathfrak{g}, f_1,f_2 \in \mathbb{C}[z,z^{-1}]$. Then $\beta(x\otimes f, x\otimes f) = (x|x)\beta_2(f,f)$. So

$$\begin{aligned}
&\beta([x_1\otimes f_1, x_2\otimes f_2], x_3\otimes f_3) + \text{cyclic permutations} \\
=&\beta([x_1,x_2]\otimes f_1f_2, x_3\otimes f_3) + \text{cyclic permutations} \\
=&([x_1,x_2]|x_3)\beta_2(f_1f_2,f_3) + \text{cyclic permutations} \\
=&([x_1,x_2]|x_3)(\beta_2(f_1f_2,f_3) + \text{cyclic permutations}) \\
=&0
\end{aligned}$$

Remark 17.17. *Can show this* β *is unique (up to scalar multiple) 2-cocycle:* $L\mathfrak{g} \to \mathbb{C}$ *(because* $\mathfrak{g} \hookrightarrow L\mathfrak{g}, x \mapsto x^{(0)} = x\otimes z^0, \beta/\mathfrak{g}\times\mathfrak{g} = 0$*).*

Definition 17.18. *The affine Lie algebra* $\hat{\mathfrak{g}}$ *is the 1-dim central extension of* $L\mathfrak{g}$ *with 2-cocycle* β*.*

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$$

as $\mathbb{C}$-linear spaces. C is central, and

$$[x^{(m)}, y^{(n)}] = [x,y]\mathfrak{g}^{(m+n)} + (x|y)m\delta_{m,-n}c$$

where $m,n \in \mathbb{Z}, x,y \in \mathfrak{g}$.

Note that $L\mathfrak{g} \subset \hat{g}$ as a subspace, not a subalgebra.

Example 17.19. *For* $\mathfrak{g} = \mathfrak{sl}_2$*, set* $(x|y) = \frac{1}{4}\kappa(x,y)$*. We have* $(e|f) = 1, (h|h) = 2, (h|e) = (h|f) = (e|e) = (f|f) = 0$*. Then* $\hat{\mathfrak{g}}$ *has linear basis* $\{e^{(m)}, f^{(m)}, h^{(m)}\}_{m\in\mathbb{Z}} \cup \{c\}$*.*

Exercise 17.20. *Verify that the relations in $\hat{\mathfrak{g}}$ are:*

$$\begin{aligned}
[h^{(m)}, h^{(n)}] &= 2\delta_{m,-n} mc \\
[h^{(m)}, e^{(n)}] &= 2e^{m+n} \\
[h^{(m)}, f^{(n)}] &= -2f^{(m+n)} \\
[e^{(m)}, e^{(n)}] &= [f^{(m)}, f^{(n)}] = 0 \\
[e^{(m)}, f^{(n)}] &= h^{m+n} + \delta_{m,-n} mc \\
[h^{(m)}, c] &= [e^{(m)}, c] = [f^{(m)}, c] = 0
\end{aligned}$$

For some general facts about $\hat{\mathfrak{g}}$:

Commuting diagram of Lie algebra homomorphisms ¡div className=”text-center”¿ ¡Image alt=”diagram” src=”/static/images/com/liealg/affine/1.png” width=180 height=100 /¿ ¡/Image¿

Triangular decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, $L\mathfrak{g} = Ln^- \oplus L\mathfrak{h} \oplus L\mathfrak{n}^+$, and $\hat{\mathfrak{g}} = L\mathfrak{n}^- \oplus \langle L\mathfrak{h}, c\rangle \oplus L\mathfrak{n}^+$.

The subalgebra $\langle L\mathfrak{b}, c\rangle$ is not abelian. $[h_i^{(m)}, h_i^{-m}] = (h_i|h_i)mc$ where $(h_i|h_i) \neq 0$ is the ∞-dim Heisenberg algebra, the ∞-dimensional analogue of the Cartan subalgebra. Two important properties:

- Nilpotent: $[H_1, [H_2, H_3]] = 0 \forall H_1, H_2, H_3 \in \langle L\mathfrak{h}, c\rangle$.
- Self-normalizing: if $X \in \hat{\mathfrak{g}}$ satisfies $[X, \langle L\mathfrak{h}, c\rangle] \subseteq \langle L\mathfrak{h}, c\rangle$, then $X \in \langle L\mathfrak{h}, c\rangle$.

Exercise 17.21. *Prove this second property using loop-triangular decomposition.*

More important for us now: ”affine Cartan subalgebra,” where $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$ is finite dimensional and abelian.

By Schur’s lemma, c acts in any irreducible $\hat{\mathfrak{g}}$-module as multiplication by a scalar.

If V is any $\hat{\mathfrak{g}}$-module such that $\exists k \in C \forall v \in V$, $c \cdot v = kv$. Then we call $k = k_V$ the **level** of V. $k_V = 0 \iff V$ is also a $L\mathfrak{g}$-module.

By surjectivity of $\tau : \hat{\mathfrak{g}} \to L\mathfrak{g}$, irreducible $L\mathfrak{g}$-modules are irreducible $\hat{\mathfrak{g}}$-modules. But $\exists$ many important irreducible ∞-dimensinoal $\hat{\mathfrak{g}}$-modules with level nonzero. (”basic representation” in terms of ”vertex operators”).

17.4 Finite Dimensional $\hat{\mathfrak{g}}$-modules

Goal: show finite dimensional $\hat{\mathfrak{g}}$-modules V have $k_V = 0$. This is similar to the lemma that stated all finite dimensional irreducible $L\mathfrak{g}$-modules are loop highest weight.

Consider $V^0 = \{v \in V | L\mathfrak{n}^+ \cdot v = 0\}$ is nonzero. Next, $\hat{\mathfrak{h}} = \bigoplus_{i=1}^r \mathbb{C}h_i \oplus \mathbb{C}c$ preserves V^0. For $v \in V^0$, we have

$$e_j^{(n)} \cdot (h_i \cdot v) = (h_i - a_{ij}) e_j^{(n)} \cdot v = 0$$
$$e_j^{(n)} \cdot (c \cdot v) = c \cdot e_j^{(n)} \cdot v = 0$$

Because $\hat{\mathfrak{h}}$ abelian, $\exists$ common joint eigenvector $v \in V^0$: $h_i \cdot v = \lambda(h_i)v, c \cdot v = kv, L\mathfrak{n}^+ \cdot v = 0$ for $\lambda \in \mathfrak{h}^*, k \in \mathbb{C}$.

First, assume V is irreducible. Because $V = \mathcal{U}(\hat{\mathfrak{g}}) \cdot v$, it suffices to show $k = 0$.

Lemma 17.22. *For all $1 \leq i \leq r, m \in \mathbb{Z}$, the subspace $\mathfrak{sl}_{2,i}^{(m)} := \mathbb{C}e_i^{(m)} \oplus \mathbb{C}f_i^{(-m)} \oplus \mathbb{C}(h_i + mc) \subseteq \hat{\mathfrak{g}}$ is a Lie algebra isomorphism to $\mathfrak{sl}_2$.*

Exercise 17.23. *Prove this.*

Let $1 \leq i \leq r, m \in \mathbb{Z}$ be arbitrary. We have $e_i^{(m)} \cdot v = 0$ for $v \in V^0$ and $(h_i + mc) \cdot v = (\lambda(h_i) + mk)v$. Consider $\mathcal{U}(\mathfrak{sl}_{2,i}^{(m)}) \cdot v$, a finite dimensional $\mathfrak{sl}_{2,i}^{(m)}$-submodule of V. It has irreducible submoudle W. By explicit description of highest weight $\mathfrak{sl}_2$-modules, W is a highest weight module, with highest weight $\lambda(h_i) + mk - 2s \in \mathbb{Z}_{\geq 0}$ where $s \in \mathbb{Z}_{\geq 0}$, which implies $\lambda(h_i) + mk \geq 0$.

Now let $m \to -\infty$. If $k \neq 0$, get contradiction. Hence $c \cdot V = 0$ if V is irreducible, $\dim V < \infty$. This implies V is an irreducible finite dimensional $L\mathfrak{g}$-module, so V is a tensor production of evaluation modules.

Remark 17.24. *There is a general result that says any finite dimensinoal highest weight $\mathfrak{g}$-module is irreducible. (consequence of complete reducibility for finite dimensional $\mathfrak{g}$-modules)*

Let's review some $\mathfrak{g}$-theory.

Let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ where $\alpha \in \mathfrak{h}^*$. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ gives a $\mathfrak{h}^*$ grading on $\mathfrak{g}$. THe root system is $\Phi := \{\alpha \in \mathfrak{h}^* | \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$. Additionally, we have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\alpha \in \Phi \implies -\alpha \in \Phi, \dim \mathfrak{g}_\alpha = 1$ for $\alpha \in \Phi$.

Example 17.25. *For $\mathfrak{g} = \mathfrak{sl}_3$, we have*

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where $\mathbb{C}[e_1, e_2] = \mathfrak{g}_{\alpha_1+\alpha_2} = \mathfrak{g}_\theta, \mathbb{C}e_1 = \mathfrak{g}_{\alpha_1}, \mathbb{C}e_2 = \mathfrak{g}_{a_2}, \mathbb{C}h_1 \oplus \mathbb{C}h_2 = \mathfrak{h}$, $\searrow = ad(f_1)$ and $\swarrow = ad(f_2)$. ω corresponds to rotating by $180°$, and $e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. ¡/Image¿

Let $(\cdot|\cdot)$ be any nonzero scalar multiple of Killing form. Then $(\cdot|\cdot)$ is nondegenerate on $\mathfrak{h}\times\mathfrak{h}$ and on $\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}$ where $\alpha\in\Phi$. We can define a linear isomorphism $\nu:\mathfrak{h}\to\mathfrak{h}^*$ by $\nu(h)(h')=(h|h')$.

There exists a unique involutive automorphism $\omega:\mathfrak{g}\to\mathfrak{g}$ ($X\mapsto -X^T$) with $\omega(e_i)=-f_i,\omega(f_i)=-e_i,\omega(h_i)=-h_i,\omega(\mathfrak{g}_\alpha)=\mathfrak{g}_{-\alpha}$, and $\omega(\mathfrak{h})=\mathfrak{h}$.

Let θ be the highest root. We can normalize $(\cdot|\cdot)$ such that $\exists e_\theta\in\mathfrak{g}_\theta, f_\theta\in\mathfrak{g}_{-\theta}, h_\theta\in\mathfrak{h}$ with the nice property that $(e_\theta|h_\theta)=1,(h_\theta|h_\theta)=2,[e_\theta,f_\theta]=h_\theta,[h_\theta,e_\theta]=2e_\theta,[h_\theta,f_\theta]=-2f_\theta,\omega(e_\theta)=-f_\theta,\omega(h_\theta)=-h_\theta$.

Example 17.26. *$e_\theta=[e_1,e_2],f_\theta=-\omega(e_\theta)=[f_2,f_2],h_\theta=[e_\theta,f_\theta]=...=h_1+h_2$. The $\mathfrak{sl}_3$ relations are $[h_i,e_i]=2e_i,[h_i,f_i]=-2f_i,[e_i,f_i]=h_i,[h_i,e_j]=-e_j,[h_i,f_j]=f_j,[e_i,f_j]=0$ for $i\neq j$. We also have*

$$\begin{aligned}\kappa(e_\theta,f_\theta)&=\kappa(e_1,[e_2,[f_2,f_1]])\\&=\kappa(e_1,[f_2,f_1])\\&=\kappa(e_1,[h_2,f_1])\\&=\kappa(e_1,f_1)\\&=\kappa(e_2,f_2)\end{aligned}$$

and $\kappa(h_\theta,h_\theta)=\kappa(h_1,h_1)+2\kappa(h_1,h_2)+\kappa(h_2,h_2)$. So we can infer that $\kappa(h_1,h_1)=\kappa(h_1,[e_1,f_1])=\kappa([h_1,e_1],h)=2\kappa(e_1,f_1)=2\kappa(e_2,f_2)=k(h_1,h_2)$ so $\kappa(h_1,h_2)=...=-\kappa(e_2,f_2)=-\kappa(e_1,f_2)\implies\kappa(h_\theta,h_\theta)=2\kappa(e_1,f_1)\implies(e_\theta,f_\theta)=1,(h_\theta,h_\theta)=2$.

Now define the elements $e_0=f_\theta\otimes z\in L\mathfrak{n}^-,f_0=e_\theta\otimes z^{-1}\in L\mathfrak{n}^+,h_0=c\cdot h_0\in\hat{\mathfrak{h}}$.

Exercise 17.27. *Show that $\mathfrak{sl}_{2,0}:=\mathbb{C}e_0\oplus\mathbb{C}f_0\oplus\mathbb{C}h_0\subset\hat{\mathfrak{g}}$ is a Lie algebra, isomorphic to $\mathfrak{sl}_2$.*

Note $\hat{\mathfrak{h}}=\bigoplus_{i=0}^r\mathbb{C}h_i$. For $0\leq i\leq r$ consider $\mathfrak{sl}_{2,i}=\mathbb{C}e_i\oplus\mathbb{C}f_i\oplus\mathbb{C}h_i$. By complete $\mathfrak{sl}_2$-reducibility, V ($\dim V<\infty,\hat{\mathfrak{g}}$ module) decomposes as a direct sum of $\mathfrak{sl}_{2,i}$-irreducible modules. So V is a $\mathfrak{sl}_{2,i}$-weight module: $V=\bigoplus_{\lambda\in(\mathbb{C}h_i)^*}V_\lambda$ where $V_\lambda=\{v\in V|h_i\cdot v=\lambda(h_i)v\}$. This tells us the action of h_i on V is diagonalizable $\forall 0\leq i\leq r$ so the action of $c=h_0+h_\theta$ is diagonalizable.

Let $W\subseteq V$ be an irreducible submodule ($\dim W\geq 1$) $c\cdot W=0$. Then V/W is finite dimensional $\hat{\mathfrak{g}}$-module, with $\dim(V/W)<\dim V$. For suitable basis, c acts on V as

$$\begin{pmatrix}\textit{action on } W & 0\\ 0 & \textit{action on } V/W\end{pmatrix}.$$

As vector spaces, $V\cong W\oplus V/W$. By induction on $\dim V$ we get $c\cdot V=0$.

17.5 Towards Kac-Moody Algebras

Proposition 17.28. *The elements $e_0,e_1,...,e_r,f_0,f_1,...,f_r\in\hat{\mathfrak{g}}$ generate $\hat{\mathfrak{g}}=L\mathfrak{g}\oplus\mathbb{C}c$.*

Proof. $[h_i] = [e_i, f_i], c \in \bigoplus_{i=0}^{r} \mathbb{C}h_i$. It suffices to prove that $\hat{\mathfrak{n}}^+ := \mathbb{C}\langle e_0, e_1, ..., e_r\rangle$ equals $\mathfrak{n}^+ \oplus g \otimes z\mathbb{C}[z]$ and $\hat{\mathfrak{n}}^- := \mathbb{C}\langle f_0, f_1, ..., f_r\rangle$ equals $\mathfrak{n}^- \oplus \mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}]$. The proofs are similar, we will focus on $\hat{\mathfrak{n}^-}$.

By induction with respect to degree in F^{-1}. Note that $e_\theta \in \mathfrak{g}$ is the highest weight vector for adjoint action of $\mathfrak{g}$. Hence for all $x \in \mathfrak{g}, \exists Y \in \mathcal{U}(\mathfrak{n}^-) : x = Y \cdot e_0$ and $x^{(-1)} = Y \cdot f_0$. In particular, $\mathfrak{h} \otimes z^{-1} \subseteq \mathcal{U}(\mathfrak{n}^-) \cdot f_0$ and $e_\theta \otimes z^{-2} \subseteq \mathcal{U}(\hat{\mathfrak{n}}^-) \cdot f_0$.

For each $m \in \mathbb{Z}_{>0}$, apply $\mathcal{U}(\mathfrak{n}^-) \subset \mathcal{U}(\hat{\mathfrak{n}}^-)$ to generate $\mathfrak{g} \otimes z^{-m}$ and apply $f_0 = e_\theta \otimes z^{-1} \in \mathcal{U}(\hat{n}^-)$ go from $\mathfrak{h} \otimes z^{-m}$ to $e_\theta \otimes z^{-(m+1)}$. □

Remark 17.29. *Let V be an irreducible finite dimensional $\hat{\mathfrak{g}}$-module. Then V is irreducible as $\hat{\mathfrak{n}}^+$ module (or $\hat{\mathfrak{n}}^-$ module). False if you replace $(\hat{\mathfrak{g}}, \hat{\mathfrak{n}}^+)$ by $(\mathfrak{g}, \mathfrak{n}^+)$.*

Example 17.30. *Take $\mathfrak{g} = \mathfrak{sl}_2$ and the $\hat{\mathfrak{g}}$-module $V = ev_a(\mathbb{C}^2) \otimes ev_b(\mathbb{C}^2)$ with $a, b \in \mathbb{C}^\times, a \neq b$. The action of $\hat{\mathfrak{n}}^-$ is: ¡div className=”text-center”¿ ¡Image alt=”diagram” src=”/static/images/com/liealg/affine/3.png” width=600 height=200 /¿ ¡/Image¿*

The (left), and the (top right bottom) form the $\mathfrak{sl}_{2,0}$ module structure, and the (right) and the (top left bottom) form the $\mathfrak{sl}_{2,1}$-module structure.

To show that V is $\hat{\mathfrak{n}}^-$ irreducible, we need to show that $\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$ is cocylic with respect to $\hat{\mathfrak{n}}$-action. ¡/Image¿

Exercise 17.31. *Show this.*

The Kac-Moody triangular decomposition is $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+$, where $\hat{\mathfrak{n}}^- = \langle f_0, f_1, ..., f_r\rangle, \hat{\mathfrak{h}} = \langle h_0, h_1, ..., h_r\rangle, \hat{\mathfrak{n}}^+ = \langle e_0, e_1, ..., e_r\rangle$ are abelian, $\dim \hat{\mathfrak{h}} = r + 1$, and $h_i = [e_i, f_i]$.

Warning: this is different from the loop-triangular decomposition.

Example 17.32 ($\hat{\mathfrak{sl}_2}$)**.** *$\hat{\mathfrak{n}^+}$ contains $f^{(2)}, f^{(1)} = e_0, h^{(2)}, h^{(1)}, e^{(2)}, e^{(1)}, e^{(0)} = e_1$, $\hat{\mathfrak{h}}$ contains h_0 and h_1, and $\hat{\mathfrak{n}}$ contains everything else.*

Given $\alpha_1 \in \mathfrak{h}^+ \implies \alpha_1(h_1) = 2$, extend to an element of $\left(\hat{\mathfrak{h}}\right)^*$ by $\alpha_1(h_0) = -2$.

We have $[h_0, e_1] = [c-h_1, e_1] = -2e_1$ and similarly $[H-0, f_1] = 2f_1$. In general, we can show that $[h_1, e^{(m)}] = 2e^{(m)}$ and $[h_0, e^{(m)}] = -2e^{(m)}$. We can also define $\alpha_0 \in \hat{\mathfrak{h}}^*$ by $\alpha_0(h_0) = 2, \alpha_0(h_1) = -2$ to get $[h_0, e_0] = [c - h_1, f_1 \otimes 2] = 2e_0$ and $[h_0, e_1] = [c - h_1, e_1 \otimes 1] = -2e_1$.]

Question: how to distinguish between the $e(m)$ where $m \in \mathbb{Z}$ using an adjoint action?

Before we define the extension, let's write down some new relations involving e_0, f_0, f_0: $[h_0, e_0] = 2e_0$, $[h_0, f_0] = -2f_0$, and $[e_0, f_0] = h_0$.

Let $1 \leq i \leq r$. We get

$$[e_i, f_0] = [e_i \otimes 1, e_\theta \otimes z^{-1}] = [e_i, e_\theta] \otimes z^{-1} = 0$$

(similar to $[e_i, f_j] = 0$ for $1 \leq i, j \leq r$ in $\mathfrak{g}$).

Similarly $[f_i, e_0] = 0$ and

$$\begin{aligned}[h_i, f_0] &= [h_i \otimes 1, e_\theta \otimes z^{-1}] \\ &= [h_1, e_\theta] \otimes z^{-1} \\ &= \theta(h_i) e_\theta \otimes z^{-1} \\ &= \theta(h_i) f_0.\end{aligned}$$

Similarly $[h_i, e_0] = -\theta(h_i) e_0$.

Lemma 17.33. *$\theta(h_i) \in \mathbb{Z}_{\geq 0}$.*

Remark 17.34. *So these relations are also similar to $\mathfrak{g}$-relations $[h_i, e_j] = a_{ij} e_j$ for $a_{ij} \in \mathbb{Z}_{\leq 0}$.*

Proof. The Weyl group of $\mathfrak{g}$ acts on $\mathfrak{h}^*$ via $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. This preserves the root system $\Phi \subset \mathfrak{h}^*$. Also $\Phi = \Phi^+ \cup (-\Phi^+)$, where Φ^+ contain the roots for $\mathfrak{n}^+$, $(-\Phi^+)$ contain the roots for $\mathfrak{n}^-$, and $\Phi^+ = \Phi \cap \text{span}_{\mathbb{Z}_{\geq 0}}(\alpha_1, ..., \alpha_r)$.

We have $s_i(\theta) = \theta - \theta(h_i)\alpha_i \in \Phi$. If $\theta(h_i) \notin \mathbb{Z}, s_i(\theta) \notin \Phi$. If $\theta(h_i) < 0$, then $s_i(\theta)$ would be higher than θ. □

Finally,

$$\begin{aligned}[h_0, e_i] &= [c - h_\theta, e_i] = -\alpha_i(h_\theta) e_i \\ [h_0, f_i] &= [c - h_\theta, f_i] = -\alpha_i(h_\theta) f_i.\end{aligned}$$

Lemma 17.35. *$\alpha_i(h_\theta) \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq r$.*

Proof. Analogous to the previous one, use Weyl group action on $\mathfrak{h}$. □

Intuitively, the "new" relations for $e_0, ..., e_r, h_0, ..., h_r, f_0, ..., f_r$ are close to the relations for $\mathfrak{g}$ itself (Chevalley-Serre presentation).

To make the extension, note:

Lemma 17.36. *The linear map $z\frac{d}{dz} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, defined by $z\frac{d}{dz}(x^{(m)}) = mx^{(m)}, z\frac{d}{dz}(c) = 0$ is a derivation. It is not inner: $z\frac{d}{dz} \neq ad(y), y \in \hat{\mathfrak{g}}$.*

Proof. To show Leibniz rule, the only nontrivial check is:

$$z\frac{d}{dz}([x^{(m)}, y^{(n)}]) = [z\frac{d}{dz}(x^{(m)}), y^{(n)}] + [x^{(m)}, z\frac{d}{dz}(y^{(n)})].$$

□

Exercise 17.37. *Prove this relation.*

Suppose $ad(y) = z\frac{d}{dz}$ *for* $y \in \hat{\mathfrak{g}}$. $ad(c) = 0$, *so* $y \in L\mathfrak{g}$. *So* $y \in \sum_{\alpha \in \Phi \cup \{0\}} \mathfrak{g}_\alpha \otimes \mathbb{C}[z, z^{-1}]$.

Because of the Killing form, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha \forall \alpha \in \Phi, h_\alpha \in \mathfrak{h}\backslash\{0\}$. *Hence,i f y has a nonzero component in* $L(\mathfrak{g}_\alpha)$ *with* $\alpha \in \Phi$, $[y, e_{-\alpha}]$ *has a nonzero component in* $L\mathfrak{h}$. *But* $z\frac{d}{dz}e_{-\alpha} = 0 \implies y \in L\mathfrak{h}$.

Note: $z\frac{d}{dz}h_i \otimes z^m = mh_i \otimes z^m$, *but* $[y, h_i \otimes z^m] \in \mathbb{C}c \implies$ *this element y does not exist.*

17.6 The Extended Affine Lie Algebra

Let's make $z\frac{d}{dz}$ *"inner."*

Definition 17.38. *Let* $\hat{\mathfrak{g}}^{ext} := \hat{\mathfrak{g}} \oplus \mathbb{C}d$ *(as vector spaces). The Lie algebra relations are defined as follows:* $\hat{\mathfrak{g}}$ *is a subalgebra,* $[d, X] = z\frac{d}{dz}(X)$ *for* $X \in \hat{\mathfrak{g}}$.

Here are some basic properties of $\hat{\mathfrak{g}}^{ext}$*:*

1. *Example of extension by derivarion, ie. a short exact sequence* $0 \to \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}^{ext} \to \mathbb{C}d \to 0$ *where d acts on* $\hat{\mathfrak{g}}$ *as a derivation.* $\hat{\mathfrak{g}}$ *is an ideal,* $\mathbb{C}d$ *is a subalgebra, so the sequence splits.*
2. *Generating set:* $e_0, e_1, ..., e_r, f_0, f_1, ..., f_r, d$.
3. *Kac-Moody triangular decomposition:* $\hat{\mathfrak{g}}^{ext} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}}^{ext} \oplus \hat{\mathfrak{n}}^+$ *(decomposition as* $\hat{\mathfrak{g}}^{ext}$*-modules) where* $\hat{\mathfrak{h}}^{ext} = \hat{\mathfrak{h}} \oplus \mathbb{C}d$ *is abelian and of dimension* $r + 2$.

Exercise 17.39. *Using this decomposition, show that* $Z(\hat{\mathfrak{g}}^{ext}) = \mathbb{C}c, [\hat{\mathfrak{g}}^{ext}, \hat{\mathfrak{g}}^{ext}] = \hat{\mathfrak{g}}$ *derived subaglgebra.*

Remark 17.40. *The center is "small" and the derived subalgebra is "large" so this is "close" to a simple Lie algebra.*

Later, we will see that $Z(\hat{\mathfrak{g}}^{ext})$, $[\hat{\mathfrak{g}}^{ext}, \hat{\mathfrak{g}}^{ext}]$ *are the only nonzero ideals.*

$\alpha \in \mathfrak{h}^*$ *extends to element of* $\left(\hat{\mathfrak{h}}^{ext}\right)^*$ *by setting* $\alpha(c) = \alpha(d) = 0 \implies \alpha_i(d) = 0$ *for* $1 \leq i \leq r$, *which is compatible with* $[c, x] = 0 = [d, x]$ *if* $x \in \mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$. *Now define* $\delta \in \left(\hat{\mathfrak{h}}^{ext}\right)^*$ *by* $\delta(X) = 0$ *if* $X \in \hat{\mathfrak{h}}$, *otherwise* $\delta(d) = 1$.

Now we define $\alpha_0 = \delta - \theta \in \left(\hat{\mathfrak{h}}^{ext}\right)^* \implies \alpha_0(d) : \delta(d) - \theta(d)$. *For* $1 \leq i \leq r$, *we get*

$$[d, e_0] = z\frac{d}{dz}e_0 = e_0 = \alpha_0(d)e_0$$

$$[d, e_i] = z\frac{d}{dz}e_i = 0 = 0 = \alpha_i(d)e_i.$$

Similar for $f_0, f_1, ..., f_r$.

This implies $\hat{\mathfrak{g}}^{ext} = bigoplus_{\lambda\in\left(\hat{\mathfrak{h}}^{ext}\right)^*}\hat{\mathfrak{g}}^{ext}_\lambda$*, where* $\hat{\mathfrak{g}}^{ext}_\lambda = \{X \in \hat{\mathfrak{g}}^{ext} | \forall H \in \hat{\mathfrak{h}}^{ext}, [H,X] = \lambda(H)X\}$ *and* $\hat{\mathfrak{g}}^{ext}_0 = \hat{\mathfrak{h}}^{ext}$.

Example 17.41. $\hat{\mathfrak{sl}_2}^{ext}$ *is really similar to the previous version.*

Lemma 17.42. *The weight space decomposition of* $\hat{\mathfrak{g}}^{ext}$ *with respect to* $\hat{\mathfrak{h}}^{ext}$ *is:*

$$\hat{\mathfrak{g}}^{ext} = \hat{\mathfrak{h}}^{ext} \oplus \bigoplus_{\substack{m\in\mathbb{Z},\\ \alpha\in\hat{\mathfrak{h}},\\ (m,\alpha)\neq(0,0)}} \hat{\mathfrak{h}}^{ext}_{\alpha+m\delta}.$$

where $\hat{\mathfrak{g}}^{ext}_0 = \hat{\mathfrak{h}}^{ext}$ *and* $\mathfrak{g}_\alpha \otimes z^m = \hat{\mathfrak{h}}^{ext}_{\alpha+m\delta}$.

Proof. By construction, it holds as linear spaces:

$$L\mathfrak{g} = \bigoplus_{\substack{m\in\mathbb{Z},\\ \alpha\in\hat{\mathfrak{h}}}} \mathfrak{g}_\alpha \otimes z^m.$$

Now add $\mathbb{C}c$ and $\mathbb{C}d$ to $\mathfrak{g}_0 \otimes z^0$ to obtain $\hat{\mathfrak{h}}^{\text{ext}}$. Moreover, let $H = h + \lambda c + \mu d \in \hat{\mathfrak{h}}^{\text{ext}}$ where $\delta(H) = \mu, \alpha(H) = \alpha(h), \lambda, \mu \in \mathbb{C}$, and $h \in \mathfrak{h}$. Suppose $x \in \mathfrak{g}_\alpha$ where $\alpha \in \mathfrak{h}^*$. Then

$$\begin{aligned}
[H, x\otimes z^m] &= [h, x\otimes z^m] + \mu[d, x\otimes z^m] \\
&= [h,x]\otimes z^m + \mu m x\otimes z^m \\
&= (\alpha + m\delta)(H)x\otimes z^m
\end{aligned}$$

which is in $\left(\hat{\mathfrak{h}}^{\text{ext}}\right)^*$. ☐

Lemma 17.43. *Let* $\mathfrak{d}$ *be a finite dimensional abelian Lie algebra over* $\mathbb{C}$*. Let* V *be an* $\mathfrak{d}$*-weight module:*

$$V = \bigoplus_{\lambda\in\mathfrak{d}^*} V_\lambda$$

where $V_\lambda = \{v \in V | \forall x \in \mathfrak{d} : x \cdot v = \lambda(x)v\}$*. Then any submodule of* V *is also an* $\mathfrak{d}$*-weight module.*

Proof. $V = \bigoplus_{\lambda\in\mathfrak{d}^*} V_\lambda$. Let $W \subseteq V$ be a submodule. Let $w \in W$ be arbitrary. Then $w = w_1 + ... + w_n$ with $w_i \in V_{\lambda_i}\backslash\{0\}$ for distinct $\lambda_i \in \mathfrak{d}^*$. Need to show: $w_i \in W$ ($\implies$ $W = \bigoplus_{\lambda\in\mathfrak{d}^*} W_\lambda$, where $W_\lambda = W \cap V_\lambda$ for $1 \leq i,j \leq n$ and $i \neq j$).

Let $\mathfrak{d}_{ij} = \{x \in \mathfrak{d} | \lambda_i(x) = \lambda_j(x)\}$. This is a proper subspace of $\mathfrak{d}$. Because $\dim(\mathfrak{d}) > \infty, \mathfrak{d} \neq \bigcup_{i\neq j} \mathfrak{d}_{ij}$. Hence $\exists x \in \mathfrak{d}, \lambda_i(x) \neq \lambda_j(x)$ for all distinct i,j. Set

$\ell_i = \lambda_i(x)$. Via the $\mathcal{U}(\mathfrak{d})$-action on W, get

$$\begin{aligned} w &= w_1 + ... + w_n \\ x \cdot w &= \ell_1 w_1 + ... + \ell_n w_n \\ x^2 \cdot w &= \ell_1^2 w_1 + ... + \ell_n^2 w_n \\ &... \\ x^{n-1} \cdot w &= \ell_1^{n-1} w_1 + ... + \ell_n^{n-1} w_n \end{aligned}$$

which arises the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \ell_1 & \ell_2 & \cdots & \ell_n \\ \vdots & \vdots & \ddots & \vdots \\ \ell_1^{n-1} & \ell_2^{n-1} & \cdots & \ell_n^{n-1} \end{pmatrix}.$$

This has nonzero determinant. So $w_1, ..., w_n$ are linear combinations of $w, x \cdot w, x^2 \cdot w, ..., x^{n-1} \cdot w \in W \implies w_1, ..., w_n \in W$. $\square$

So

$$\{\textit{ideals of } \hat{\mathfrak{g}}^{ext}\} = \{\textit{submodules of adjoint action}\}.$$

Proposition 17.44. *All nonzero ideals of $\hat{\mathfrak{g}}^{ext}$ contain a nonzero element of $\hat{\mathfrak{h}}^{ext}$.*

Proof. Let $i \subseteq \hat{\mathfrak{g}}^{\text{ext}}$ be an ideal such that $\hat{\mathfrak{h}}^{\text{ext}} = 0$. We need to show $i = 0$. Assume $i \neq 0$. Combining the previous two lemmas:

$$i = \left(i \cap \hat{\mathfrak{h}}^{\text{ext}}\right) \oplus \bigoplus_{\substack{m \in \mathbb{Z}, \\ \alpha \in \hat{\mathfrak{h}}, \\ (m,\alpha) \neq (0,0)}} \left(i \cap \hat{\mathfrak{g}}^{\text{ext}}_{\alpha + m\delta}\right).$$

$\exists (\alpha, m) \in (\hat{\mathfrak{h}}^* \times \mathbb{Z}) \backslash \{(0,0)\}$ such that $i \cap \hat{\mathfrak{g}}^{\text{ext}}_{\alpha + n\delta} \neq 0$. Let $x \otimes z^m \in i$ for $x \in \mathfrak{g}_\alpha \backslash \{0\}$.

By nondegeneracy of $(\cdot|\cdot)|_{\mathfrak{g} \times \mathfrak{g}}, \exists y \in \mathfrak{g}_{-\alpha}$ such that $(x|y) \neq 0 \implies [x, y] \in \mathfrak{h} \backslash \{0\}$. Hence $(x \otimes z^m | y \otimes z^{-m}) = (x|y) \neq 0$, the pairing in $\hat{\mathfrak{g}}$.

So $[x \otimes z^m, y \otimes z^{-m}] = [x, y] + (x|y)mc$ in $\hat{\mathfrak{g}} \subset \hat{\mathfrak{g}}^{\text{ext}}$ because $i \cap$
$hat\mathfrak{h}^{\text{ext}} = 0$. Because $\hat{\mathfrak{h}}^{\text{ext}} =$
$hat\mathfrak{h} \oplus \mathbb{C}d = \hat{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ as vector spaces, and we get a contradiction. $\square$

Now, we can essentially conclude that $\hat{\mathfrak{g}}^{ext}$ is a so-called Kac-Moody algebra.

So Kac-Moody algebras are a subset of the simple finite dimensional algebras; the finite dimensional reps are well-understood. Additionally, they are also a subset of extended affine Lie algebras, where $\exists$ interesting highest weight reps of ∞ dimension.

There are two important classes of reps of $\hat{\mathfrak{g}}$:

1. *The reps of $L\mathfrak{g}$*
2. *The reps of $\hat{\mathfrak{g}}^{ext}$.*

18 Kac-Moody Lie Algebras

We will now work more abstractly, with a large class of Lie algebras with uniform definition: $\mathfrak{g}$ arises a Cartan matrix, and we can recover $\mathfrak{g}$ using the Chevalley-Serre presentation. We will generalize the $A \to \mathfrak{g}$ direction.

18.1 Realizations of Square Matrices

Let I be a finite set (ie. $\{1, 2, ..., r\}$ for finite case, $\{0, 1, ..., r\}$ for affine case), which we will call the ***index set****. Let $A = (a_{ij})_{i,j \in I}$ be a square matrix with entries in $\mathbb{C}$. If $J \subseteq I$ is a subset, $A_J = (a_{ij})_{i,j \in J}$ is the* ***principal submatrix****.*

Definition 18.1. *A (minimal)* ***realization*** *of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that*

- *$\hat{\mathfrak{h}}$ is a $\mathbb{C}$-linear space of dimension $\dim(\mathfrak{h}) = 2|I| - rank(A) = |I| + corank(A)$.*
- *$\Pi = \{\alpha_i\}_{i \in I}$ is $L_0 I_0$ subset of $\mathfrak{h}^*$, where the elements are "simple roots" formed by the "root basis"*
- *$\Pi^\vee = \{h_i\}_{i \in I}$ is $L_0 I_0$ subset of $\mathfrak{h}$, where the elements are the "simple coroots" formed by the "coroot basis"*
- *$\alpha_j(h_i) = a_{ij} \forall i, j \in I$.*

Note: $\det A \neq 0 \iff rank(A) = |I| \iff span(\Pi) = \mathfrak{h}^* \iff span(\Pi^\vee) = \mathfrak{h}$.

Two realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ ***isomorphic*** *if $\exists$ linear isomorphism $\varphi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $\varphi(\Pi_1^\vee) = \Pi_2^\vee, \varphi^*(\Pi_2) = \Pi_1$.*

Proposition 18.2. *Every A has a realization, unique up to isomorphism. The realizations of A_1 and A_2 are isomorphic $\iff A_1, A_2$ have the same index set I and are related by permutations of I.*

Proof. Let $r = \text{rank}(A), n = |I|$. A has an invertible submatrix A_J where $|J| = r$. Map I bijectively to $\{1, 2, ..., n\}$. Reordering rows and columns, we get matrix $\tilde{A} = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ where the top left block is or dimension $r \times r$, the bottom right is of dimension $(n-r) \times (n-r)$, and $\det(A_{11}) \neq 0$. Let $C = \left(\begin{array}{cc|c} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & \text{Id}_{n-r} \\ \hline 0 & \text{Id}_{n-r} & 0 \end{array}\right)$, which has $\det C = \pm\det(A_{11}) \neq 0$. Set $\mathfrak{h} = \mathbb{C}^{2n-r}$.

For $1 \leq i \leq n$, define h_i to be the i-th row of $C \in \mathfrak{h}$, and define $\alpha_i \in \mathfrak{h}^*$ by $\alpha_i(x_1, ..., x_{2n-r}) = x_i$. We can now check that this defines a realization of $\tilde{A}$. Now apply the inverse map $\{1, 2, ..., n\} \to I$ to get a realization of A. $\square$

Exercise 18.3. *Show uniqueness statements.*

Given A_1, A_2 with realizations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$, $A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$ has the realization

$$(\mathfrak{h}_1 \oplus \mathfrak{h}_2, (\Pi_1 \times \{0\}) \cup (\{0\} \times \Pi_2), (\Pi_1^\vee \times \{0\}) \cup (\{0\} \times \Pi_2^\vee))$$

*Call A **decomposable** if (after permuting I) $A = A_J \oplus A_{I \setminus J}$ where $\subsetneq J \subsetneq I$. Call A **indecomposable** otherwise. Two decomposable examples are $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and two indecomposable examples are $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$.*

Set $Q = span_{\mathbb{Z}}\Pi = \{\sum_{i \in I} k_i \alpha_i | k_i \in \mathbb{Z}\}$ the root lattice and $Q^+ = span_{\mathbb{Z}}\Pi = \{\sum_{i \in I} k_i \alpha_i | k_i \in \mathbb{Z}_{\geq 0}\}$. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q$, define $ht(\alpha) = \sum_{i \in I} k_i \alpha_i$ the height. The partial ordering of $\mathfrak{h}^*$ is given by: $\lambda \geq \mu :\iff \lambda - \mu \in Q^+$, $\lambda > \mu :\iff \lambda \geq \mu$ and $\lambda \neq \mu$.

18.2 Auxiliary Lie Algebra

Given $A = (a_{ij})_{i,j \in I}$ and realization $(\mathfrak{h}, \Pi, \Pi^\vee)$, define $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{g}}(A, \mathfrak{h}, \Pi, \Pi^\vee)$ as follows:

- *Generators: $\{e_i, f_i\}_{i \in I}, \mathfrak{h}$, the **Chevalley generators**.*
- *Relations: $[e_i, f_j] = \delta_{ij} h_i, [h, h'] = 0, [h, e_i] = \alpha_i(h) e_i$, and $[h_i, f_i] = -\alpha_i(h) f_i$ where $i, j \in I, h, h' \in \mathfrak{h}, i \in I$.*

Remark 18.4.

1. *For now, there are no Serre relations like $ad(e_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$.*
2. *Up to isomorphism, $\tilde{\mathfrak{g}}$ only depends on A, not on $(\mathfrak{h}, \Pi, \Pi^\vee)$.*
3. *Later on, we make more assumptions on A.*
4. *Involutive Lie algebra automorphism $\tilde{\omega} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is given by $\tilde{\omega}(e_i) = -f_i, \tilde{\omega}(f_i) = -e_i, \tilde{\omega}(h) = -h$ for $i \in I, h \in \mathfrak{h}$.*

Set $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, I = \{0, 1\}, \dim(\mathfrak{h}) = 3, \Pi^\vee = \{h_0, h_1\}, \Pi = \Pi = \{\alpha_0, \alpha_1\} \subset \mathfrak{h}^*, \mathfrak{h} = \mathbb{C}h_0 \oplus \mathbb{C}h_1 \oplus \mathbb{C}d$ with $\alpha_0(h_0) = 2, \alpha_0(h_1) = -2, \alpha_0(d) = 1, \alpha_1(h_0) = -2, \alpha_1(h_1) = 2$, and $\alpha_1(d) = 0$. Let $\mathfrak{g}$ be a Lie algebra generated by $e_0, e_1, f_0, f_1, \mathfrak{h}$, with $\mathfrak{h} \subseteq$ as an abelian Lie subalgebra. Furthermore, we have relations $[e_i, f_j] = \delta_{ij} h_i, [h_0, e_0] = 2e_0, [h_0, e_1] = -2e_1, [d, e_0] = e_0, [d, e_1] = 0$ and similar relations for h_1.

Let's recover some relations of $\hat{\mathfrak{sl}}_2^{ext}$. We will see $\hat{\mathfrak{sl}}_2^{ext} \cong$ some quotient of $\mathfrak{g}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ of $\tilde{\mathfrak{g}}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Here are some basic properties:

Theorem 18.5. *Set $\tilde{\mathfrak{n}}^+ = \mathbb{C}\langle\{e_i\}_{i\in I}$, $\tilde{\mathfrak{n}}^- = \mathbb{C}\langle\{f_i\}_{i\in I}$.*

1. *$\tilde{\mathfrak{n}}^+$ are freely generated by the indicated generators.*
2. *Triangular decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+$ (as $\mathfrak{h}$-modules).*
3. *Root space decomposition $\tilde{\mathfrak{n}}^\pm = \bigoplus_{\alpha\in Q^+\setminus\{0\}}$, $V_\alpha = \{v \in V | h{\cdot}v = \alpha(h)v \forall h \in \mathfrak{h}\}$ for any $\mathfrak{h}$-module V, $\mathfrak{h} = \tilde{\mathfrak{g}}_0$, and $\tilde{\mathfrak{g}}_\alpha < \infty$ for all $\alpha \in Q$.*
4. *The set $\{i \subseteq \tilde{\mathfrak{g}}(A) | i$ ideal, $i \cap \mathfrak{h} = 0\}$ has unique maximal element $\mathfrak{r}$, and $\mathfrak{n} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}^+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}^-)$ (as ideals).*

Proof. We provide a sketch: Set $V = \text{span}_{\mathbb{C}}\{v_i\}_{i\in I}$. Given $\lambda \in \mathfrak{h}^*$, define $\tilde{\mathfrak{g}}$-rep on $T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus ....$

- $f_i \cdot a = v_i \otimes a$ for $a \in T(v), i \in I$.
- $h \cdot 1 = \lambda(h)$ for $h \in \mathfrak{h}, e_i \cdot 1 = 0$ for $i \in I$.
- By induction with respect to $s \in \mathbb{Z}_{>0}$, $h{\cdot}(v_j \otimes a) = -\alpha_j(h) v_j \otimes a + v_j \otimes (h{\cdot}a)$ and $e_i \cdot (v_j \otimes a) = \delta_{ij} h_i \cdot a + v_j \otimes (e_i \cdot a)$ for $a \in V^{\otimes(s-i)}, i, j \in I, h \in \mathfrak{h}$.

□

Exercise 18.6. *Prove it defines a rep of $\tilde{\mathfrak{g}}$ on $T(V)$.*

1. *Assignment $f_i \mapsto v_i$ $(i \in I)$ defines an algebra homomorphism $\mathcal{U}(\tilde{\mathfrak{n}}^-) \twoheadrightarrow T(V)$. $T(V)$ is free, and $\mathcal{U}(\hat{\mathfrak{n}}^-) \cong T(V)$. The PBW theorem gives $\tilde{\mathfrak{n}}^-$ is freely generated by $\{f_i\}_{i\in I}$. Use $\tilde{\omega}$ to go to $\tilde{\mathfrak{n}}^+$.*
2. *Use $\tilde{\mathfrak{g}}$-relations to see a Lie product of $s \in \mathbb{Z}_{\geq 0}$ elements of $\{e_i f_i\}_{i\in I} \cup \mathfrak{h}$ lies in $\tilde{\mathfrak{n}}^- + \mathfrak{h} + \tilde{\mathfrak{n}}^+$. Suppose $0 = x^- + h + x^+$ where $x^\pm \in \tilde{\mathfrak{n}}^+, h \in \mathfrak{h}$. We have $0 = (x^- + h + x^+) \cdot 1 = x^- \cdot 1 + \lambda(h) \implies \lambda(h) = 0$ for any $\lambda \in \mathfrak{h}^* \implies h = 0$. $T(V) \cong \mathcal{U}(\tilde{\mathfrak{n}}^-) \implies x^- \mapsto x^- \cdot 1$ is the canonical embedding $\tilde{\mathfrak{n}}^- \hookrightarrow \mathcal{U}(\tilde{n}^-) \implies x^- = 0 \implies x^+ = 0$. So the $\tilde{\mathfrak{g}}$-relations $\implies$ decomposition as $\mathfrak{h}$-modules.*
3. *Using $[h, e_i], [h, f_i]$ we get the root space decomposition for $\tilde{n}^\pm$. By combinatorial argument, $\dim \tilde{\mathfrak{g}}_{\pm\alpha} \leq |I|^{ht(\alpha)} < \infty$ for $\alpha \in Q^+\setminus\{0\}$. Additionally, we get $\dim(\mathfrak{h}) = |I| + rank(A) < \infty$, and $\mathfrak{h} \subseteq \tilde{\mathfrak{g}}_0$ is an equality by part 2.*
4. *$\mathfrak{h}$ is abelian and finite dimensional $\implies$ submodules of $\mathfrak{h}$-module $\tilde{\mathfrak{g}}$ have root space decomposition $\implies \forall i \subset \tilde{\mathfrak{g}}$ ideal $i = \bigoplus_{\alpha\in Q}(\tilde{\mathfrak{g}}(A)_\alpha \cap i)$. Set $\mathfrak{r} :=$ sum of all ideals i such that $i \cap \mathfrak{h} = 0$. Then $\mathfrak{r} \cap \mathfrak{h} = 0$ and $\mathfrak{r}$ is maximal. Part 2 impies $\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}^+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}^-)$ as $\mathfrak{h}$-modules.*

Example 18.7. *Show $[f_i, \mathfrak{r} \cap \tilde{\mathfrak{n}}^+] \subseteq \tilde{\mathfrak{n}}^+$, deduce $\mathfrak{r} \cap \tilde{\mathfrak{n}}^+$ is an ideal of $\hat{\mathfrak{g}}$.*

This implies $\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}^+) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}^-)$ *as ideals of* $\hat{\mathfrak{g}}$.

18.3 Kac-Moody Algebras

From part 4 of the previous theorem, define $\mathfrak{g} = \mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}$ *and* $\mathfrak{n}^{\pm}(A) := \mathfrak{n}^{\pm}(A)/(\tilde{\mathfrak{n}}^{\pm}(A) \cap \mathfrak{r})$.

Note: all ideals of $\mathfrak{g}$ *has nonzero intersection with* $\mathfrak{h}$*, and we will use the same notation* $\{e_i, f_i\}_{i\in I}, \mathfrak{h}$ *for generators of* $\mathfrak{g}$*. Relations of* $\tilde{\mathfrak{g}}$ *give relations for* $\mathfrak{g}$.

Remark 18.8. *If* A *is a Cartan matrix,* $\mathfrak{g}(A)$ *will just be the finite dimensional simple Lie algebra* $\mathfrak{g}$ *with* A *as its Cartan matrix.*

The earlier theorem implies:

- $\mathfrak{g}'(A) := [\mathfrak{g}(A), \mathfrak{g}(A)]$ *is generated by* $\{e_i, f_i\}_{i\in I}$ *and* $h_i = [e_i, f_i]$.
- $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathfrak{h}(A)$
- $\mathfrak{g}'(A) \cap \mathfrak{h}(A) = span_{\mathbb{C}}\Pi^\vee =: \mathfrak{h}'(A)$ *where* $span_{\mathbb{C}}\Pi^\vee = \bigoplus_{i\in I} \mathbb{C}h_i$*. So* $\mathfrak{g}'(A) \cap \mathfrak{g}(A)_\alpha = \mathfrak{g}(A)_\alpha$ *if* $\alpha \neq 0$.
- $\tilde{\omega}(\mathfrak{r}) \subseteq \mathfrak{r} \implies \tilde{\omega}$ *descends to involution* $\omega : \mathfrak{g}(A) \to \mathfrak{g}(A), \omega(e_i) = -f_i, \omega(f_i) = -e_i$*, and* $\omega\backslash\mathfrak{h} = -id(\mathfrak{h})$.
- $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *with* $\mathfrak{n}^{\pm} = \bigoplus_{\alpha\in Q^+\setminus\{0\}} \mathfrak{g}(A)_{\pm\alpha}$*, where* $\mathfrak{g}(A)_{\pm\alpha} = \tilde{\mathfrak{g}}(A)_{\pm\alpha}$*. The* Q*-graded Lie algebra is* $[\mathfrak{g}(A)_\alpha, \mathfrak{g}(A)_\beta] \subseteq \mathfrak{g}(A)_{\alpha+\beta}$.
- $mult(\alpha) := \dim(\mathfrak{g}(A)_\alpha) = \dim(\mathfrak{g}(A)_{-\alpha})$ *for* $\alpha \in Q^+$*. The basic estimate is* $mult(\alpha) \leq |I|^{ht(\alpha)}$ *and the finite case is* $mult(\alpha) = 1$.
- *Root system:* $\Phi := \{\alpha \in Q | \alpha \neq 0, mult(\alpha) > 0\}$ *where* $\Phi = \Phi^+ \cap (-\Phi^+), \Phi^+ = \Phi \cap Q^+$*. We have* $\mathfrak{n}^+(A)_\alpha = span_{\mathbb{C}}\{[e_{i1}, [e_{i2}, \ldots, [e_{i(s-1)}, e_{is}]\ldots]] \mid \alpha_{i1} + \cdots + \alpha_{is} = \alpha\}$ *for* $s \in \mathbb{Z}_{>1}$*. This implies* $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ *and* $\mathfrak{g}_{s\alpha_i} = 0$*. We can proceed similarly for* $\mathfrak{n}^-$.

Consider the class of all triples $(\mathfrak{g}, \mathfrak{h}, \Pi, \Pi^\vee)$ *such that*

- $\mathfrak{g}$ *Lie algebra over* $\mathbb{C}$*,* $\mathfrak{h} \subset \mathfrak{g}$ *finite dimensional abelian subalgebra*
- $\Pi^\vee = \{h_i\}_{i\in I} \subset \mathfrak{h}$ *and* $\Pi = \{\alpha_i\}_{i\in I} \subseteq \mathfrak{h}^*$ *are linearly independent*
- $\mathfrak{g}$ *is graded by* $\mathfrak{h} \cup \{e_i, f_i\}_{i\in I}$ *subject to the relations.*
- *All nonzero ideals* $i \subset \mathfrak{g}$ *contain a nonzero element of* $\mathfrak{h}$.
- $A := (\alpha_j(h_i))_{i,j\in I}$ *satisfies* $\dim \mathfrak{h} = |I| + corank(A)$.

The notion of isomorphism is as follows: $(\mathfrak{g}_1, \mathfrak{h}_1, \Pi_1, \Pi_1^\vee) \sim (\mathfrak{g}_2, \mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ *iff:* $\exists$ *Lie algebra isomorphism* $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ *such that* $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2, \varphi(\Pi_1^\vee) = \Pi_2^\vee, \varphi^*(\Pi_2) = \Pi_1$.

The notion of $\oplus$ *is direct sum of* $\mathfrak{g}, \mathfrak{h}$*, and the disjoint union of* $\Pi, \Pi^\vee$.

Proposition 18.9 ([@kac theorem 1.4). *]*

The asignment $A \mapsto (\mathfrak{g}(A), \mathfrak{h}(A), \Pi, \Pi^\vee)$ *and the assignment* $(\mathfrak{g}, \mathfrak{h}, \Pi, \Pi^\vee) \mapsto A := (\alpha_j(h_i))_{i,j \in I}$ *defines a bijection*

$$\{\textit{square matrices over } \mathbb{C}\}/\textit{relabelling index sets} \xrightarrow{\sim} \{\textit{tuples satisfying above conditions}\}/\sim$$

compatible with $\oplus$. *]*

Goal 1*: describe the ideals.*

Proposition 18.10 ([@kac theorem 1.6). *]*

1. $Z(\mathfrak{g}) = \{x \in \mathfrak{h} | \forall j \in I, \alpha_j(x) = 0\}$.
2. $\dim(Z(\mathfrak{g})) = \textit{corank}(A)$.
3. $Z(\mathfrak{g}) \subset \mathfrak{h}'$.

]

Proof.

1. If $x \in Z(\mathfrak{h})$, then $[h, x] = 0 \forall h \in \mathfrak{h} \implies x \in \mathfrak{g}_0 = \mathfrak{h}$. Hence $0 = [x, e_j] = \alpha_j(x) e_j \implies \alpha_j(x) = 0$. Conversely, if $\alpha_j(x) = 0$ for $x \in \mathfrak{h}$, $[x, e_j] = [x, f_j] = 0 = [x, h] \forall h \in \mathfrak{h}$.
2. Since $\{\alpha_j\}_{j \in I}$ is linearly independent, $\dim(Z(\mathfrak{g})) = \dim(\mathfrak{h}) - |I| = \text{corank}(A)$.
3. Let $\{c^{(s)}\}_{1 \leq s \leq \text{corank}(A)}$ be a basis for $\text{Ker}(A^t) = \text{Coker}(A)$. Then one checks $\{\sum_{i \in I} c_i^{(s)} h_j\}_{1 \leq s \leq \text{corank}(A)}$ is a linearly independent subset of $Z(\mathfrak{g}) \cap \mathfrak{h}' \implies \dim(Z(\mathfrak{g}) \cap \mathfrak{h}') \geq \text{corank}(A) \implies Z(\mathfrak{g}) \subset \mathfrak{h}'$.

□

Call A ***connected*** *if* $\forall j, k \in I$ *such that* $j \neq k, \exists i_0, i_1, ..., i_\ell \in I, j = i_0, k = i_\ell$, *and* $a_{i_0 i_1}, a_{i_1 i_2}, ..., a_{i_{\ell-1} i_\ell} \neq 0$. *The* ***distance*** *between j and k is the minimum of such* ℓ.

Example 18.11. $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ *is not connected because there is no path down from 2 to 1.* $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ *is connected.*

Proposition 18.12 ([@kac prop 1.7). *]*

1. *If i is an ideal of* $\mathfrak{g}$, *then (*$\mathfrak{g}$ *is "almost simple") either* $i \subseteq Z(\mathfrak{g})$ *or* $g' \subseteq i$.
2. $\mathfrak{g}(A)$ *is simple* $\iff$ *det*$A \neq 0$.

]

Proof.

1. Assume $i \subsetneq Z(\mathfrak{g})$. Need to show $\forall k \in I, e_k, f_k \in i$. Because $i \cap \mathfrak{h} \neq 0$, choose $h \in i \cap \mathfrak{h}, h \notin Z(\mathfrak{g})$. $\exists j \in I$ such that $\alpha_j(h) = 0 \implies [h, e_j] = \alpha_j(h)e_j, [h, f_j] = -\alpha_j(h)f_j \implies e_j, h_j, f_j \in i$. Use connectedness and adjoint action of $\mathfrak{h}'$ to show $e_h, f_h \in i$ for all k. (induction with respect to distance).
2. By the previous prop, $\dim(Z(\mathfrak{g})) = \operatorname{corank}(A)$. Also, $\operatorname{codim}(\mathfrak{g}') = \operatorname{corank}(A)$. Hence if A invertible, $Z(\mathfrak{g}) = 0, \mathfrak{g}' = \mathfrak{g}$.

□

Exercise 18.13. *If $\forall j, k \in I, a_{jk} = 0 \implies a_{kj} = 0$, then show A indecomposable $\iff$ A connected. Hint: connectedness is equivalence relation on I. Reorder I accordingly.*

For suitable A, we can deduce the ***Serre relations****.*

Lemma 18.14 ([@kac lemma 1.5)**.** *] Let $x \in \mathfrak{n}^-$ such that $\forall k \in I, [e_k, x] = 0$ then $x = 0$.]*

Remark 18.15. *By appplying ω we get similar result for $\mathfrak{n}^+$.*

Proof. Let $F = \bigoplus_{j \in I} \mathbb{C} f_j \subset \mathfrak{n}^-$ subspace. Let $i = \sum_{\ell, m \in \mathbb{Z}_{\geq 0}} (\operatorname{ad} F)^\ell (\operatorname{ad} \mathfrak{h})^m x \subseteq \mathfrak{n}^-$.

Claim: i is an ideal.

Because $i \cap \mathfrak{h} = 0$, $i = 0$, so $x = 0$.

Exercise 18.16. *Prove that i is an ideal.*

□

Goal 2: *Assume A is indecomposable. Decide for which A we get Serre relations in $\mathfrak{g}(A)$: if $i, j \subset I$, $i \neq j$, $ad(e_i)^{1-a_{ij}}(e_j) = 0 = ad(f_i)^{\lambda a_{ij}}(f_j)$ and relations with $i \iff j$ but no other relations involving just e_i, e_j, or just f_i, f_j. Clearly, we need $a_{ij} \in \mathbb{Z}_{\geq 0}$ for all i, j with $i \neq j$. ($a_{ij} = 1 \implies e_j = 0 = f_j \implies h_j = 0$). Use ω to get relations for e_i, e_j from relations for f_i, f_j. View $\mathfrak{g}$ as a module over $\mathcal{U}(\mathfrak{g}_i)$ (adjoint action) $\mathfrak{g}_i = \mathbb{C}\langle e_i, f_i, h_i \rangle$. By the lemma, the desired relations are obtained if*

$$\forall k \in I, [e_k, f_i^{1-a_{ij}} \cdot f_j] = 0$$

where the $\cdot$ is the action of $\mathcal{U}(\mathfrak{g}_i)$.

We also require $f_i^\ell \cdot f_j \neq 0 \forall \ell < 1 - a_{ij}$. Only nontrivial cases: $k = i$ or $k = j$.

For $k = i$:

Exercise 18.17. *Show in* $\mathcal{U}(\mathfrak{g}_i)$*, we have* $[e_i, f_i^m] = mf_i^{m-1}(h_i + \frac{1-m}{2}a_{ii})$ *for* $m \in \mathbb{Z}_{>0}$*. ¡/1¿*

Hence

$$\begin{aligned}[e_i, f_i^{1-a_{ij}} \cdot f_j] &= e_i \cdot (f_i^{1-a_{ij}} \cdot f_j) \\ &= [e_i, f_i^{1-a_{ij}}] \cdot f_j \\ &= (1-a_{ij}) f_i^{-a_{ij}} \left(h_i + a_{ij}\frac{a_{ii}}{2}\right) \cdot f_j \\ &= (1-a_{ij}) a_{ij} \left(\frac{a_{ii}}{2} - 1\right) f_i^{-a_{ij}} \cdot v.\end{aligned}$$

By indecomposability, $\exists j$ *such that* $a_{ij} \neq 0 \implies a_{ii} = 2$ *(*$\implies \mathfrak{g}_i \cong \mathfrak{sl}_2$*).*

For $k = j$*:* $[e_j, f_i^{1-a_{ij}} \cdot f_j] = ad(f_i)^{1-a_{ij}}(h_j)$*. If* $a_{ij} = 0$*, this equals* $[f_i, h_j] = a_{ji} f_i$*, need* $a_{ji} = 0$*. If* $a_{ij} < 0$*, this equals* $ad(f_i)^{-a_{ij}}(a_{ji} f_i) = 0$*.*

Definition 18.18. *Call* $A = (a_{ij})_{i,j \in I}$ *a* ***generalized Cartan matrix*** *if* $a_{ii} = 2, i \neq j \implies a_{ij} \in \mathbb{Z}_{\leq 0}$*, and* $a_{ij} = 0 \implies a_{ji} = 0$*. Call* $\mathfrak{g}(A)$ *a* ***Kac-Moody algebra****. ¡/0¿*

Remark 18.19. *1. For suitable* A*, we will see that there is a presentation of* $\mathfrak{g}(A)$ *involving Serre relations.*

2. If A *is a generalized Cartan matrix then for all* $J \subseteq I$*,* $A_J := (a_{ij})_{i,j \in J}$ *is a generalized Cartan matrix.*

18.4 Types of Kac-Moody Algebras and Generalizations

Finnite type*: If* A *is a generalized Cartan matrix and* $det(A_J) > 0$ *for all* $J \subseteq I$*, then it is an ordinary Cartan matrix, called* ***general Cartan matrix of finite type****.*

Example 18.20. $(2), \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \mathfrak{sl}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{so}_5/\mathfrak{sp}_4, \mathfrak{g}_2$*, etc.*

Since $\mathfrak{g}(A)$ *is simple (*A *is invertible) we have the Lie algebra homomorphism*

$$\varphi : (\textit{finite dimensional simple Lie algebras with Cartan matrix } A) \to \mathfrak{g}(A)$$

with $ker\varphi = 0, im\varphi = \mathfrak{g}(A)$*, and mappings* $e_i \mapsto e_i, f_i \mapsto f_i, h_i \mapsto h_i$*. This implies* φ *is an isomorphism and* $\dim(\mathfrak{g}(A)) < \infty$*.*

There are two important generalizations:

1. *Relax conditions on generalized Cartan matrix to* $a_{ii} \in \{2\} \cup \mathbb{Z}_{\leq 0}, a_{ij} \leq 0$ *if* $i \neq j$*,* $aij \in \mathbb{Z}$ *if* $a_{ii} = 2$*, and* $a_{ij} = 0 \implies a_{ji} = 0$ *(not neccesarily integer). Then we can define generalization of* $\mathfrak{g}(A)$ *called* ***Borcherds***

algebra. *The Serre relations* $ad(e_i)^{1-a_{ij}}(e_j) = 0$ *are imposed only if* $a_{ii} = 2$. *(If* $a_{ij} = 0$ *you can only impose* $[e_i, e_j] = 0$ *and similarly for* f_i, h_i*.) This was used by Borcherds in the proof of the Conway Norton conjectures on the representation theory of the "monster" simple group.*

2. *Allow* $a_{ij} \in \mathbb{Z}_{\geq 0}$ *for certain* i, j *with* $i \neq j$ *provided that* a_{ij} *and* a_{ji} *have the same sign. The Serre relations are replaced by*

$$\begin{aligned} ad(e_i)^{1+a_{ij}}(f_j) &= 0 \\ ad(f_i)^{1+a_{ij}}(e_j) &= 0 \\ ad(e_i)(e_j) &= 0 \\ ad(f_i)(f_j) &= 0. \end{aligned}$$

The resulting algebras are called ***generalized intersection matrix algebras***. *These also arise as fixed point subalgebras of Cartan involutions on Kac-Moody algebras.*

If $A = (a_{ij})_{i,j \in I}$ *is a generalized Cartan matrix such that all principal minors* $A_J = (a_{ij})_{i,j \in J}$ *with* $J \subseteq I$ *have* $det(A_J) > 0$ *then call A "of finite type," then* $\mathfrak{g}(A)$ *is finite dimensional and semisimple. The classication is in terms of* ***Dynkin diagrams***. *The ordered multigraph* $\Gamma(A)$ *such that* $\{vertices\} = I \forall i, j \in I, i \neq j$:

INSERT DIAGRAMS

Definition 18.21. *A generalized Cartan matrix* $A = (a_{ij})_{i,j \in I}$ *is called of* ***affine type*** *if*

- $detA = 0$
- $detA_J > 0 \forall J \subsetneq I \iff detA = 0$ *and* $\forall i \in I, A_{I \setminus \{i\}}$ *is of finite type.*

We can use the classifications of generalized Cartan matrices of finite type to gegt classificiation generalized Cartan matrices of affine type.

Example 18.22. $|I| = 1, A = (2)$ *is of finite type* ($\mathfrak{sl}_2$).

Exercise 18.23. *Show all generalized Cartan martices A with* $|I| = 2$ *of affine type are*

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Add $\overset{i}{\cdot}\!=\!\overset{j}{\cdot} \iff a_{ij} = a_{ji} = -2$ *and* $\overset{i}{\cdot}\!\Rrightarrow\!\overset{j}{\cdot} \iff a_{ij} = -1, a_{ji} = -4$.

Assume A is indecomposable. To help with the classification, note the following about generalized Cartan matrices of finite type:

- *Only one branch point can occur with only three branches (also length of the two branches is constrained)*

- *Only one non-simple edge can occur* $\cdot \Longrightarrow \cdot$ *or* $\cdot \Rrightarrow \cdot$ *(length of one of the two branches is constrained)*
- *No cycles can occur*

Thus, the Dynkin diagrams of affine type are:

INSERT DIAGRAM

We have the "Dynkin" notation with : subscript $=$ *rank*$(A) = |I| - 1 = \dim \mathfrak{h}(A) - 2$.

We also have the "Kac" notation $X_r^{(m)}$ *which refers more precisely to the construction of the affine Kac-Moody algebra as the extesnion of the untwistied/twisted loop algebra of a finite dimensional simple Lie algebra* $\mathfrak{g}_{fin}$ *where* $X =$ *Lie type of* $\mathfrak{g}_{fin}, r =$ *rank of* $\mathfrak{g}_{fin}$*, and* $m =$ *order of automorphism of* $\mathfrak{g}_{fin}$.

Recall the construction of ext affine Lie algebras: we have $\mathfrak{g}(A) = \mathfrak{g}_{fin}$ *a finite dimensional simple Lie algebra, which gives* $\hat{\mathfrak{g}}_{fin}^{ext} = (\mathfrak{g}_{fin} \otimes \mathbb{C}[z, z^{-1}]) \otimes \mathbb{C}c \oplus \mathbb{C}d$. *We have*

	A	$\hat{A}$
$\mathfrak{sl}_2$	(2)	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$
$\mathfrak{sl}_3$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$
$\mathfrak{sp}_4$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$

We dfeined a new $\mathfrak{sl}_2$ *triple* $\{e_0, f_0, h_0\} \subset \hat{\mathfrak{g}}_{fin} \subset \hat{\mathfrak{g}}_{fin}^{ext}$ *which gives a simple root of* $a_0 = \delta - \theta$*, where* δ *gives* $\mathfrak{sl}_{\mathfrak{h}_{fin}} = 0, \delta(c) = 0, \delta(d) = 1$*, and* θ *is the highest root of* $\mathfrak{g}_{fin}$. *We also have new generators* $\{e_i, f_i, h_i\}_{i=0}^r \cup \{d\}$ *that satisfy the Kac-Moody relations, and that all ideals of* $\hat{\mathfrak{g}}_{fin}^{ext}$ *contain element of* $\hat{\mathfrak{h}}_{fin}^{ext} = \mathfrak{h}_{fin} \oplus \mathbb{C}c \oplus \mathbb{C}d$. *Set* $\hat{A} = (\alpha_j(h_i))_{i,j=0}^r$. *We can verify*

$$\dim \hat{\mathfrak{h}}_{fin}^{ext} = |I| + corank\left(\hat{A}\right).$$

Exercise 18.24. *Show that rank* $\left(\hat{A}\right) = r$*, using linear independence of* $\{\alpha_i\}_{i=0}^r$.

So we have a trichotomy of generalized Cartan matrices into finite, affine, and indefinite types, where the affine and indefinite types are together called the infinite type. In [@kac] and [@carter] these three types are defined in a different way, using the action of A *on vectors of nonnegative real numbers.*

Proposition 18.25 ([@kac proposition 4.7). *] The two definitions are equivalent if A is* ***symmetrizable****:* $\exists(\epsilon_i)_{i\in I} \in (\mathbb{Q}^\times)^I$ *such that* $\forall i,j \in I, \epsilon_i \cdot a_{ij} = \epsilon_j \cdot a_{ji}$ *(exists a diagonal invertible matrix D such that DA is symmetric).]*

Exercise 18.26. *1. Show, wlog, can assume* $\{e_i\}$ *are setwise coprime integers.*

2. Find such $(\epsilon_i)_{i\in I}$ *for*

$$\begin{pmatrix} 2 & -1 \\ -m & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

where $m \in \mathbb{Z}_{>0}$.

Here is a collection of facts:

Proposition 18.27. *1. A generalized Cartan matrix A is symmetrizable* $\iff \forall\{i_1, ..., i_k\} \subseteq I, a_{i_1 i_2} a_{i_2 i_3} ... a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} ... a_{i_1 i_k}$.

2. $\epsilon_i > e_j \iff$ *orientation in finite/affine Dynkin diagrams.*

3. If A is of finite or affine type, then it is symmetrizable.

18.5 Invertible Symmetric Bilinear Form on $\mathfrak{g}(A)$

Assume A is indecomposable, and A is symmetrizable ($\implies i, j, \frac{1}{\epsilon_j} a_{ij} = \frac{1}{\epsilon_i} a_{ji}$*). Define a symmetric bilinear form on* $span_{\mathbb{C}}\{h_i\}_{i\in I} = \mathfrak{h}'$*:*

$$(h_i|h_j) := \frac{1}{\epsilon_j} a_{ij}.$$

Extend to a symmetric bilinear form on $\mathfrak{h}(A)$ *by choosing a complement* $\mathfrak{h}''$ *(*$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$*) and set* $(x|h_j) = \epsilon_j^{-1} \alpha_j(x) \forall x \in \mathfrak{h}(A)$.

Proposition 18.28 ([@carter 16.1). *] This form is nondegenerate:* $\forall y \in \mathfrak{h}, (x|y) = 0 \implies x = 0$. *]*

Example 18.29. *For* $\hat{\mathfrak{sl}}_2^{ext}$, $\hat{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ *with* $\epsilon_0 = \epsilon_1 = 1$*, and* $(h_0|h_0) = 2 = (h_1|h_1), (h_0|h_1) = -2$.

Choose $\alpha \in \hat{\mathfrak{h}}^{ext} \backslash (\mathbb{C}h_0 \oplus \mathbb{C}h_1)$ *such that* $\alpha_0(d) = 1, \alpha_1(d) = 0 \implies (d|h_0) = 1, (d|h_1) = 0$*, and* $(d|d) = 0$.

We have a canonical linear isomorphism $\nu : \mathfrak{h}(A) \to \mathfrak{h}(A)^\star$*:*

$$\nu(h)(h') = (h|h')$$

with $h, L' \in \mathfrak{h}(A)$.

Exercise 18.30. *Show* $\forall i \in I, \alpha_i = \epsilon_i \nu(h_u)$.

This gives nondegenerate bilinear form on $\mathfrak{h}(A)^*$ *by* $(\nu(h)|\nu(h')) := (h|h')$ *where* $h, h' \in \mathfrak{h}(A)$ *which satisfies*

$$\begin{aligned}(\alpha_i|\alpha_j) &= (\epsilon_i \nu(h_i)|\epsilon_j \nu(h_j)) \\ &= \epsilon_i \epsilon_j (h_i|h_j) \\ &= \epsilon_i a_{ij} \\ &= \epsilon_j a_{ji}.\end{aligned}$$

This implies $(\alpha_i|\alpha_i) = 2\epsilon_i$ *(*e_i *proportional to "squared length" of* α_i*) and symmetrizability corresponds to having a consistent assignment of length of simple roots.*

Proposition 18.31 (@carter theorem 16.2)**.** $\mathfrak{g}(A)$ *has a nondegenerate invariant symmetric bilinear form.*

Proof. We will sketch the proof. Start with a principal grading of $\mathfrak{g}(A)$: $\mathfrak{g} = \bigoplus_{h \in \mathbb{Z}} g[h]$ where $\mathfrak{g}[h] = \bigoplus_{\alpha \in Q, \mathrm{ht}(\alpha)=h} \mathfrak{g}_\alpha$, $\mathrm{ht}\left(\sum_{i \in I} m_i \alpha_i\right) = \sum_{i \in I} m_i \in \mathbb{Z}$. Recursively define $(\cdot|\cdot)$ by induction with respect to $|h| \in \mathbb{Z}_{\geq 0}$. For $h = 0$: Cartan subalgebra $\mathfrak{h}(A)$.

For $|h| = 1$, set $\mathfrak{g}[-1, 0, 1] := \mathfrak{g}[-1] \oplus \mathfrak{g}[0] \oplus \mathfrak{g}[1]$, where $g[1] = \bigoplus_{i \in I} \mathbb{C} e_i$ and $\mathfrak{g}[-1] = \bigoplus_{i \in I} \mathbb{C} f_i$. Extend $(\cdot|\cdot)$ from $\mathfrak{g}[0] = \mathfrak{h}$ to $\mathfrak{g}[-1, 0, 1]$:

$$(f_i|e_j) = (e_j|f_i) = \frac{1}{\epsilon_i}\delta_{ij}$$
$$(e_i|h) = (h|e_i) = (f_i|h) = (h|f_i) = 0$$

For invariance, one example:

$$(h_i|[e_j, f_k]) = (h_i|\delta_{jk} h_j) = \delta_{jk} \frac{1}{\epsilon_j} a_{ij}$$
$$([h_i, e_j]|f_k) = (a_{ij} e_j | f_k) = a_{ij} \frac{1}{\epsilon_j} \delta_{jk}.$$

For nondegeneracy: let $i = \{x \in \mathfrak{g} | \forall y \in \mathfrak{g}, (x|y) = 0\}$.

Exercise 18.32. *Use invariance of* $(\cdot|\cdot)$ *to show* i *is an ideal.*

Now it follows: if $i \neq 0, \exists h \in \mathfrak{h} \backslash \{0\}, h \in i$. But $(\cdot|\cdot)|_{\mathfrak{h} \times \mathfrak{h}}$ is nondegenerate, so $i = 0$. ☐

Remark 18.33. *1. The form constructed here* $(e_i|f_j) = \frac{1}{\epsilon_i}\delta_{ij}$ *is called the* ***standard invariant form****. This is not equal to Killing form of* $\mathfrak{g}(A)$ *if* A *is of finite type.*

2. *Later, we will use $(\cdot|\cdot)$ to define the generalization of the Casimir element, acting on suitable representations.*

Exercise 18.34. *Show that $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ if $\alpha \neq -\beta$ and deduce $(\cdot|\cdot)|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is nondegenerate.*

Proposition 18.35. *Suppose $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ both nonzero. Then $[x,y] = (x|y)\nu^{-1}(\alpha) \in \mathfrak{h}\backslash\{0\}$.*

Proof. Let $h \in \mathfrak{h}$ be arbitrary. Consider

$$\begin{aligned}([x,y] - (x|y)\nu^{-1}\alpha|h) &= ([x,y]|h) - (x|y)(\nu^{-1}(\alpha)|h) \\ &= (x|[y,h]) - (x|y)\alpha(h) \\ &= 0.\end{aligned}$$

So with nondegeneracy, $[x,y] = (x|y)\nu^{-1}(\alpha)$. ☐

19 Representation Theory of Kac-Moody Algebras

19.1 Integrable Modules of $\mathfrak{g}(A)$

*Let V be a vector space over $\mathbb{C}$. Call $a \in End(V)$ **locally nilpotent** if $\forall v \in V, \exists m \in \mathbb{Z}_{\geq 0}$ such that $a^m(v) = 0$. Then $exp(a) = Id_V + a + \frac{1}{2}a^2 + ...$ is well-defined, invertible, and linearly independent. Additionally, we have $exp(a)^t = exp(ta)$ for $t \in \mathbb{Z}$.*

Remark 19.1. *For $\dim(V) < \infty$: nilpotent $\iff$ locally nilpotent.*

Example 19.2. *$\frac{d}{dx} : \mathbb{C}[x] \to \mathbb{C}[x]$ is locally nilpotent, but not nilpotent.*

Exercise 19.3. *Let $\mathfrak{g}$ be any Lie algebra over $\mathbb{C}$ with generating set $\{y_j\}$. Suppose $x \in \mathfrak{g}$ such that $\forall j, \exists m_j \in \mathbb{Z}_{\geq 0}$ with $ad(x)^{m_j}(y_j)$.*

1. *Show that $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is locally nlipotent. Hint: show $ad(x)^m([y,x]) = \sum_{\ell=0}^{m} \binom{m}{\ell} [ad(x)^\ell(y), ad(x)^{m-\ell}(z)] \forall y, z \in \mathfrak{g}$.*

2. *Show $exp(ad(x)) : \mathfrak{g} \to \mathfrak{g}$ is a well-defined Lie algebra automorphism.*

3. *If $\mathfrak{g} = \mathfrak{g}(A)$, A a generalized Cartan matrix, show that $ad(e_i)$ and $ad(f_i)$ are locally nilpotent for all $i \in I$.*

Let X be any associative $\mathbb{C}$-algebra. Fix $a \in X$. Define $ad(a) := \ell_a - r_a : X \to X$ a linear map, where ℓ_a and r_a are left and right mutliplication by a, respectively. We have $\ell_a = ad(a) + r_a$, so by the binomial theorem, $a^k x = \sum_{s=0}^{k} \binom{k}{s} ad(a)^s(x) a^{k-s} \forall x \in X, k \in \mathbb{Z}_{\geq 0}$, which implies that $exp(a) \cdot x = exp(ad(a))(x) \cdot exp(a)$.

Define $Ad(y)$ by conjugation by y. Then $Adexp(a) = exp(ad(a)) \in GL(X)$. We take $X = End(V)$, where V is a $\mathfrak{g}(A)$-module where e_i and f_i act locally nilpotently. View $exp(e_i), exp(f_i)$ as elements in a group associated to $\mathfrak{g}(A)$. This gives the ***Kac-Moody group*** *$G(A)$.*

Fix $i \in I$ and take $\mathfrak{g}_i = \langle e_i, f_i \rangle \cong \mathfrak{sl}_2$. The interesting element of $GL(V)$: $T_i = T_i^{(V)} = exp(e_i) \circ exp(-f_i) \circ exp(e_i)$. THe identity is $e_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $T_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$. Additionally, we have $n_i = Ad(T_i) = exp(ad(e_i)) \circ exp(-ad(f_i)) \circ exp(ad(e_i)) \in Aut_{Lie}(\mathfrak{g}(A))$.

Exercise 19.4. *Show (in $\mathfrak{g}_i$): $n_i(e_i) = -f_i, n_i(f_i) = -e_i, n_i(h_i) = -h_i$.*

Suppose $\exists$ finite dimensional irreducible submodule $U \subseteq V$ (as $\mathfrak{g}$-module) where $\dim(U) = n, U = \bigoplus_{s=0}^{n-1} f_i^s \cdot u^t$ where u^0 is the highest weight vector $e_i(u^+) = 0$ (similarly, define $u^- := f_i^{n-1}(u^+)$ as the lowest weight vector), n_i is a surjective Lie algebra homomorphism $\mathfrak{g}_i \to \mathfrak{g}_i$, and

$$\begin{aligned} T_i u^- &= T_i f_i^{n-1}(u^+) \\ &= n_i(f_i)^{n-1} T_i(u^+) \\ &= (scalar) e_i^{n-1} T_i(u^+) \end{aligned}$$

Definition 19.5. *A $\mathfrak{g}(A)$-weight module is a $\mathfrak{g}(A)$-module V such that $V = \bigoplus_{\lambda \in \mathfrak{h}(A)^*} V_\lambda$ where $V_\lambda = \{v \in V | h \cdot v = \lambda(h) v \forall h \in \mathfrak{h}(A)\}$ is the weight space. Furthermore, define $SuppV = \{\lambda \in \mathfrak{h}(A)^* | V_\lambda \neq 0\}$ and $mult_V(\lambda) = \dim V_\lambda \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A $\mathfrak{g}(A)$-weight module V is called* ***integrable*** *if e_i, f_i act locally nilpotently for each $i \in I$.*

Example 19.6.

- *$\mathfrak{g}(A)$ itself is integrable under the adjoint action.*
- *A is generalized Cartan matrix of finite type, $\mathfrak{g}(A)$ is finite dimensional semisimple, finite dimensional $\mathfrak{g}(A)$-modules are integrable modules.*

Proposition 19.7. *If V is an integrable $\mathfrak{g}(A)$-module then $\forall i \in I, V$ decomposes as a direct sum of finite dimensional irreducible $\mathfrak{g}_i$-modules, which are $\mathfrak{h}(A)$-invaraint.*

Remark 19.8. *Hence, the action of $\mathfrak{g}_i \cong \mathfrak{sl}_2$ can be "integrated" to a action of $SL_2(\mathbb{C})$ via exp.*

Proof. We provide a sketch, see [@kac proposition 3.6] or [@carter proposition 19.13] for more details.

It suffices to show that each $v \in V_\lambda$ (where $\lambda \in \text{Supp}(v)$) lies in a direct sum of $\mathfrak{g}_i$-modules. Using relations, we can show that $\mathcal{U} = \text{span}_{\mathbb{C}}\{f_i^k e_i^\ell \cdot v\}_{k,\ell \in \mathbb{Z}_{\geq 0}}$ is a $\langle \mathfrak{g}, \mathfrak{h} \rangle$-module. Local nilpotency of $e_i, f_i \implies \dim(\mathcal{U}) < \infty$. Weyl complete reducibility gives $\mathcal{U} = \bigoplus_j$ irreducibile finite dimensional $\mathfrak{g}_i$-module $\mathcal{U}_j$. Because $f_i^k e_i^\ell \cdot v$ are weight vectors, we can show $\mathcal{U}_j$ is a $\mathfrak{h}$-module. ∎

The invertible map T_i acts on any integrable module V. $\forall\lambda \in SuppV$, the set

$$M_i(\lambda) = (\lambda + \mathbb{Z}\alpha_i) \cap SuppV$$

has natural involution, preserving multiplicities, given by

$$\lambda + t\alpha_i \mapsto \lambda + (-\lambda(h_i) - t)\alpha_i$$

for $t \in \mathbb{Z}$.

19.2 Weyl Group of $\mathfrak{g}(A)$

Assume A is symmetrizable and indecomposable. Fix $i \in I$. We have

$$n_i = Ad(T_i) = exp(ad(e_i)) \circ exp(-ad(f_i)) \circ exp(ad(e_i)) \in Aut_{Lie}(\mathfrak{g}).$$

Proposition 19.9. *n_i preserves $\mathfrak{h} = \mathfrak{h}(A)$. More precisely,*

$$n_i(h) = h - \alpha_i(h)h_i \qquad (\star)$$

for $h \in \mathfrak{h}$.

Proof. Computation, see [@carter proposition 16.11]. □

*Let $s_i := n_i|_{\mathfrak{h}}$. From $(\star)$ it follows that $s_i^2 = id_{\mathfrak{h}}, s_i(h_i) = -h_i$, and $s_i(h) = h \iff \alpha_i(h) = 0 \iff (h|h_i) = 0$. The **Weyl group** of $\mathfrak{g}(A)$ is $W = \langle s_i | i \in I\rangle \subset GL(\mathfrak{h})$.*

Let $i \in I, x, y \in \mathfrak{h}$. Then

$$\begin{aligned}(s_i x|s_i y) &= (x - \alpha_i(x)h_i|y - \alpha_i(y)h_i)\\ &= (x|y) - \alpha_i(x)(h_i|y) - \alpha_i(y)(x|h_i) + \alpha_i(x)\alpha_i(y)(h_i|h_i)\\ &= (x|y) - \alpha_i(x)\epsilon_i^{-1}\alpha_i(y) - \alpha_i(y)\epsilon_i^{-1}\alpha_i(x) + 2\epsilon_i^{-1}\alpha_i(x)\alpha_i(y)\\ &= (x|y)\end{aligned}$$

which tells us that $(\cdot|\cdot)$ is W-invariant.

Define the action of W on $\mathfrak{h}^*$ via: for $\lambda \in \mathfrak{h}^*$, $w(\lambda) = \lambda \circ w^{-1}$, ie. $w(\lambda)(h) = \lambda(w^{-1}(h))$. Compatible with isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ induced by $(\cdot|\cdot)$, $w \circ \nu = \nu \circ w$.

Exercise 19.10. *Show, using $(\star)$, that $s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ where $i \in I, \lambda \in \mathfrak{h}^*$, and verify the W-invariance of $(\cdot|\cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$.*

Proposition 19.11. *$n_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)} \forall\alpha \in \Phi, i \in I$. So*

$$\begin{aligned}[h, n_i(x)] &= n_i([s_i(h), x])\\ &= n_i([h - \alpha_i(h)h_i, x])\\ &= (\alpha(h) - \alpha_i(h)\alpha(h_i))n_i(x)\\ &= s_i(\alpha)(h)n_i(x).\end{aligned}$$

This implies $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w(\alpha)}, w(\Phi) = \Phi$ *for* $w \in W$.

We can study the order m_{ij} *of* $s_i s_j \in W$ *where* $i, j \in I, i \neq j$ *using the linear independence of*